Chapter 3 :- Gate-Level Minimization

* The complexity of digital logic gates is directly proportional to the boolean expression from which the function is implemented.

$$F_1 = x + \bar{x}y$$

$$= (x + \bar{x}) \cdot (x + y)$$

$$= x + y$$

$$x + \bar{x}y \equiv x + y$$

Less simplex

we need to find the simplest boolean expression to describe the function

Minimize the function using algebra is a walkword approach (Lacks of specific rules to predict the next step.

So we need another way for minimization which is

Map Method :- straight forward

- Truth table is unique (befor and after minimization)

* Karnaugh map (K-map)

It is a diagram made of squares where each square represented one minterm.

2 variable map :-

2 Variables $\longrightarrow$ 2^2 minterms = 4 minterms

x	y	
0	0	m_0
0	1	m_1
		m_2
1	0	m_3
1	1	

x \ y	0	1
0	$\bar{x}\bar{y}$ m_0	$\bar{x}y$ m_1
1	$x\bar{y}$ m_2	xy m_3

Example :- Minimize the following function :

$$F(x,y) = \bar{x}y + x\bar{y} + xy$$

$m_1 + m_2 + m_3$

x \ y	0	1
0		1
1	1	1

$\rightarrow y$

x

$x + y$

using algebra

$F = \bar{x}y + x(y+\bar{y})$

$= x + \bar{x}y$

$= (x+\bar{x})\cdot(x+y)$

$= x + y$

Example:- minimize the following expression

$$F = \bar{x}\bar{y} + \bar{x}y + x\bar{y}$$

$$m_0 + m_1 + m_2$$

$x \backslash y$	0	1
0	1	1
1	1	

$\rightarrow \bar{x}$

$\downarrow \bar{y}$

$$F = \bar{x} + \bar{y}$$

$F = \bar{x}(y+\bar{y}) + x\bar{y}$

$F = \bar{x} + x\bar{y}$

$= (\bar{x}+x)\cdot(\bar{x}+\bar{y})$

$= \bar{x}+\bar{y}$

Example:- minimize the following expression

$$F(x,y) = \bar{x}y + x\bar{y}$$

$$m_1 + m_2$$

$x \backslash y$	0	1
0		$\boxed{1}$
1	$\boxed{1}$	

$\rightarrow \bar{x}y$

$x\bar{y}$

$$F = \bar{x}y + x\bar{y}$$

Example: minimize the following expression:

$$F = \bar{A}\bar{B} + \bar{A}B + A\bar{B} + AB$$

A \ B	0	1
0	1	1
1	1	1

$\rightarrow 1$

$$F = \bar{A}(B+\bar{B}) + A(\bar{B}+B)$$

$$= \bar{A} + A = 1$$

* Three Variables map

3 variables $\longrightarrow$ $2^3 = 8$ minterms

F (x, y, z)

x \ yz	00	01	11	10
0	m0	m1	m3	m2
1	m4	m5	m7	m6

xy \ z	0	1
00	m0	m1
01	m2	m3
11	m6	m7
10	m4	m5

Example:- Minimize the following function using K-map

$$F(x,y,z) = \sum(0,1,3,7)$$

$$= m_0 + m_1 + m_3 + m_7$$

$\bar{x}\bar{y}$

$x \backslash yz$	00	01	11	10
0	1	1	1	
1			1	

$y \cdot z$

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

$$F = \bar{x}\bar{y} + y \cdot z$$

Example:- Minimize the following expression

$$F(x,y,z) = \sum(0,1,2,3)$$

$x \backslash yz$	00	01	11	10
0	1	1	1	1
1				

$\rightarrow \bar{x}$

$$F = \bar{x}$$

Example:- Minimize the following function:-

$$F(A,B,C) = \sum(0,1,2,3,4,6)$$

x \ yz	00	01	11	10
0	1	1	1	1
1	1			1

$\rightarrow \bar{A}$

$\bar{C}$

$$F = \bar{A} + \bar{C}$$

Example:- Minimize the following function

$$F(A,B,C) = m_3 + m_5 + m_6 + m_7$$

A \ BC	00	01	11	10
0			1	
1		1	1	1

$\rightarrow AB$

AC $\quad$ BC

$$\therefore F = AB + AC + BC$$

* Four Variables Maps

$F(x, y, z, w) \Rightarrow 4$ variables $\Rightarrow 2^4 = 16$ miterms $= 16$ squares

xy \ zw	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Example :- $F(A, B, C, D) = \Sigma(0, 1, 4, 5, 7)$

