

Chapter -3-

Z-Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \Rightarrow \text{Fourier Transform}$$

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x(n)| |e^{-j\omega n}|$$

F.T. Converges if $\sum_{n=-\infty}^{\infty} |x(n)|$ converges (i.e. finite)

OR $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$ (absolutely summable)

* If seq. $x(n)$ is absolutely summable, Then F.T. of $x(n)$ converges (i.e. exist).

* Stability of DT-system $\rightarrow$ F.T. of Unit sample response ($h(n)$) of the system $H(e^{j\omega})$ converges.

* Unit sample response of system is absolutely summable $\Rightarrow$ system is stable.

Example: ① $x(n] = \left(\frac{1}{2}\right)^n u(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)| = 2 \Rightarrow \text{F.T. converges (exist)}$$

② $x(n) = 2^n u(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)| = \infty \Rightarrow \text{F.T. is not exist.}$$

* exponential seq. which decays exponentially (To right or to left)
 $\Rightarrow$ It's F.T. exist and The exponential seq. which grows exp.
 $\Rightarrow$ It's F.T. doesn't exist.

* There is a way around it by using Multiplying growing exp. by a decaying exponential (fast decaying) $\Rightarrow$

$\Rightarrow$ product has F.T.

$$X_r(j\omega) = \sum_{n=-\infty}^{\infty} [x(n) \tilde{r}^{-n}] e^{-j\omega n} \quad \tilde{r}^{-n} \rightarrow \text{decaying complex exponential}$$

$\Rightarrow$ We pick r so that $(x(n)r^{-n})$ is absolutely summable.

$$X_r(e^{j\omega}) = \sum_{-\infty}^{\infty} x(n) \underbrace{(re^{j\omega})^{-n}}_{\substack{\text{new complex variable called } z \\ z = re^{j\omega}}} \rightarrow$$

$$|z|=r, \angle z = \omega$$

* going back to example (2) if we multiply 2^n by a decaying exponential (faster than 2^n) any choice of r which is greater than 2. $\Rightarrow$ so the product $2^n r^{-n}$ converges.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad z\text{-Transform}$$

* how z-transform related to Fourier Transform (if it is exist)

$$X(e^{j\omega}) = X(z) \Big|_{z=e^{j\omega}} \quad \text{or when magnitude of } z = 1$$

* When z-transform converges?

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty \Rightarrow \text{for}$$

[illegible]

Example: $x(n) = \left(\frac{1}{2}\right)^n u(n)$. [called Right-sided exponential seq.]

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \bar{z}^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\bar{z}\right)^n$$
$$= \frac{1}{1 - \frac{1}{2}\bar{z}} = \frac{z}{z - \frac{1}{2}}$$

Converges When $\sum \left|\left(\frac{1}{2}\bar{z}\right)^n\right| < \infty \Rightarrow \left|\frac{1}{2}\bar{z}\right| < 1 \Rightarrow |z| > \frac{1}{2}$

So, ~~at~~ $X(z)$ converges when $|z|$ or r greater than $\frac{1}{2}$.

