Ch2 : Limits and Continuity

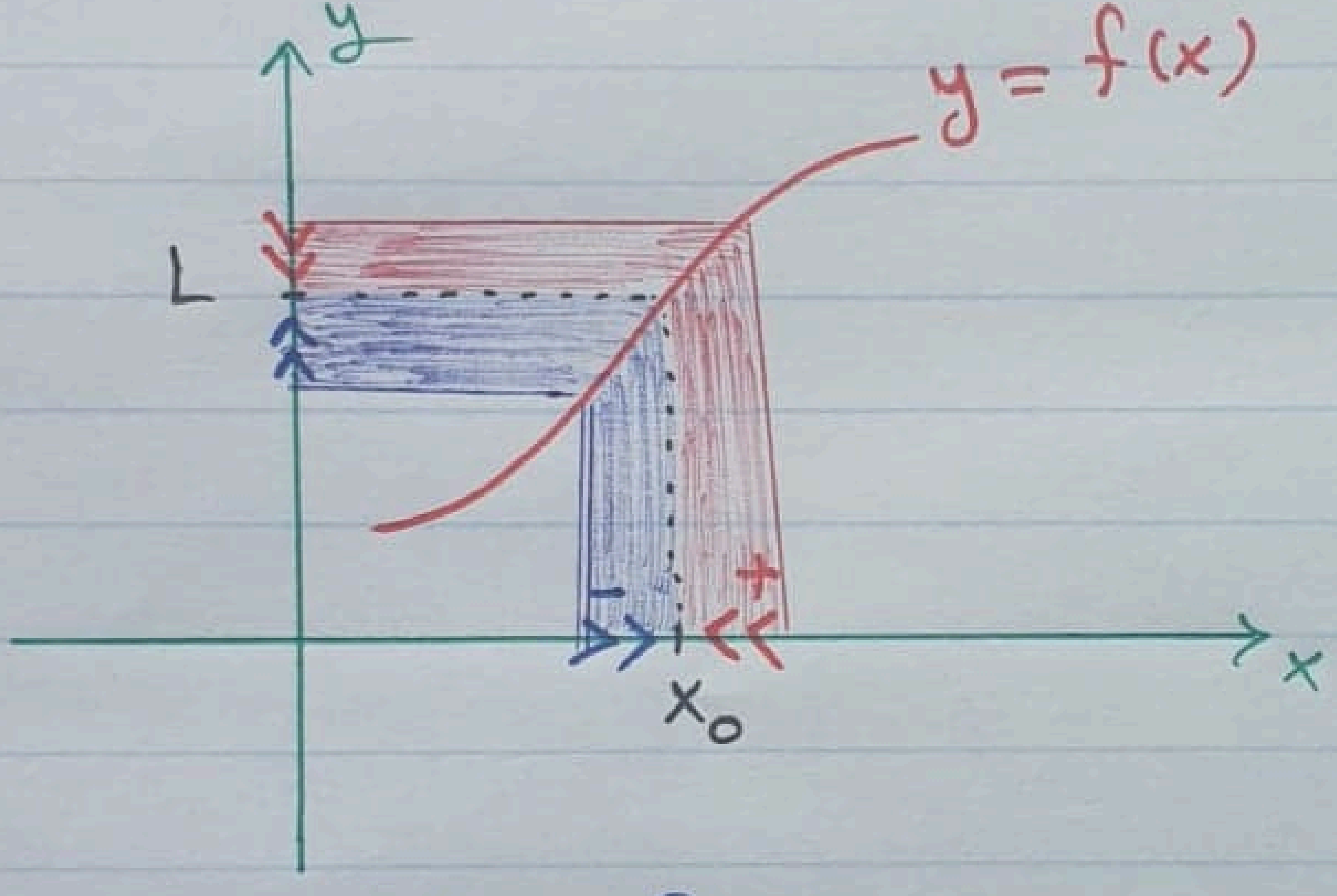
Def The function $f(x)$ has Limit L as x approaches x_0 if