AB \ CD	00	01	11	10
00	1	1		
01	1	1	1	
11				
10				

$\bar{A}\bar{C}$

$\bar{A}BD$

$F = \bar{A}\bar{C} + \bar{A}BD$

Example :- Minimize using K-map $F(A, B, C, D) = \Sigma(0, 2, 4, 6, 8, 10)$

AB \ CD	00	01	11	10
00	1			1
01	1			1
11				
10	1			1

$\bar{B}\bar{D}$

$\bar{A}\bar{D}$

$F = \bar{A}\bar{D} + \bar{B}\bar{D}$

Example:- $F(A,B,C,D) = \sum(0,1,2,3,5,7,13,15,9,11)$

AB \ CD	00	01	11	10
00	1	1	1	1
01		1	1	
11		1	1	
10		1	1	

$\to \bar{A}\bar{B}$

$\to D$

$F = \bar{A}\bar{B} + D$

Example:- $F(A,B,C,D) = \sum(0,2,4,6,8,10,12,14)$

AB \ CD	00	01	11	10
00	1			1
01	1			1
11	1			1
10	1			1

$\to \bar{D}$

$F = \bar{D}$

Example:- $F(A,B,C,D) = \sum(0,2,5,7,8,10,13,15)$

AB \ CD	00	01	11	10
00	1			1
01		1	1	
11		1	1	
10	1			1

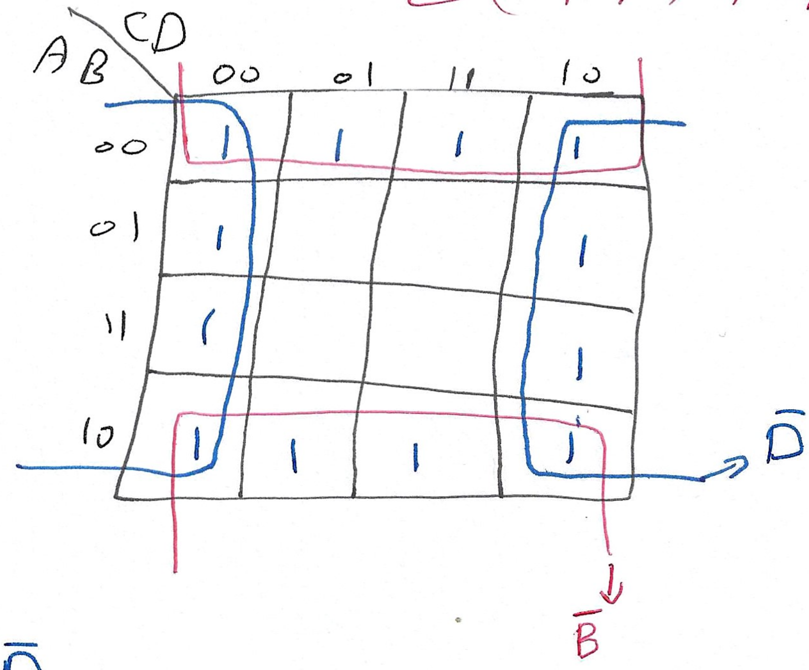
$\to BD$

$\bar{B}\bar{D}$

$F = BD + \bar{B}\bar{D}$

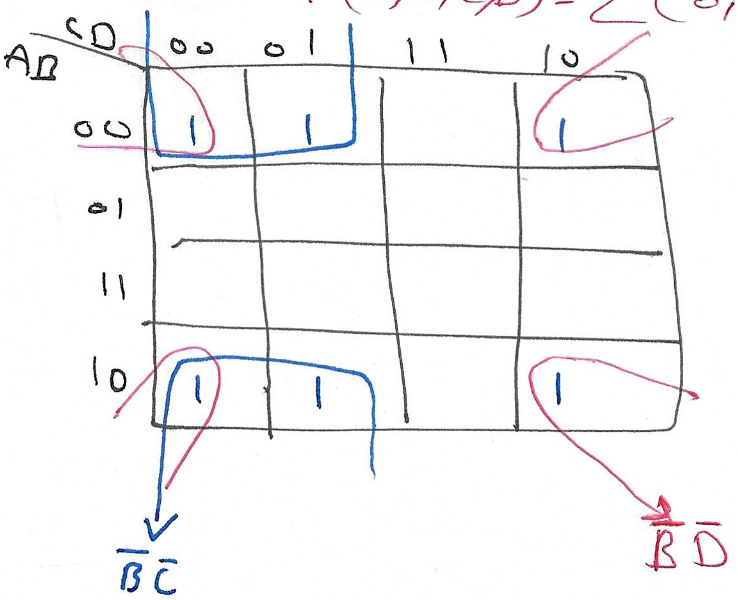
$= \overline{(B \oplus D)}$ XNOR

Example :- $F(A,B,C,D) = \sum(0,1,2,3,4,6,8,9,10,11,12,14)$



$F = \bar{B} + \bar{D}$

Example :- $F(A,B,C,D) = \sum(0,1,2,8,9,10)$



$F = \bar{B}\bar{C} + \bar{B}\bar{D}$

* Five Variables Map

$F(A,B,C,D,E) \Rightarrow 5$ Variable $\Rightarrow 2^5 = 32$ minterms

A = 0

BC \ DE	00	01	11	10
00	m0	m1	m3	m2
01	m4	m5	m7	m6
11	m12	m13	m15	m14
10	m8	m9	m11	m10

A = 1

BC \ DE	00	01	11	10
00	m16	m17	m19	m18
01	m20	m21	m23	m22
11	m28	m29	m31	m30
10	m24	m25	m27	m26

	A	B	C	D	E	F
A = 0	0	0	0	0	0	m0
	0	0	0	0	1	m1
	0	0	0	1	0	m2
	0	0	0	1	1	m3
	0	0	1	0	0	m4
	0	0	1	0	1	m5
	0	0	1	1	0	m6
	0	0	1	1	1	m7
	0	1	0	0	0	m8
	0	1	0	0	1	m9
	0	1	0	1	0	m10
	0	1	0	1	1	m11
	0	1	1	0	0	m12
	0	1	1	0	1	m13
	0	1	1	1	0	m14
	0	1	1	1	1	m15
A = 1	1	0	0	0	0	m16
	1	0	0	0	1	m17
	1	0	0	1	0	m18
	1	0	0	1	1	m19
	1	0	1	0	0	m20
	1	0	1	0	1	m21
	1	0	1	1	0	m22
	1	0	1	1	1	m23
	1	1	0	0	0	m24
	1	1	0	0	1	m25
	1	1	0	1	0	m26
	1	1	0	1	1	m27
	1	1	1	0	0	m28
	1	1	1	0	1	m29
	1	1	1	1	0	m30
	1	1	1	1	1	m31