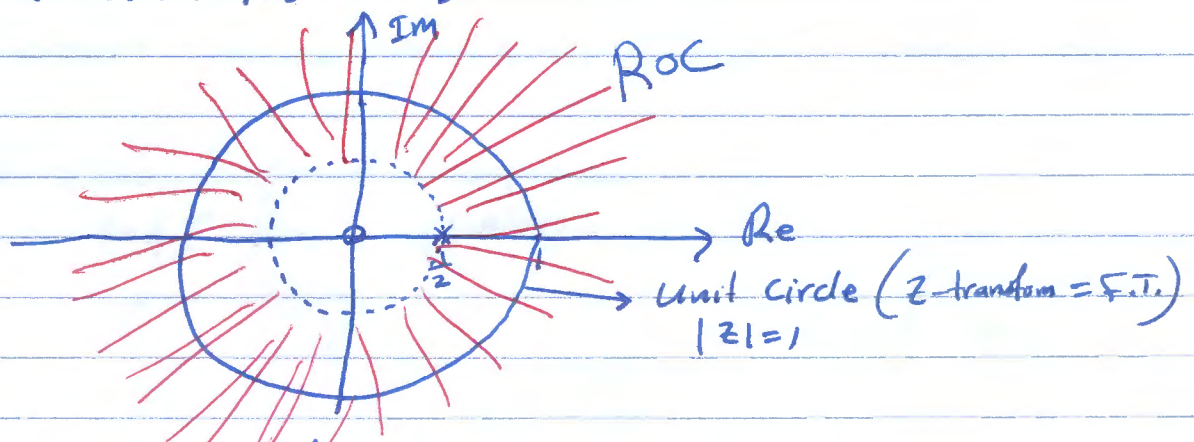
* all values of $|z|$ or r which makes $X(z)$ converges are called Region of Convergence (ROC).

So, ROC of $X(z)$ is $|z| > \frac{1}{2}$.

Does F.T. ^{of $X(z)$} Converges? $\Rightarrow$ F.T. converges (exist) only if z -transform($X(z)$) converges at $|z|=1$. $\Rightarrow$ i.e. ROC contains Unit circle. ($|z|=1$).

* Exponential sequences or sum of exponential sequence always have z -transform as a ratio of polynomials of $\bar{z}$ or z . (i.e. rational function of $\bar{z}$ or z).

* z -plane



- Zero's of transform $\Rightarrow$ roots of numerator polynomial.

- pole's of transform $\Rightarrow$ " " denominator " "

* Same as zeros and poles of Laplace-transform of the continuous signals.

Zeros $\rightarrow$ denoted as circles $\circ$
 poles $\rightarrow$ " " cross $\times$

Example: $x(n) = -(\frac{1}{2})^n u(-n-1)$ $[(\frac{1}{2})^n \text{ but for -ve values of } n]$
 [this is called left-sided exponential sequence]

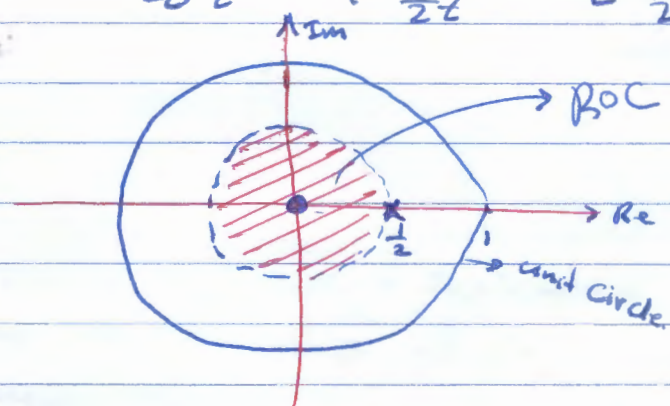
$$X(z) = - \sum_{-\infty}^{-1} (\frac{1}{2})^n u(-n-1) z^{-n} = - \sum_{n=1}^{\infty} (\frac{1}{2})^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} (\frac{1}{2})^{-n} z^n$$

So, $X(z)$ converges if $|\frac{1}{2} z| < 1 \Rightarrow |z| < \frac{1}{2}$

$$\Rightarrow X(z) = 1 - \frac{1}{1 - (\frac{1}{2})^{-1} z} = \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}, \quad |z| < \frac{1}{2}$$

zeros: at $z=0$
 poles: at $z=\frac{1}{2}$



* ROC of $X(z)$ doesn't contain Unit circle $\Rightarrow$ F.T. doesn't exist.