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$

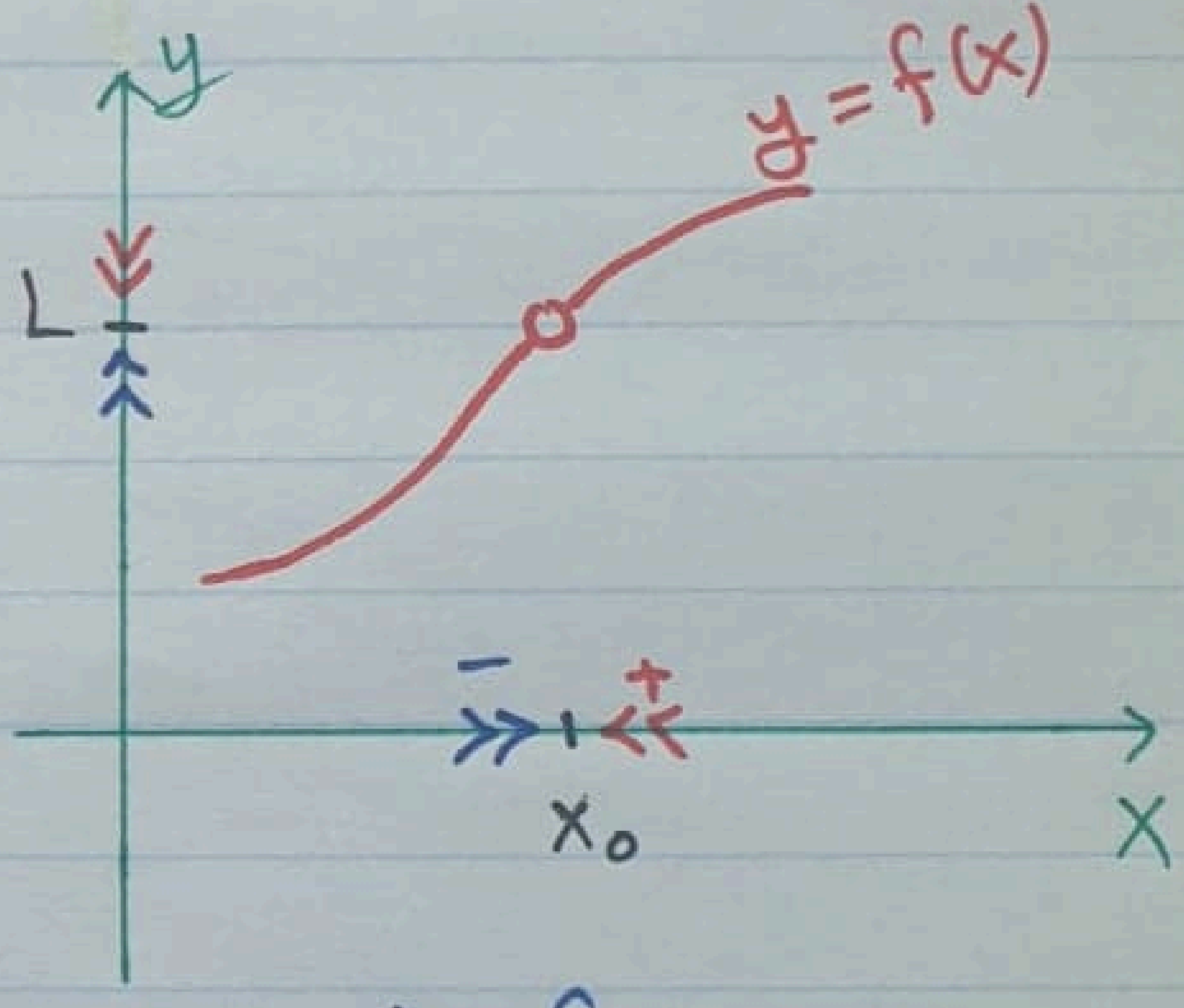
and we write $\lim\limits_{x \to x_0} f(x) = L$