please note that m0 is adjacent to m16
m1 is adjacent to m17
and so on

Example :- Minimize $F(A,B,C,D,E) = \Sigma(0,2,4,6,9,13,21,23,25,29,31)$

$A = 0$

BC \ DE	00	01	11	10
00	1			1
01	1			1
11		1		
10		1		

$\rightarrow \bar{A}\bar{B}\bar{E}$

$A = 1$

BC \ DE	00	01	11	10
00				
01		1	1	
11		1	1	
10		1		

$\rightarrow AC\bar{E}$

$B\bar{D}E$

$$F = B\bar{D}E + \bar{A}\bar{B}\bar{E} + AC\bar{E}$$

Example:- $F(A,B,C,D,E) = \sum(0,1,8,9,16,17,22,23,24,25)$

$A=0$

BC \ DE	00	01	11	10
00	1	1		
01				
11				
10	1	1		

$A=1$

BC \ DE	00	01	11	10
00	1	1		
01			1	1
11				
10	1	1		

$\rightarrow A\bar{B}CD$

$\bar{C}\bar{D}+$

$F = \bar{C}\bar{D} + A\bar{B}CD$

* Product of Maxterms

Example:- Simplify $F(A,B,C,D) = \prod(3,4,6,7,11,12,13,14,15)$

AB \ CD	00	01	11	10
00			0	
01	0		0	0
11	0	0	0	0
10			0	

$(\bar{A}+\bar{B})$

$(\bar{B}+D)$

$(\bar{C}+\bar{D})$

$$F = (\bar{A}+\bar{B}) \cdot (\bar{B}+D) \cdot (\bar{C}+\bar{D})$$

Example:- $F(A,B,C,D) = \prod(0,4,5,6,7,8,12)$

AB \ CD	00	01	11	10
00	0			
01	0	0	0	0
11	0			
10	0			

$(A+\bar{B})$

$(C+D)$

$$F = (C+D) \cdot (A+\bar{B})$$

$\bar{F}$

AB \ CD	00	01	11	10
00	1			
01	1	1	1	1
11	1			
10	1			

$\bar{A}B$

$\bar{C}\bar{D}$

$$\bar{F} = \bar{C}\bar{D} + \bar{A}B$$

$$F = \bar{\bar{F}} = \overline{(\bar{C}\bar{D} + \bar{A}B)}$$

$$= (C+D) \cdot (A+\bar{B})$$

* Don't Care Conditions:-

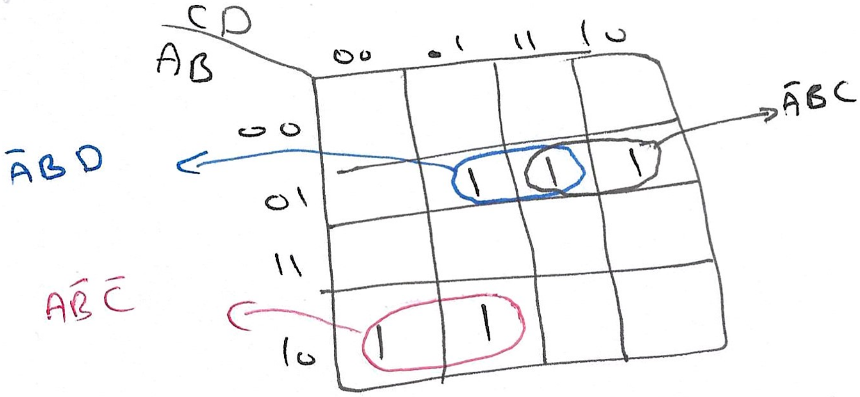
A Function table may contain entries for wich

1- The input values of the variables will never occur

2- The output value of the function is never used

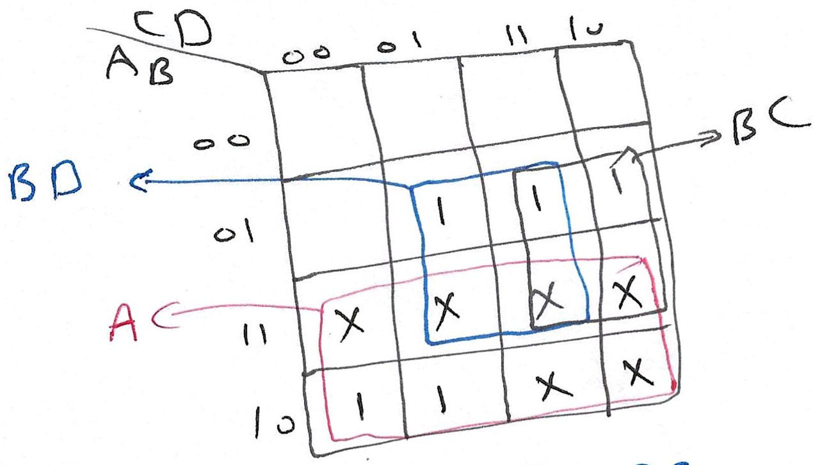
Example:- Consider a function (f) defined over BCD inputs where the function output is (0) if the BCD input is (0-4), and the function output is (1) if the BCD input is (5-9), The function output is (X) if the input is (10-15)

$$F = \underbrace{\Sigma_m(5,6,7,8,9)}_{\text{minterms}} + \underbrace{\Sigma_d(10,11,12,13,14,15)}_{\text{don't care}}$$



without using don't care $F = \bar{A}BD + A\bar{B}\bar{C} + \bar{A}BC$

A	B	C	D	F
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	x
1	0	1	1	x
1	1	0	0	x
1	1	0	1	x
1	1	1	0	x
1	1	1	1	x