Example: Two-sided Exponential Sequence

$$x(n) = (-\frac{1}{3})^n u(n) - (\frac{1}{2})^n u(-n-1)$$

$x(n) \rightarrow -\infty$ as $n \rightarrow \infty$

$$(-\frac{1}{3})^n u(n) \xrightarrow{z} \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{3} \text{ ROC}$$

$$- (\frac{1}{2})^n u(-n-1) \xrightarrow{z} \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| < \frac{1}{2} \text{ ROC}$$

Thus, by linearity of z-transform

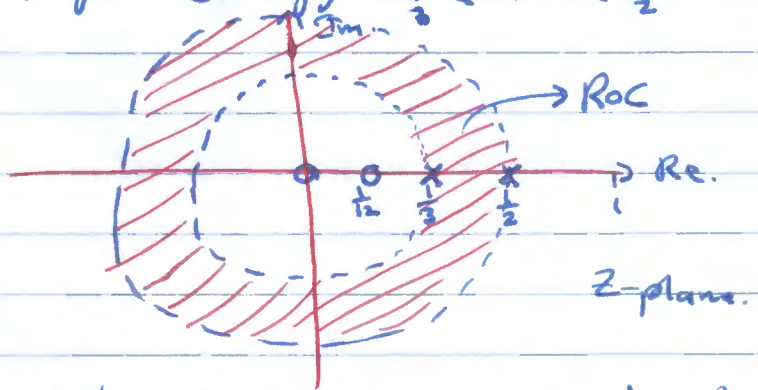
$$X(z) = \frac{1}{1 + \frac{1}{3} z^{-1}} + \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad \frac{1}{3} < |z|, \quad |z| < \frac{1}{2}$$

$$X(z) = \frac{z(1 - \frac{1}{12}z^{-1})}{(1 + \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{2z(z - \frac{1}{12})}{(z + \frac{1}{3})(z - \frac{1}{2})}$$

* ROC of $X(z)$ is annular region (ring) $\frac{1}{3} < |z| < \frac{1}{2}$

zeros: at $z=0$ and $z = \frac{1}{12}$

poles: at $z = -\frac{1}{3}$, $z = \frac{1}{2}$



* Note that! ROC doesn't contain unit circle, so Seq. $x(n)$ doesn't have F.T.

* From the previous examples, $\Rightarrow$ Infinitely long exponential Seq's have z -transforms that can be expressed as a rational function of either z^{-1} or z . The case when the sequence has finite length also has a simple form. if $x(n)$ is nonzero only in the interval

$$N_1 \leq n \leq N_2$$

$X(z) = \sum_{n=N_1}^{N_2} x(n) z^{-n}$ converges if each of the terms $|x(n)z^{-n}|$ is finite.

$$\text{i.e. } |x(n)z^{-n}| < \infty.$$

* For example, if $x(n) = \delta(n) + \delta(n-5)$, then $X(z) = 1 + z^{-5}$ which is finite for $|z| > 0$.

Example:

$$x(n) = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (a z^{-1})^n$$

$$= \frac{1 - (a z^{-1})^N}{1 - a z^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1 - \alpha} \quad N_2 \geq N_1$$

ROC?

$$\sum_{n=0}^{N-1} |a z^{-1}|^n < \infty$$

* Since there are only finite number of nonzero terms, the sum will be finite as long as $(a z^{-1})$ is finite $\Rightarrow |a| < \infty$ and $z \neq 0$.

* Assume $|a|$ is finite $\Rightarrow$ ROC includes the entire z -plane except the origin ($z=0$)

* For $N=16$ and a is real and $0 < |a| < 1$

$\Rightarrow X(z)$ has N zeros at:

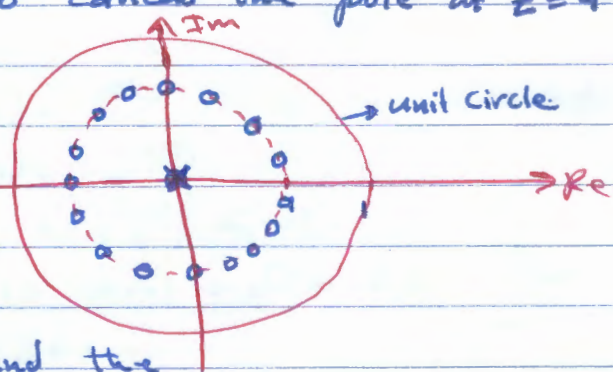
$$z_k = a e^{j \frac{2\pi k}{N}}, \quad k=0, 1, 2, \dots, N-1$$

