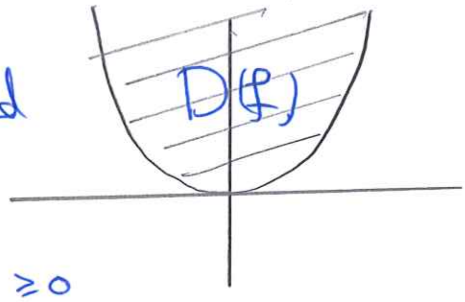
14.1 Domains and Ranges.

(1) $Z = f(x,y) = \sqrt{y - x^2}$

$D_f = \{(x,y) : y - x^2 \geq 0\} = \{(x,y) : y \geq x^2\}$

Domain is closed and unbounded

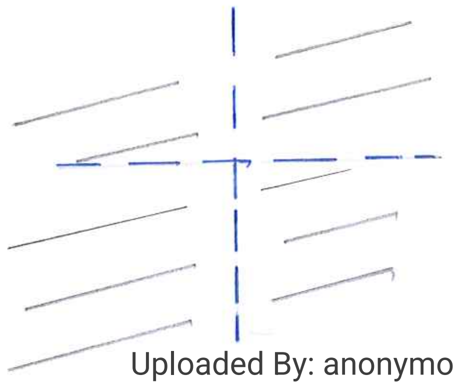


$R_f = [0, \infty)$, $[\; y - x^2 \geq 0$
$z = \sqrt{y - x^2} \geq 0 \Rightarrow z \geq 0 \;]$.

(2) $f(x,y) = \dfrac{1}{xy} = Z$

$D_f = \{(x,y) : x \neq 0 \text{ and } y \neq 0\}$

Domain is unbounded and open



$R_f = (-\infty, 0) \cup (0, \infty)$

(3) $f(x,y) = e^{-x^2-y^2} = \dfrac{1}{e^{x^2+y^2}} > 0$

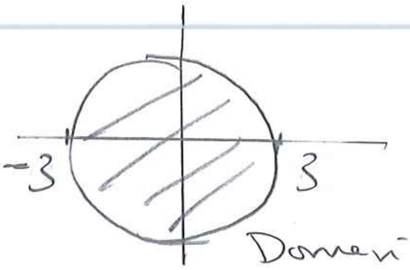
D_f = Entire plane which is clopen (closed & open)

$R_f = (0, 1]$ $\left\{ \; x^2 + y^2 \geq 0 \Rightarrow 0 < e^{-x^2-y^2} \leq e^0 = 1 \right.$
$\Rightarrow 0 < z < 1$

(4) $f(x,y) = \sqrt{9 - x^2 - y^2}$

$D_f = \{(x,y) : 9 - x^2 - y^2 \geq 0\} = \{(x,y) : x^2 + y^2 \leq 9\}$

Domain is closed and bounded.



Range : $0 \leq x^2 + y^2 \leq 9$

$-9 \leq -x^2 - y^2 \leq 0$

$0 \leq 9 - x^2 - y^2 \leq 9$

$\therefore \quad 0 \leq \underbrace{\sqrt{9 - x^2 - y^2}}_{z} \leq 3$

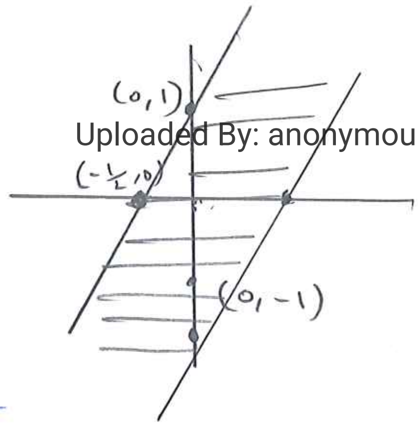
$\Rightarrow$ Range : $[0, 3]$

Recall
$D(\sin^{-1}(x)) = [-1, 1]$
Range $(\sin^{-1} x) =$
$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(5) $f(x,y) = 4 \sin^{-1}(y - 2x)$

$D_f = \{(x,y) : -1 \leq y - 2x \leq 1$

$= \{(x,y) : 2x - 1 \leq y \leq 2x + 1$



Domain is unbounded and closed.

Range : $-\frac{\pi}{2} \leq \sin^{-1}(y - 2x) \leq \frac{\pi}{2}$

$-4\frac{\pi}{2} \leq 4 \sin^{-1}(y - 2x) \leq 4\frac{\pi}{2}$

$\Rightarrow \quad -2\pi \leq \underbrace{4 \sin^{-1}(y - 2x)}_{z} \leq 2\pi$

$\Rightarrow$ Range $[-2\pi, 2\pi]$

(6) $z = f(x,y) = 9 - x^2 - y^2 = 9 - (x^2+y^2)$

$D_f :=$ Entire plane.