Notes

- $\lim$: limit
- $x \to x_0$: x approaches x_0
- $x \to x_0$ does not mean $x = x_0$
- 1^- one from left



$$\lim_{x \to x_0} f(x) = L$$



$$\lim_{x \to x_0} f(x) = L$$

Exp 1) $\lim\limits_{x \to 0} (x^2 - 1) = 0^2 - 1 = -1$ تعويض مباشر

2) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x \to 2} (x+2) = 4$

or $\lim\limits_{x \to 2} \dfrac{2x}{1} = 2(2) = 4$ since $\dfrac{0}{0}$ ✓

بزبط نشتق كمان $\dfrac{\infty}{\infty}$ ✓

L'Hopital's Rule

$\boxed{2}$ $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-x} = \lim\limits_{x\to 1} \dfrac{(x-1)(x+2)}{x(x-1)}$ $\left(\frac{0}{0}\right)$

$= \lim\limits_{x\to 1} \dfrac{x+2}{x} = \dfrac{\boxed{1}+2}{1} = 3$

$\underline{\text{or}}$ $\lim\limits_{x\to 1} \dfrac{2x+1}{2x-1} = \dfrac{2+1}{2-1} = 3$ since $\frac{0}{0}$

$\boxed{3}$ $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1} \cdot \dfrac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$

$\lim\limits_{x\to -1} \dfrac{x^2+8-9}{(x+1)\left[\sqrt{x^2+8}+3\right]} = \lim\limits_{x\to -1} \dfrac{x^2-1}{(x+1)\left[\sqrt{x^2+8}+3\right]}$

$= \lim\limits_{x\to -1} \dfrac{(x+1)(x-1)}{(x+1)\left[\sqrt{x^2+8}+3\right]} = \lim\limits_{x\to -1} \dfrac{x-1}{\sqrt{x^2+8}+3}$

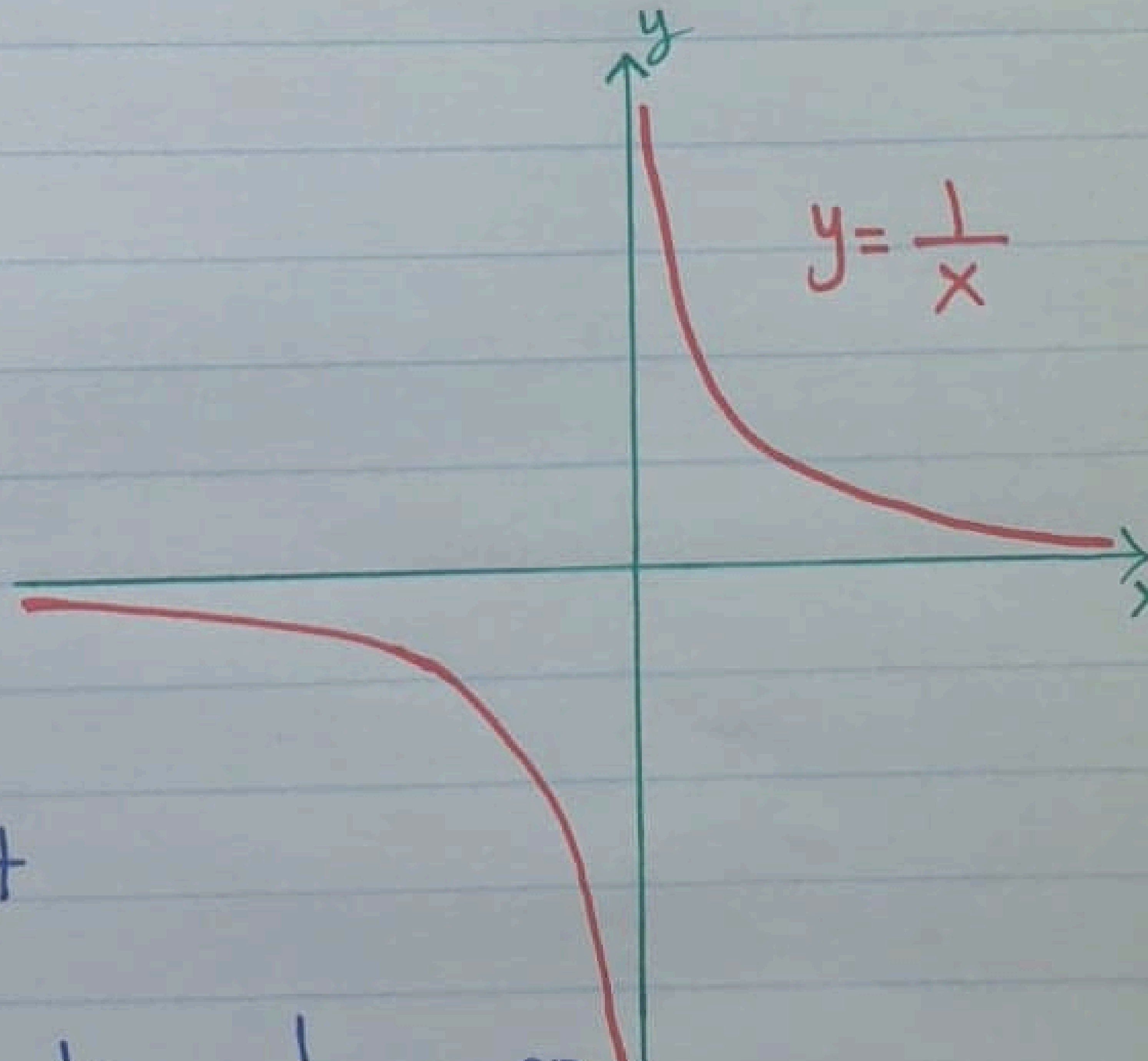
$= \dfrac{-1-1}{\sqrt{1+8}+3} = \dfrac{-2}{\sqrt{9}+3} = \dfrac{-2}{6} = \dfrac{-1}{3}$