with don't care $F = A + BD + BC$

Example:- Simplify $F = \sum_m(1,3,7,11,15) + \sum_d(0,2,5)$

AB \ CD	00	01	11	10
00	X	1	1	X
01		X	1	
11			1	
10			1	

$\bar{A}D$

CD

AB \ CD	00	01	11	10
00	X	1	1	X
01		X	1	
11			1	
10			1	

$\bar{A}\bar{B}$

CD

$F = \bar{A}D + CD$ or $F = \bar{A}\bar{B} + CD$

not all don't cares need to be covered

Example:- Simplify $F = \sum_m(1,3,7,11,15) + \sum_d(0,2,5)$

obtain a product of sums minimal expression

$F(POS) = \bar{\bar{F}}(SOM)$

$\bar{F}(SOM) = \sum_m(4,6,8,9,10,12,13,14) + \sum_d(0,2,5)$

AB \ CD	00	01	11	10
00	X			X
01	1	X		1
11	1	1		1
10	1	1		1

$\bar{D}$

$A\bar{C}$

$\bar{F} = A\bar{C} + \bar{D}$

$F = \bar{\bar{F}} = \overline{(A\bar{C}) + \bar{D}}$

$= D.(\bar{A} + C)$

$F = \prod(4,6,8,9,10,12,13,14) + \prod_d(0,2,5)$

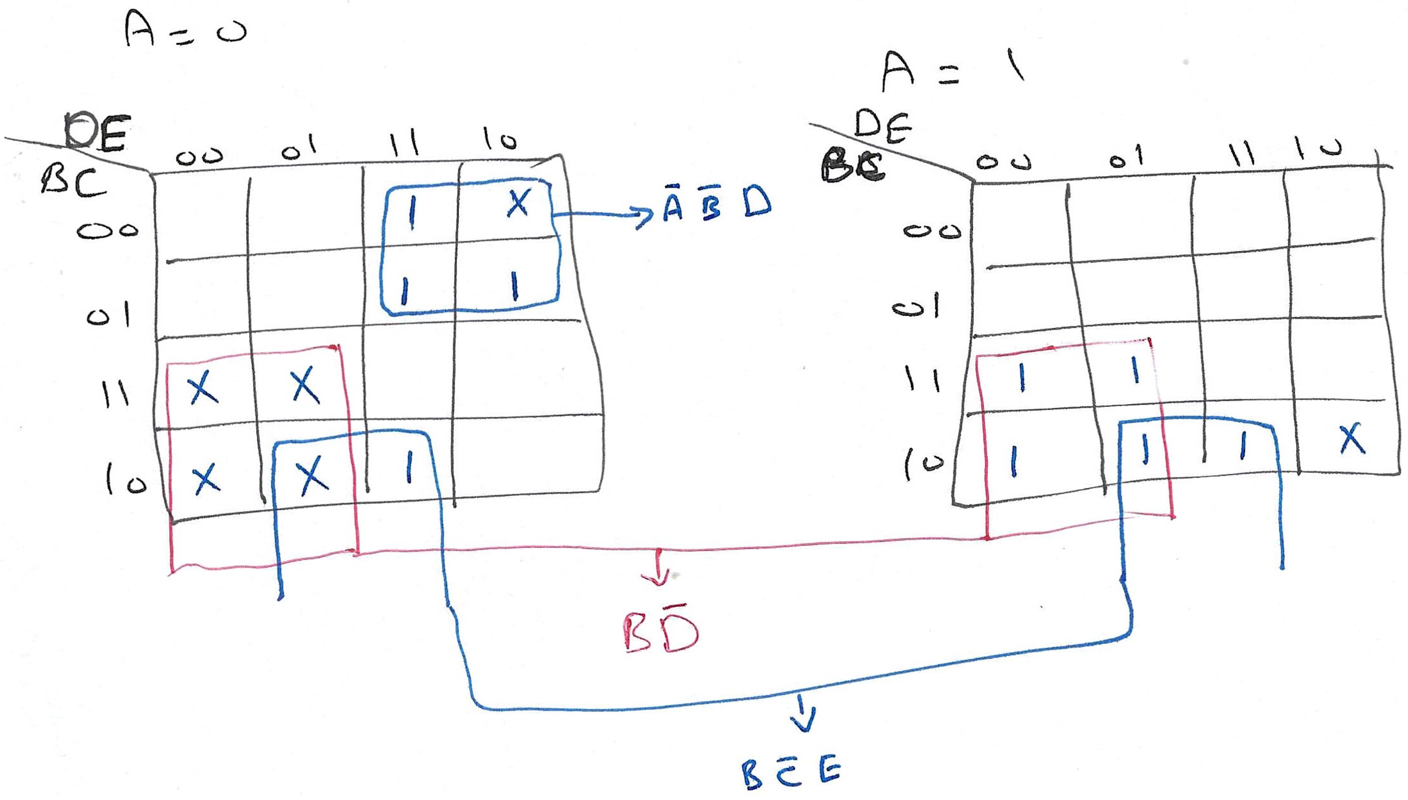
AB \ CD	00	01	11	10
00	X			X
01	0	X		0
11	0	0		0
10	0	0		0

$\bar{A} + C$

D

$F = D.(\bar{A} + C)$

Example:- $F(A,B,C,D,E) = \Sigma_m(3,6,7,11,24,25,27,28,29) + \Sigma_d(2,8,9,12,13,26)$



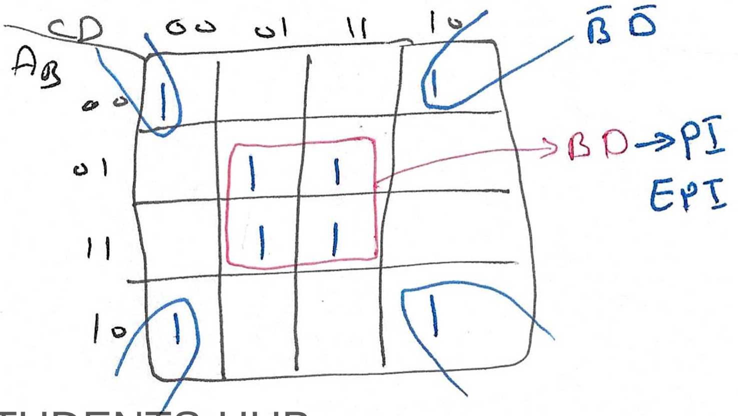
$F = \bar{A}\bar{B}D + B\bar{C}E + B\bar{D}$

* Prime Implicant and Essential Prime Implicant

Prime Implicant (PI):- a product term obtained by combining the maximum number of adjacent squares in the k-map

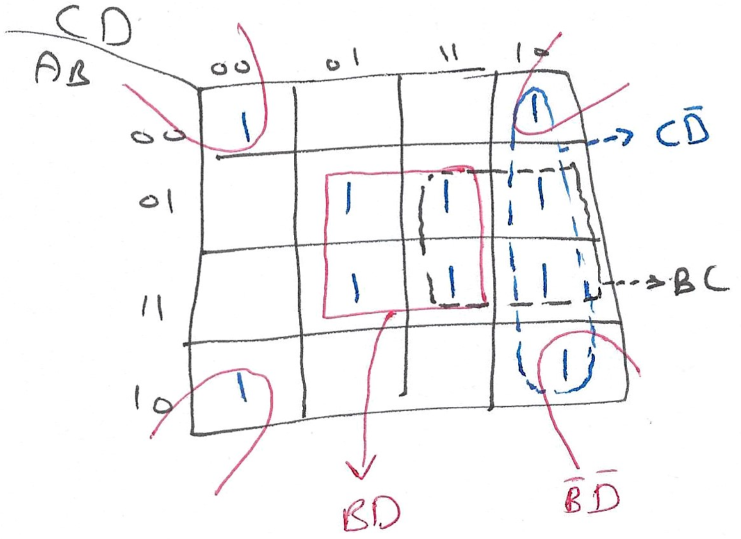
Essential prime Implicant (EPI):- is a prime implicant that covers at least one minterm not covered by the other prime implicants