$R_f : \quad z = 9 - (\underbrace{x^2+y^2}_{\geq 0}) \leq 9$

$$x^2+y^2 \geq 0 \Rightarrow -x^2-y^2 \leq 0$$

$$z = 9 - x^2 - y^2 \leq 9$$

$\text{Range}(f) = (-\infty, 9]$

(7) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$D_f = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \geq 0\} =$ Entire space

$R_f = [0, \infty)$

(8) $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

$D_f = \{(x,y,z) : (x,y,z) \neq 0\}$

$R_f = (0, \infty)$

(9) $w = xy \ln z$

$D_w = \{(x,y,z) ; z > 0\} =$ Half space

$R_w = (-\infty, \infty)$.

14.2 ① $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \left(\frac{0}{0}\right)$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{(\sqrt{x} - \sqrt{y})} \cdot \left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right) =$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{(x-y)}} = \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) = \boxed{0}$$

(Q16)

Example ② $\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x} = \left(\frac{0}{0}\right)$

$$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{xy(x-1) + 4x(x-1)}$$

$$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{x(x-1)[y+4]} = \lim_{(x,y)\to(2,-4)} \frac{1}{x(x-1)} = \boxed{\frac{1}{2}}$$

Example ③ $\lim_{(x,y)\to(0,0)} y^2 \sin\left(\frac{1}{x}\right)$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \quad , x \ne 0$$

$$-y^2 \le y^2 \sin\left(\frac{1}{x}\right) \le y^2 \quad , x \ne 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} -y^2 \le \underbrace{\lim_{(x,y)\to(0,0)} y^2 \sin\left(\frac{1}{x}\right)} \le \lim_{(x,y)\to(0,0)} y^2$$

By Sandwich thm $= 0$

(Q21)

Example: 4 $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \left(\frac{0}{0}\right)$

Let $u = x^2+y^2$, $(x,y)\to(0,0)$ then $u \to 0$

$$\lim_{u\to 0} \frac{\sin(u)}{u} \overset{L.H}{=} \lim_{u\to 0} \frac{\cos u}{1} = 1$$

Example 5 : $\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2} = \left(\frac{0}{0}\right)$

polar coordinates: $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$ & $(x,y)\to(0,0) \Rightarrow r\to 0$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta - r\cos\theta . r^2\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0} \frac{r^3\cos\theta\,(\cos^2\theta - \sin^2\theta)}{r^2}$$

$$= \lim_{r\to 0} r \underbrace{\cos\theta\,(\cos 2\theta)}_{\text{bounded}} = 0$$

Q67) $\lim\limits_{(x,y)\to(0,0)} \ln\left(\dfrac{3x^2+3y^2-x^2y^2}{x^2+y^2}\right)$ [polar coordinates]

$= \ln\left(\lim\limits_{r\to 0} \dfrac{3r^2\cos^2\theta+3r^2\sin^2\theta - r^4\sin^2\theta\cos^2\theta}{r^2}\right)$

$= \ln\left(\lim\limits_{r\to 0} 3 - r^2\sin^2\theta\cos^2\theta\right) = \boxed{\ln 3}$.

Q64) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x}{x^2+y^2+x} = \lim\limits_{r\to 0} \dfrac{2r\cos\theta}{r^2+r\cos\theta}$

$= \lim\limits_{r\to 0} \dfrac{2\cos\theta}{r+\cos\theta} = 2$, if $\cos\theta \neq 0$

$= DNE$ if $\cos\theta = 0$

Example: $\lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2y^2}{x^4+y^4} = \lim\limits_{r\to 0} \dfrac{4r^4\sin^2\theta\cos^2\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$

$= \lim\limits_{r\to 0} \dfrac{4\sin^2\theta\cos^2\theta}{\cos^4\theta+\sin^4\theta} = \dfrac{4\sin^2\theta\cos^2\theta}{\sin^4\theta+\cos^4\theta}$

So, the limit DNE, since the limit varies for different θ.

Two - Path Test for Nonexistence of a limit.

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \to (x_0, y_0)$, then:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ does not exist.}$$

$\begin{cases} y = kx \\ y = kx^2 \end{cases}$ Two paths.

Example(1): Show that $f(x,y) = \dfrac{2x^2 y}{x^4 + y^2}$ has no limit as $(x,y) \to (0,0)$.

sol: Along $y = x$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^3}{x^4+x^2} = \lim_{x\to 0} \frac{2x}{x^2+1} = 0$$

Along $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^4}{x^4+x^4} = \boxed{1}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4+y^2} \text{ DNE.}$$

Example: $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^3+3y^3}$.