$\boxed{4}$ $\lim\limits_{x\to \infty} \dfrac{1}{x} = 0$

$\boxed{5}$ $\lim\limits_{x\to -\infty} \dfrac{1}{x} = 0$

$\boxed{6}$ $\lim\limits_{x\to 0} \dfrac{1}{x} =$ DNE

Does Not Exist



since $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$ but $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$

7. $$\lim_{\theta \to 0} \frac{1-\cos\theta}{\sin 2\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \qquad \left(\frac{0}{0}\right)$$

$$\lim_{\theta \to 0} \frac{1-\cos^2\theta}{(\sin 2\theta)(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{(2\cos\theta)(1+\cos\theta)} = \frac{0}{(2)(1)(1+1)} = 0$$

or $$\lim_{\theta \to 0} \frac{0 - -\sin\theta}{2\cos 2\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{2\cos 2\theta} = \frac{0}{(2)(1)} = 0$$

8. $$\lim_{x \to \infty} \left(\sqrt{x^2+1} - \sqrt{x^2-x}\right) \cdot \frac{\sqrt{x^2+1} + \sqrt{x^2-x}}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$\lim_{x \to \infty} \frac{(x^2+1) - (x^2-x)}{\sqrt{x^2+1} + \sqrt{x^2-x}} = \lim_{x \to \infty} \frac{x+1}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$\lim_{x \to \infty} \frac{1 + \frac{1}{x}}{\frac{\sqrt{x^2+1}}{x} + \frac{\sqrt{x^2-x}}{x}}$$

