6.3 Step Functions

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities. As we saw in the mixing problem of Section 7.1, jump discontinuities occur naturally in any physical situation that involves switching. Finding the Laplace transforms of such functions is straightforward; however, we need some theory for inverting these transforms. To facilitate this, Oliver Heaviside introduced the following step function.

Unit Step Function

Definition 5. The **unit step function** $u(t)$ is defined by

$$(1) \qquad u(t) := \begin{cases} 0, & t < 0, \\ 1, & 0 < t. \end{cases}$$

(Any Riemann integral, like the Laplace transform, of a function is unaffected if the integrand's value at a single point is changed by a finite amount. Therefore, we do not specify a value for $u(t)$ at $t = 0$.)

By shifting the argument of $u(t)$, the jump can be moved to a different location. That is,

$$(2)\quad u_a(t) = u(t-a) = \begin{cases} 0, & t-a<0, \\ 1, & 0<t-a \end{cases} = \begin{cases} 0, & t<a \\ 1, & a<t \end{cases}$$

has its jump at $t = a$. By multiplying by a constant M, the height of the jump can also be modified:

$$Mu(t-a) = \begin{cases} 0, & t<a, \\ M, & a<t. \end{cases}$$

$$5u(t-3) = \begin{cases} 0, & t<3 \\ 5, & t>3 \end{cases}$$

See Figure 7.8.

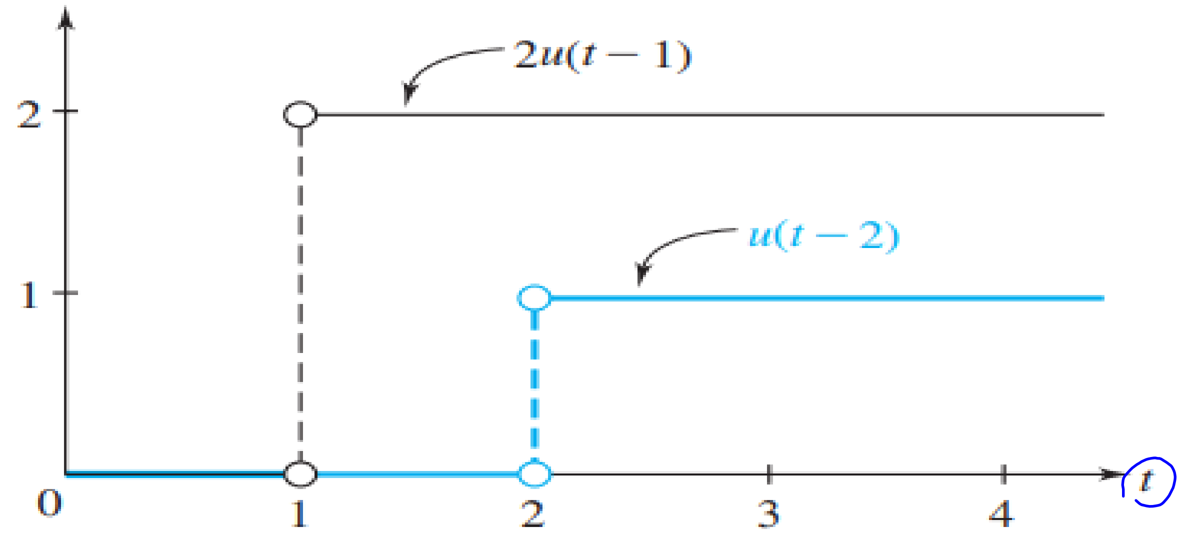

Figure 7.8 Two-step functions expressed using the unit step function

Rectangular Window Function

Definition 6. The **rectangular window function** $\Pi_{a,b}(t)$ is defined by[†]

(3)
$$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a, \\ 1, & a < t < b, \\ 0, & b < t. \end{cases}$$