Example:- $F(A,B,C,D) = \Sigma(0,2,5,7,8,10,13,15)$



In this example we have two prime Implicants and they are both essential

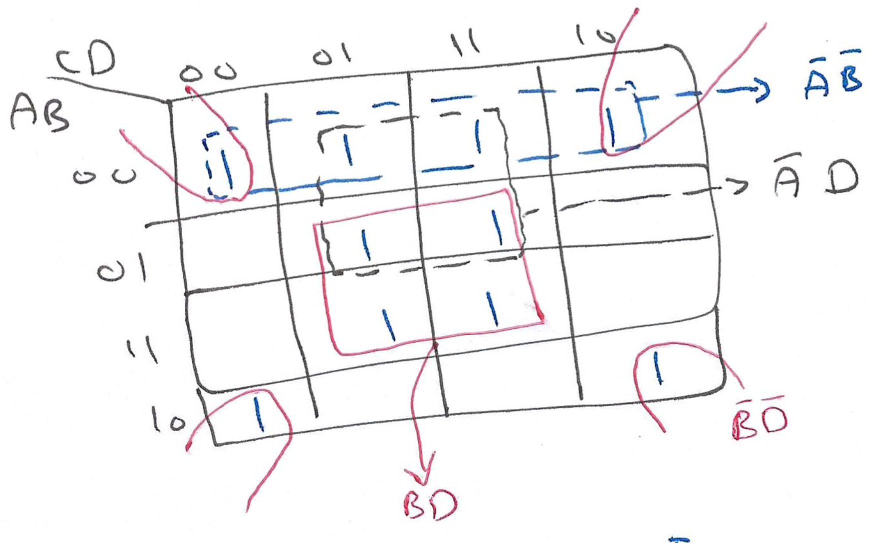
Example: $F(A,B,C,D) = \Sigma(0,2,5,6,7,8,10,13,14,15)$



$F = BD + \bar{B}\bar{D} + C\bar{D} \leftarrow PI$

$F = \underset{\substack{\uparrow \\ EPI}}{BD} + \underset{\substack{\uparrow \\ EPI}}{\bar{B}\bar{D}} + \underset{\substack{\uparrow \\ PI}}{BC}$

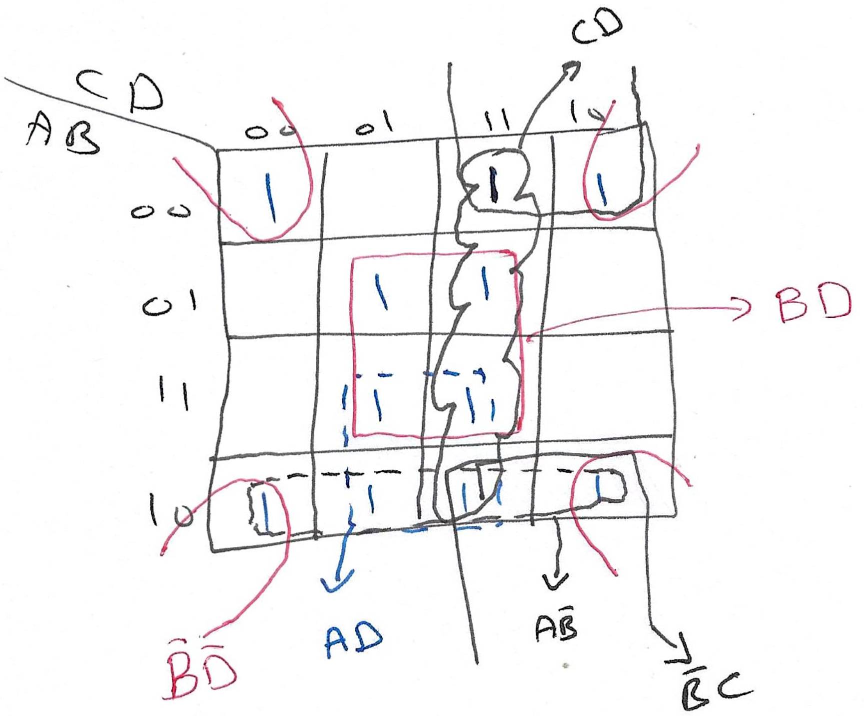
Example: $F(A,B,C,D) = \Sigma(0,1,2,3,5,7,8,10,13,15)$



$F = BD + \bar{B}\bar{D} + \bar{A}\bar{B} \leftarrow PI$

$F = \underbrace{BD + \bar{B}\bar{D}}_{EPI} + \bar{A}D \leftarrow PI$

Example : $F(A,B,C,D) = \Sigma(0,2,3,5,7,8,9,10,11,13,15)$



So we have 6 prime Implicants

two are essential

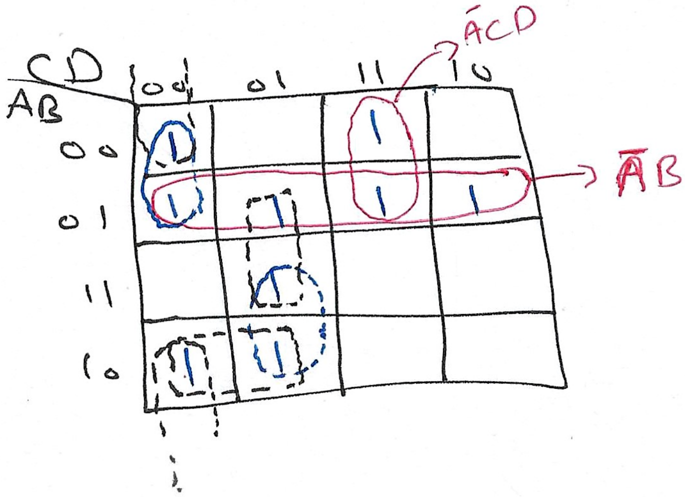
$F = BD + \bar{B}\bar{D} + A\bar{B} + CD$