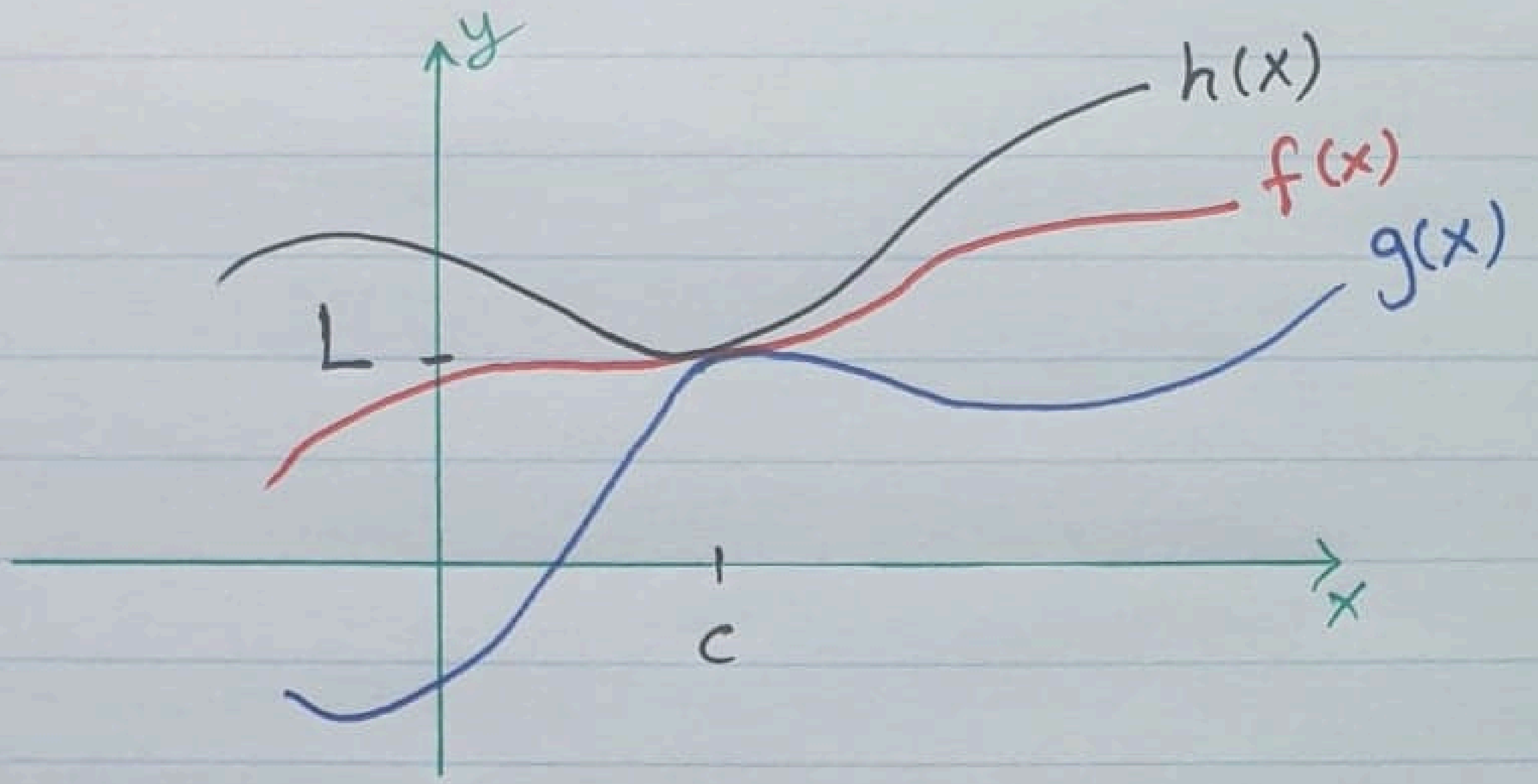
$$\lim_{x \to \infty} \frac{1 + \frac{1}{x}}{\sqrt{\frac{x^2+1}{x^2}} + \sqrt{\frac{x^2-x}{x^2}}}$$

$$\lim_{x \to \infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}} = \frac{1+0}{\sqrt{1+0} + \sqrt{1-0}} = \frac{1}{2}$$

Th (Sandwich Th)

Suppose $g(x) \le f(x) \le h(x) \quad \forall\, x \in I$ s.t

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad \text{Then} \quad \lim_{x\to c} f(x) = L$$