The function $\Pi_{a,b}(t)$ is displayed in Figure 7.9

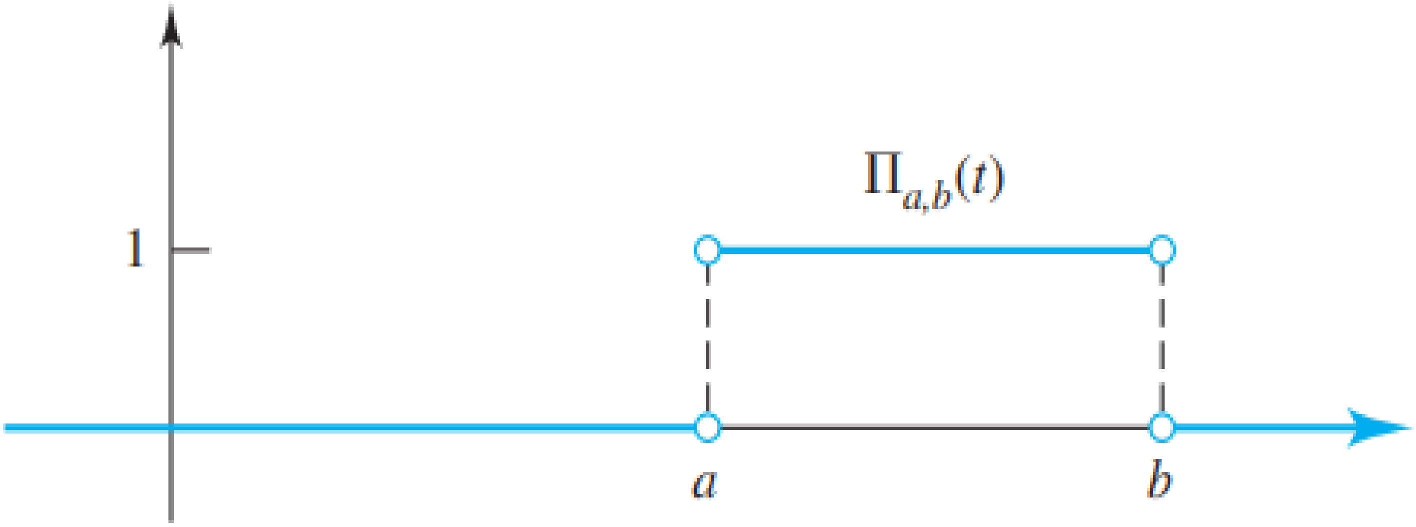

Figure 7.9 The rectangular window

Example 1 Write the function

$$(4) \qquad f(t) = \begin{cases} 3, & 0 < t < 2, \\ 1, & 2 < t < 5, \\ t, & 5 < t < 8, \\ t^2/10, & 8 < t \end{cases}$$

in terms of window and step functions.

$$f(t) = 3\,\Pi_{0,2} + \Pi_{2,5} + t\,\Pi_{5,8} + \frac{t^2}{10}\,u(t-8)$$

Solution Clearly, from the figure we want to window the function in the intervals (0, 2), (2, 5), and (5, 8), and to introduce a step for $t > 8$. From (5) we read off the desired representation as

(5) $$f(t) = 3\Pi_{0,2}(t) + 1\Pi_{2,5}(t) + t\Pi_{5,8}(t) + (t^2/10)u(t-8). \blacklozenge$$

Translation in s

Theorem 3. If the Laplace transform $\mathscr{L}\{f\}(s) = F(s)$ exists for $s > \alpha$, then

(1) $$\mathscr{L}\{e^{at}f(t)\}(s) = F(s-a) = \mathscr{L}\{f(t)\} \quad s' \to s'-a$$

for $s > \alpha + a$.

Proof. We simply compute

$$\begin{aligned}\mathscr{L}\{e^{at}f(t)\}(s) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\ &= F(s-a)\,. \quad \blacklozenge\end{aligned}$$

Translation in t

Theorem 8. Let $F(s) = \mathcal{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

(8) $$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s),$$

and, conversely, an inverse Laplace transform† of $e^{-as}F(s)$ is given by

(9) $$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a).$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\} \quad u(t-a)$$

$t \to t-a$

Proof. By the definition of the Laplace transform,

$$\textbf{(10)} \qquad \mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt\,,$$

where, in the last equation, we used the fact that $u(t-a)$ is zero for $t < a$ and equals 1 for $t > a$. Now let $v = t - a$. Then we have $dv = dt$, and equation (10) becomes

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-as} e^{-sv} f(v)\,dv$$

$$= e^{-as}\int_0^\infty e^{-sv} f(v)\,dv = e^{-as}F(s)\,. \quad \blacklozenge$$

Remark:

(8) $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s) = e^{-as}\mathcal{L}\{f(t)\}$

In practice it is more common to be faced with the problem of computing the transform of a function expressed as $g(t)u(t-a)$ rather than $f(t-a)u(t-a)$. To compute $\mathcal{L}\{g(t)u(t-a)\}$, we simply identify $g(t)$ with $f(t-a)$ so that $f(t) = g(t+a)$. Equation (8) then gives

(11) $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$.

Example 2 Determine the Laplace transform of $t^2u(t-1)$.

$$\mathcal{L}\{t^2u(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\}$$

Solution To apply equation (11), we take $g(t) = t^2$ and $a = 1$. Then

$$g(t+a) = g(t+1) = (t+1)^2 = t^2 + 2t + 1 .$$

Now the Laplace transform of $g(t+a)$ is

$$\mathcal{L}\{g(t+a)\}(s) = \mathcal{L}\{t^2 + 2t + 1\}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} .$$

So, by formula (11), we have

$$\mathcal{L}\{t^2 u(t-1)\}(s) = e^{-s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right\} .$$ ◆

Example 3 Determine $\mathcal{L}\{(\cos t)u(t-\pi)\}$.