$F = BD + \bar{B}\bar{D} + A\bar{B} + \bar{B}C$

$F = BD + \bar{B}\bar{D} + AD + \bar{B}C$

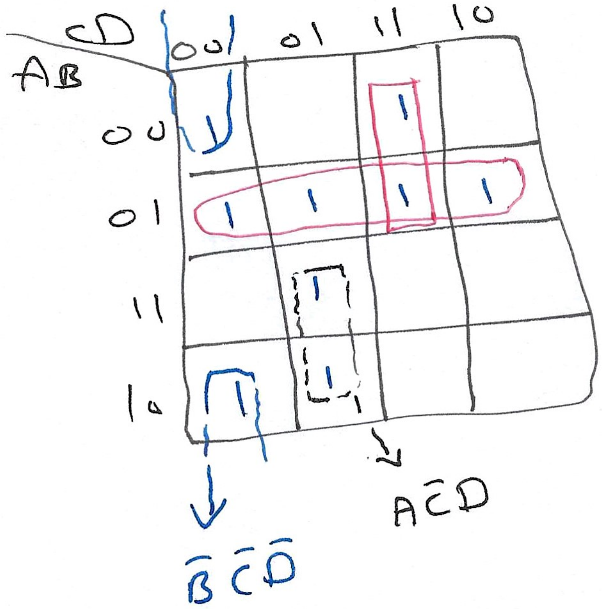
$F = BD + \bar{B}\bar{D} + AD + CD$

Example: $F(A,B,C,D) = \sum(0,3,4,5,6,7,8,9,13)$



So we have 7 prime implicants

two are Essential



$F = \bar{A}B + \bar{A}CD + A\bar{C}D + \bar{B}\bar{C}\bar{D}$

NAND / NOR Implementation

NAND and NOR gates are said to be universal gates bacause any logic ~~circuit~~ circuit can be implemented with it.

Digital circuits are frequently constructed with NAND or NOR gates rather than with AND and OR gates for the following reasons:-

1. use fewer transistors
   - → Less cost
   - → Less delay
   - → Less space

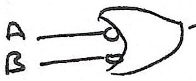
2. NAND and NOR are easier to fabricate.

* NAND gate

$F = (A.B)'$

$= \bar{A} + \bar{B}$

$\equiv$



$F = \bar{A} + \bar{B}$

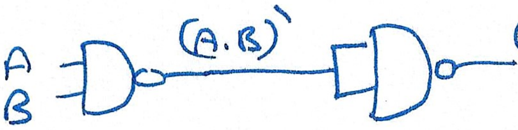
NAND

A	B	F
0	0	1
0	1	1
1	0	1
1	1	0

① NAND gate as inverter

$\overline{(A.A)} = \bar{A}$

② NAND gate as AND

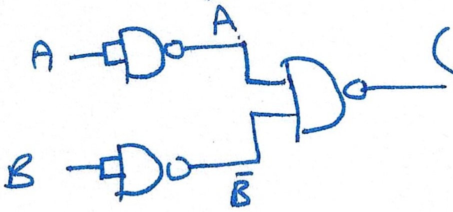


$(A.B)'$

$(A.B)'' = A.B$

AND

③ NAND gate as OR

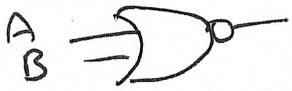


$\bar{A}$, $\bar{B}$

$\overline{(\bar{A}.\bar{B})} = A + B$

OR

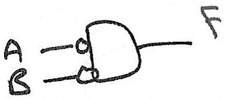
* NOR gate



$F = \overline{(A+B)}$

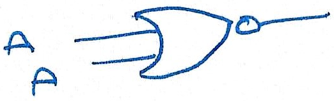
$= \bar{A} \cdot \bar{B}$

$\equiv$

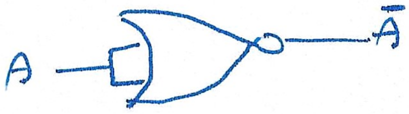


A	B	F
0	0	1
0	1	0
1	0	0
1	1	0

① NOR as inverter

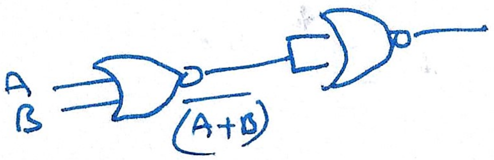


$F = \overline{(A+A)} = \bar{A}$



inverter

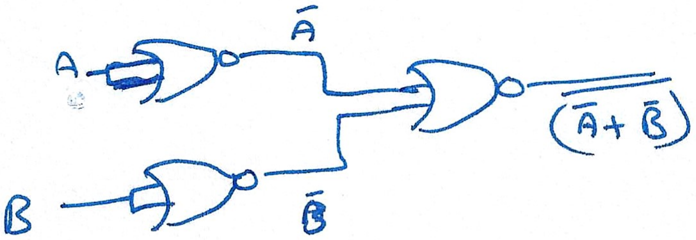
② NOR as OR



$\overline{\overline{(A+B)}} = A + B$

OR

③ NOR as AND



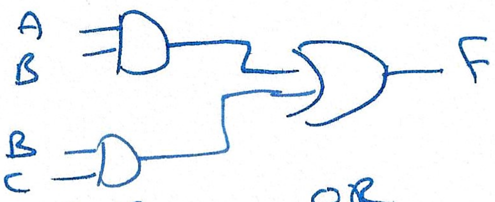
$\overline{(\bar{A}+\bar{B})} = A \cdot B$

AND

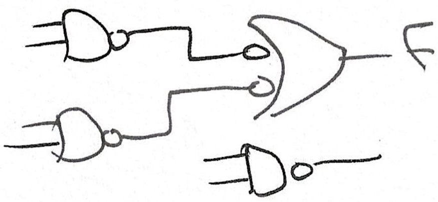
*AND - OR Implementation

$F = A \cdot B + B \cdot C$

(SOP)