I : Interval
s.t : such that

$c \in I$ or
$c \notin I$

Exp Assume $2 - x^2 \le f(x) \le x^2 + 2$. Find $\lim_{x\to 0} f(x)$

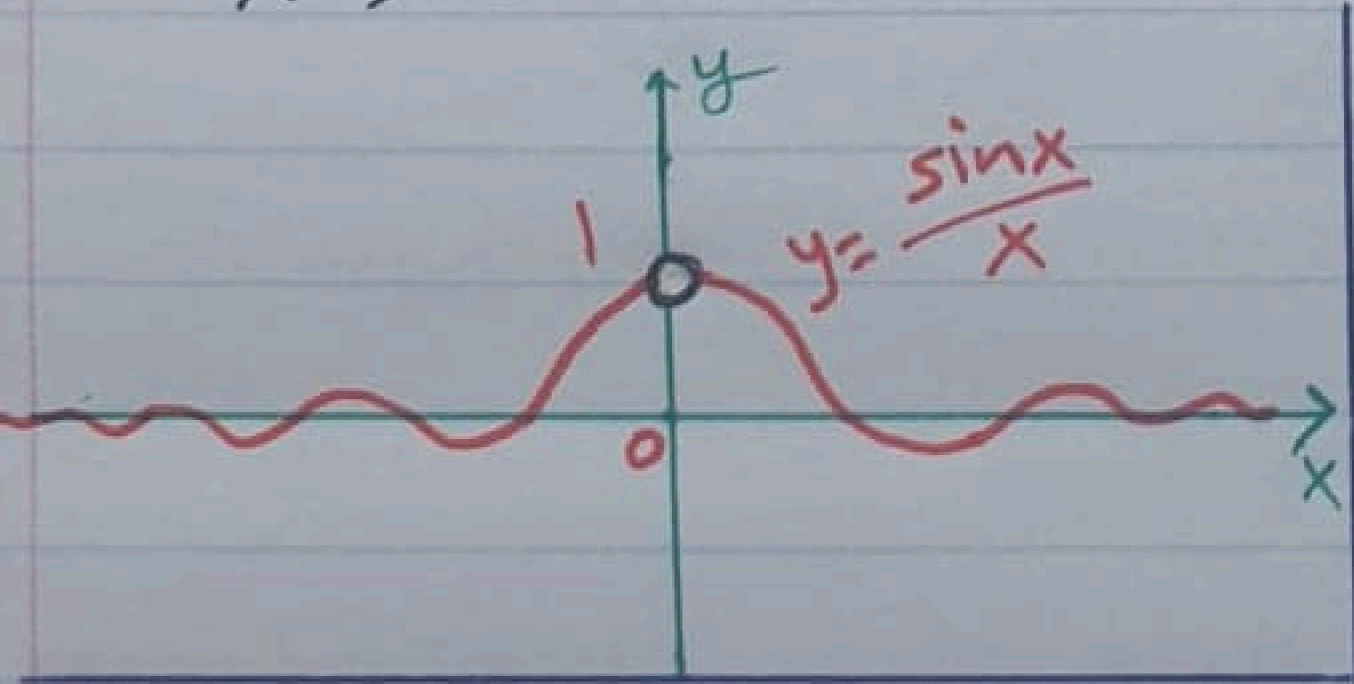
$$\lim_{x\to 0} (2 - x^2) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0} (x+2)$$

$$2 \le \lim_{x\to 0} f(x) \le 2$$

Hence, $\lim_{x\to 0} f(x) = 2$ by Sandwich Th.

Exp $\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$ by S.Th

$$-1 \le \sin x \le 1$$



$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$-\lim_{x\to\infty} \frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$$

Exp $\lim_{x\to 0} \dfrac{\sin x}{x} = 1 \quad \left(\dfrac{0}{0}\right)$

$\lim_{x\to 0} \dfrac{\cos x}{1} = \cos 0 = 1$