

(Q) Let p be a fixed point of $g(x)$.

(a) show that if $g'(p) = g''(p) = \dots = g^{(k-1)}(p) = 0$ and $g^{(k)}(p) \neq 0$, then the fixed point iteration of $g(x)$ will converge to p with $R = k$ and $A = \left| \frac{g^{(k)}(p)}{k!} \right|$

(b) Use part (a) to show that if p is a simple root of $f(x)$, then Newton's iteration will converge to p with $R = 2$ and $A = \left| \frac{f''(p)}{2f'(p)} \right|$

Proof: (a) Apply Taylor's expansion of $g(x)$ about $x = p$

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \dots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

$$\text{but } g(p) = p \text{ and } g'(p) = g''(p) = \dots = g^{(k-1)}(p) = 0$$

$$\rightarrow g(x) = p + \frac{g^{(k)}(c)}{k!}(x-p)^k$$

$$\text{Substitute } x = p_n \rightarrow g(p_n) = p + \frac{g^{(k)}(c)}{k!}(p_n - p)^k, \text{ c between } p_n \text{ and } p$$

$$\text{but } g(p_n) = p_{n+1} \rightarrow p_{n+1} - p = \frac{g^{(k)}(c)}{k!}(p_n - p)^k \rightarrow \frac{p_{n+1} - p}{(p_n - p)^k} = \frac{g^{(k)}(c)}{k!} \rightarrow \frac{|E_{n+1}|}{|E_n|^k} = \left| \frac{g^{(k)}(c)}{k!} \right|$$

now take limit as $n \rightarrow \infty$ and considering that $c \approx p$ when $n \rightarrow \infty$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{|E_{n+1}|}{|E_n|^k} = \left| \frac{g^{(k)}(p)}{k!} \right| \rightarrow R = k \text{ and } A = \left| \frac{g^{(k)}(p)}{k!} \right|$$

(b) We know that Newton's iteration is a special case of FPI with $g(x) = x - \frac{f(x)}{f'(x)}$

Therefore, based on part (a), we only need to prove that $g'(p) = 0$ but $g''(p) \neq 0$

Recall that p is a simple root of $f(x)$ mean $f(p) = 0$ and $f'(p) \neq 0$

$$\text{now, } g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2} \rightarrow g'(p) = 0$$

$$\text{now, } g''(x) = \frac{(f'(x))^2 [f(x)f'''(x) + f''(x)f'(x)]}{(f'(x))^4} = \frac{f(x)f'''(x) + f''(x)f'(x)}{(f'(x))^2} \rightarrow g''(p) = \frac{f''(p)}{f'(p)} \neq 0$$

$$\rightarrow R = 2 \text{ and } A = \left| \frac{g''(p)}{2!} \right| = \left| \frac{f''(p)}{2f'(p)} \right|$$