AND OR

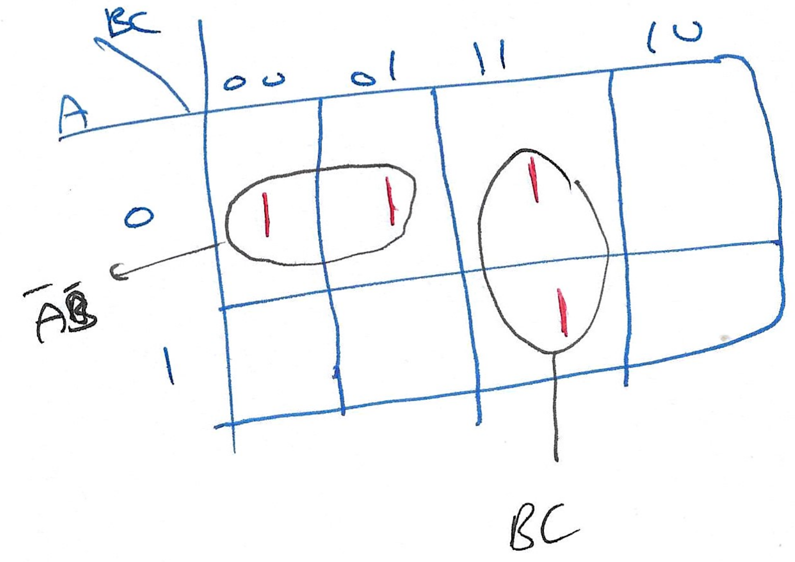


NAND NAND

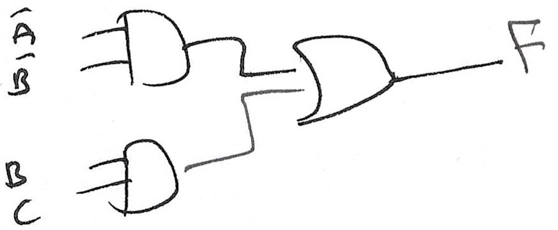
Example:- Implement the following Function with NAND gates