Solution Here $g(t) = \cos t$ and $a = \pi$. Hence,

$$g(t+a) = g(t+\pi) = \cos(t+\pi) = -\cos t\,,$$

and so the Laplace transform of $g(t+a)$ is

$$\mathscr{L}\{g(t+a)\}(s) = -\mathscr{L}\{\cos t\}(s) = -\frac{s}{s^2+1}\,.$$

Thus, from formula (11), we get

$$\mathscr{L}\{(\cos t)u(t-\pi)\}(s) = -e^{-\pi s}\frac{s}{s^2+1}\,. \quad ◆$$

Example 4 Determine $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

Solution To use the translation property (9), we first express e^{-2s}/s^2 as the product $e^{-as}F(s)$. For this purpose, we put $e^{-as} = e^{-2s}$ and $F(s) = 1/s^2$. Thus, $a = 2$ and

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t\,.$$

It now follows from the translation property that

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}(t) = f(t-2)u(t-2) = (t-2)u(t-2)\,.$$

Example 5 The current I in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t)\,; \qquad I(0) = 0\,, \qquad I'(0) = 0\,, \tag{12}$$

where

$$g(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \\ 0\,, & 2 < t\,. \end{cases}$$

Determine the current as a function of time t.

Solution Let $J(s) := \mathscr{L}\{I\}(s)$. Then we have $\mathscr{L}\{I''\}(s) = s^2 J(s)$.

Writing $g(t)$ in terms of the rectangular window function $\Pi_{a,b}(t) = u(t-a) - u(t-b)$, we get

$$g(t) = \Pi_{0,1}(t) + (-1)\Pi_{1,2}(t) = u(t) - u(t-1) - [u(t-1) - u(t-2)]$$
$$= 1 - 2u(t-1) + u(t-2) ,$$

and so

$$\mathscr{L}\{g\}(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} .$$

Thus, when we take the Laplace transform of both sides of (12), we obtain

$$\mathscr{L}\{I''\}(s) + 4\mathscr{L}\{I\}(s) = \mathscr{L}\{g\}(s)$$
$$s^2 J(s) + 4J(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$
$$J(s) = \frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)} .$$

To find $I = \mathscr{L}^{-1}\{J\}$, we first observe that

$$J(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s) ,$$

where

$$F(s) := \frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) .$$

Computing the inverse transform of $F(s)$ gives

$$f(t) := \mathscr{L}^{-1}\{F\}(t) = \frac{1}{4} - \frac{1}{4}\cos 2t .$$

Hence, via the translation property (9), we find

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\{F(s) - 2e^{-s}F(s) + e^{-2s}F(s)\}(t) \\ &= f(t) - 2f(t-1)u(t-1) + f(t-2)u(t-2) \\ &= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left[\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right]u(t-1) \\ &\quad + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2) . \end{aligned}$$

EXAMPLE 1

Find the solution of the differential equation

$$2y'' + y' + 2y = g(t), \tag{1}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \text{and} \quad t \ge 20. \end{cases} \tag{2}$$

Assume that the initial conditions are

$$y(0) = 0, \qquad y'(0) = 0. \tag{3}$$

The Laplace transform of Eq. (1) is

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$$
$$= (e^{-5s} - e^{-20s})/s.$$

Introducing the initial values (3) and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{4}$$

To find $y = \phi(t)$, it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \tag{5}$$

where

$$H(s) = \frac{1}{s(2s^2 + s + 2)}. \tag{6}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{7}$$

Finally, to determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}. \tag{8}$$

Upon determining the coefficients, we find that $a = \frac{1}{2}$, $b = -1$, and $c = -\frac{1}{2}$. Thus

$$H(s) = \frac{1/2}{s} - \frac{s + \frac{1}{2}}{2s^2 + s + 2} = \frac{1/2}{s} - \left(\frac{1}{2}\right) \frac{\left(s + \frac{1}{4}\right) + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

$$= \frac{1/2}{s} - \left(\frac{1}{2}\right) \left[\frac{s + \frac{1}{4}}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} + \frac{1}{\sqrt{15}} \frac{\sqrt{15}/4}{\left(s + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2} \right]. \tag{9}$$

Then, by referring to lines 9 and 10 of Table 6.2.1, we obtain

$$h(t) = \tfrac{1}{2} - \tfrac{1}{2}\left[e^{-t/4} \cos(\sqrt{15}\, t/4) + (\sqrt{15}/15) e^{-t/4} \sin(\sqrt{15}\, t/4) \right]. \tag{10}$$