Along $y = kx$, $x \neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^3+3y^3} = \lim_{x\to 0} \frac{2x(kx)^2}{x^3+3(kx)^3}$$

$$= \lim_{x\to 0} \frac{2k^2x^3}{x^3(1+3k^3)} = \frac{2k^2}{1+3k^3}$$

If $k = 0$, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

If $k = 1$, $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ DNE by Two-Path Test.

Example: $\lim_{(x,y)\to(1,-1)} \left(\frac{xy+1}{x^2-y^2}\right)$

Along $x = 1 \Rightarrow \lim_{y\to -1} \frac{y+1}{1-y^2} = \lim_{y\to -1} \frac{1+y}{(1+y)(1-y)}$

$$= \boxed{\frac{1}{2}}$$

$\left(\frac{2}{3}\right)$ (43)

Along $y=-1 \implies \lim_{x\to 1} \frac{-x+1}{x^2-1} = \lim_{x\to 1} \frac{(1-x)}{(x-1)(x+1)}$

$$= \frac{-1}{x+1} = \frac{-1}{2}$$

$\therefore \lim_{(x,y)\to(1,-1)} f(x,y)$ DNE by Two-path test.

Example: $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$

Along $x=1$: $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1} = \lim_{y\to 1} \frac{y^2-1}{y-1} = \boxed{2}$

Along $y=x$: $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1} = \lim_{(x,y)\to(1,1)} \frac{x^3-1}{x-1} = \boxed{3}$

$\therefore \lim_{(x,y)\to(1,1)} f(x,y)$ DNE.

Continuity:

Def: A function $f(x,y)$ is continuous at the point (x_0, y_0) if

1) f is defined at (x_0, y_0)

2) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

3) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

A function is continuous if it is continuous at every point of its domain.

Example: Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

is continuous at every point except the origin.

1) $f(0,0) = 0$ defined.

③ $\lim_{(x,y)\to(0,0)} f(x,y) \underset{=\!=}{\overset{\text{Along } y=x}{}} \lim_{x\to 0} \frac{2x^2}{2x^2} = \boxed{1}$

Along $y=x^2$: $\lim_{x\to 0} \frac{2x^3}{x^2+x^6} = \boxed{0}$

$\therefore$ limit DNE as $(x,y)\to(0,0)$.

$\Rightarrow$ $f(x,y)$ is cont. everywhere except $(0,0)$

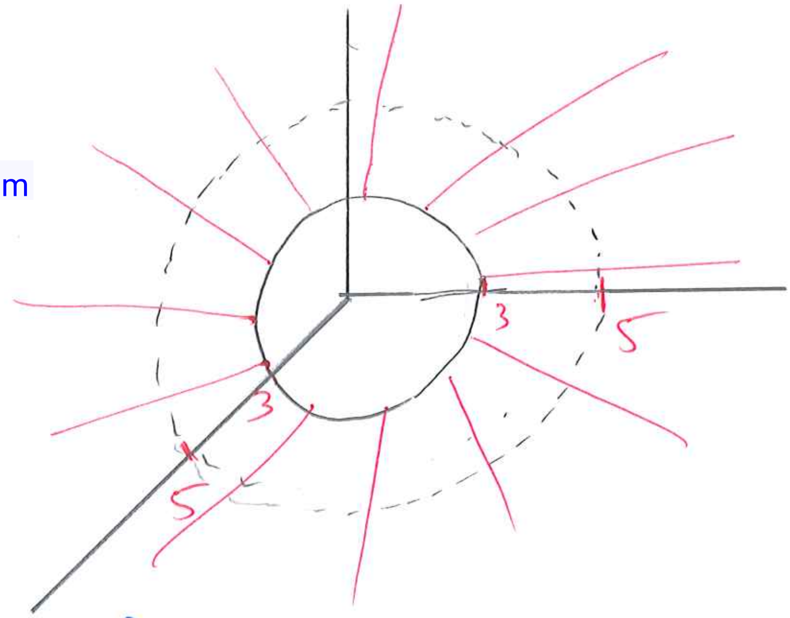
Q40

Example: Where $f(x,y,z) = \frac{1}{4-\sqrt{x^2+y^2+z^2-9}}$

is Continuous.

$D_f = \{(x,y,z) : x^2+y^2+z^2-9 \geq 0 \text{ \& } 4-\sqrt{x^2+y^2+z^2-9} \neq 0\}$

$= \{(x,y,z) : x^2+y^2+z^2 \geq 9 \text{ \& } x^2+y^2+z^2 \neq 25$



f is continuous on D_f.