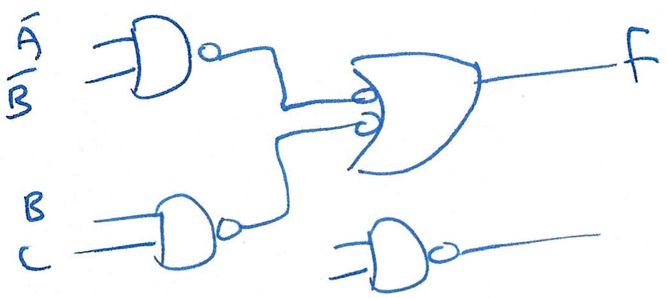
$$F(A,B,C) = \Sigma(0,1,3,7)$$

$m_0 \quad m_1 \quad m_3 \quad m_7$



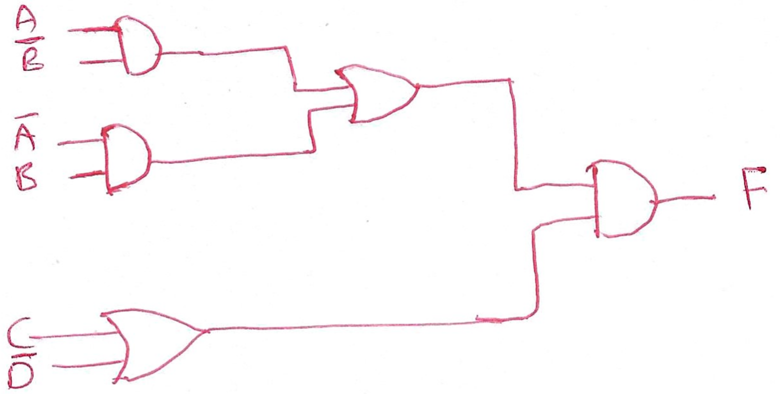
$$F = \bar{A}\bar{B} + BC$$

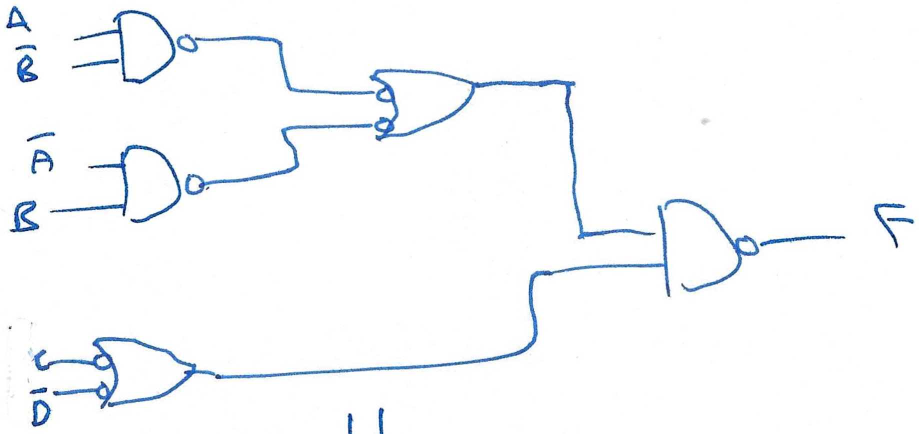




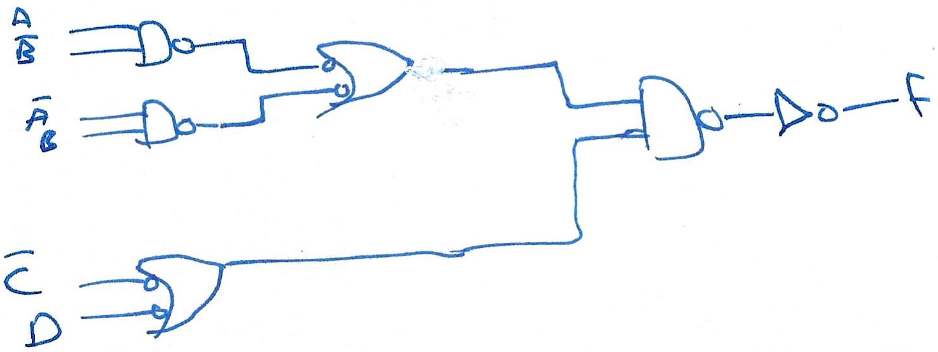
NAND NAND

Example: $F(A\bar{B} + \bar{A}B)(C+\bar{D})$



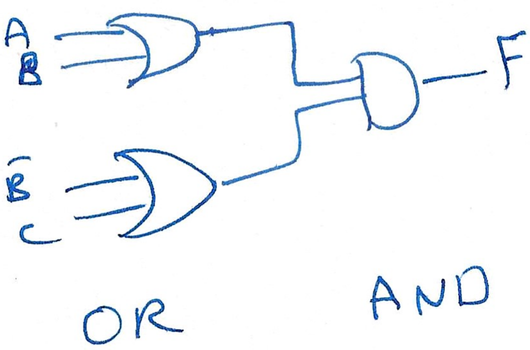


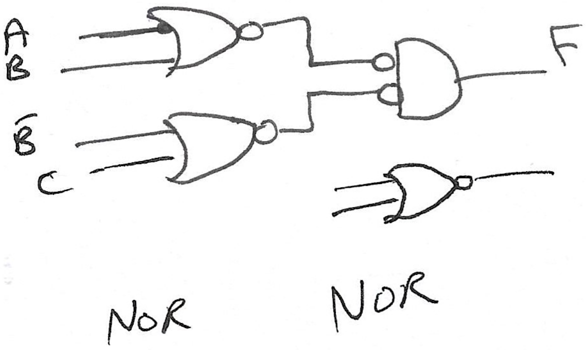
$\Downarrow$



* OR-AND Implimintation

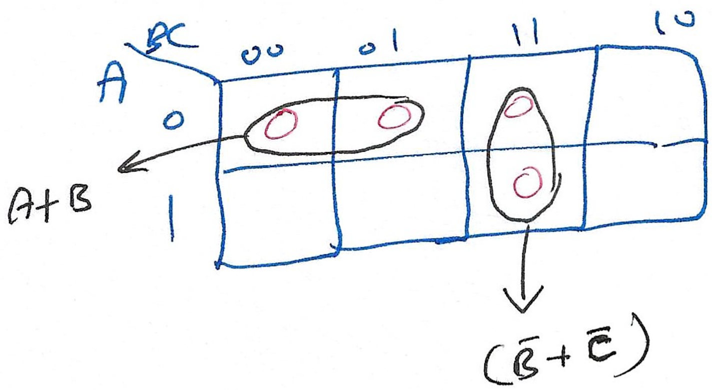
$F = (A+B) \cdot (\bar{B}+C)$ Product of sum



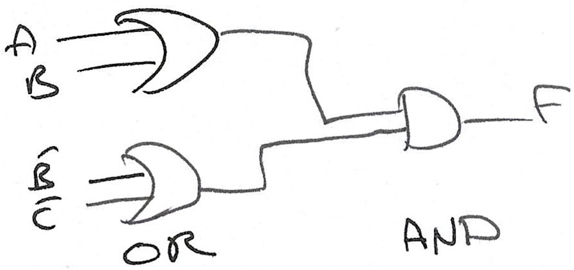


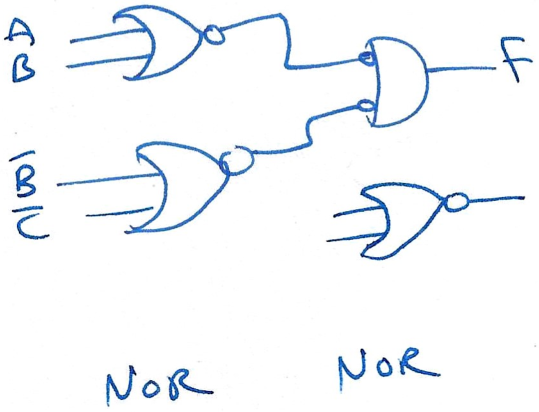
Example:- Implimint the following function using NOR-NOR gates

$F(A,B,C) = \Pi(0,1,3,7)$



$F = (A+B) \cdot (\bar{B}+\bar{C})$





Example:- $F(A,B,C,D) = \Sigma(2,4,6,10,12)$

$d(A,B,C,D) = \Sigma(0,8,9,13)$

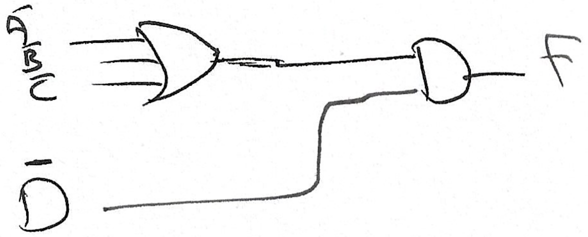
Impliment the function using NOR gates

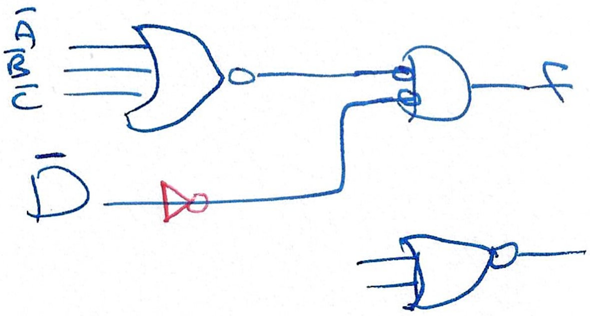
AB \ CD	00	01	11	10
00	X	0	0	1
01	1	0	0	1
11	1	X	0	0
10	X	X	0	1

$\rightarrow (\bar{A}+\bar{B}+\bar{C})$

$\bar{D}$

$F = \bar{D} \cdot (\bar{A}+\bar{B}+\bar{C})$





* Parity Generators:

Parity: Extra bit added to the message to check If the recieved message is Correct (Single error)

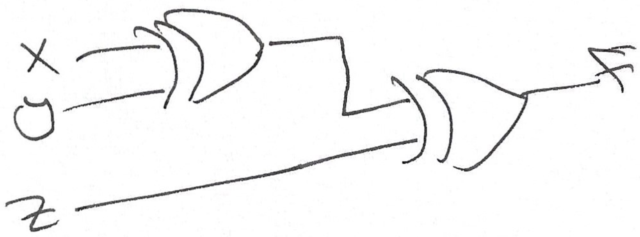
Parity → Even
Parity → odd

Example: Design even parity generator and Checker for 3-bit codes

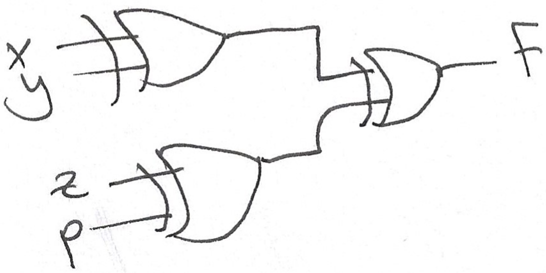
odd parity ⟶ even Function

even parity ⟶ odd Function

X	y	z	P
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1



x	y	z	p	Check
0	0	0	0	0
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
⋮				
1	1	1	1	0



Example :- $F = (A\bar{B} + \bar{A}B)E(C+\bar{D})$