Which satisfy $z^N = a^N$

When $a=1$, these complex values are the N^{th} roots of unity (at unit circle). The zero at $k=0$ cancels the pole at $z=1$

poles: $z=0$, $z=1$

Cancelled by
zero at $k=0$



$\Rightarrow$ one pole at $z=0$, and the remaining zeros at $z_k = a e^{j \frac{2\pi k}{N}}, \quad k=1, 2, 3, \dots, N-1$

Some Common Z-transform Pairs

Sequence	Z-transform	ROC
① $\delta(n)$	1	All z
② $u(n)$	$\frac{1}{1-z^{-1}}$	$ z > 1$
③ $-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$ z < 1$
④ $\delta(n-m)$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
⑤ $a^n u(n)$	$\frac{1}{1-az^{-1}}$	$ z > a $
⑥ $-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$ z < a $
⑦ $na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
⑧ $-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
⑨ $\cos \omega_0 n u(n)$	$\frac{1 - [\cos \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}}$	$ z > 1$
⑩ $\sin \omega_0 n u(n)$	$\frac{(\sin \omega_0) z^{-1}}{1 - (2 \cos \omega_0) z^{-1} + z^{-2}}$	$ z < 1$
⑪ $[r^n \cos \omega_0 n] u(n)$	$\frac{1 - [r \cos \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + r^2 z^{-2}}$	$ z > r$
⑫ $[r^n \sin \omega_0 n] u(n)$	$\frac{(r \sin \omega_0) z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + r^2 z^{-2}}$	$ z > r$
⑬ $\begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - a z^{-1}}$	$ z > 0$

Properties of ROC for Z-transform

- [1] ROC is a ring or disk in z-plane centred at the origin i.e. bounded by a pole or (∞ or 0).

$$0 \leq r_r < |z| < r_l \leq \infty$$

- [2] F.T. converges if and only if ROC of Z-transform includes unit circle.
- [3] ROC cannot contain any poles.

- [4] If $x(n)$ is a finite-duration seq. i.e. $x(n) = 0$ except in $-\infty < N_1 \leq n \leq N_2 < \infty$ interval, then ROC is the entire z-plane except possibly $z=0$ or $z=\infty$.

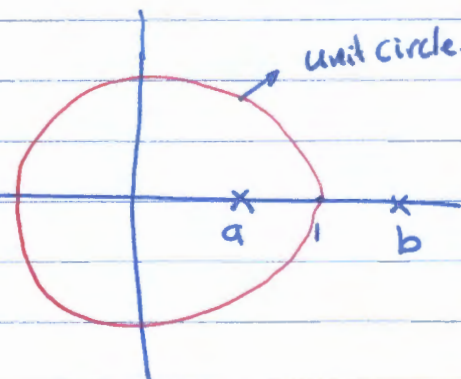
- [5] If $x(n)$ is right-sided seq., $x(n) = 0, n < 0$
 $\Rightarrow$ ROC is outside of the outermost (i.e. largest magnitude) finite pole to (and possibly including) $z=\infty$.

- [6] Left-sided seq. $x(n) = 0$ for $n > 0$. $\Rightarrow$
ROC extends inward from the innermost (i.e. smallest magnitude) nonzero pole to (and possibly including) $z=0$.

- [7] Two-sided seq. $\Rightarrow$ ROC is a ring in z-plane, bounded on the interior and exterior by a pole (and not containing any poles).

- [8] ROC must be connected region.

ROC? Which sided? F.T. converges?



* There are three choices of ROC

<u>ROC</u>	<u>sided</u>	<u>F.T. Convergence</u>
1. $ z < a$	left	No
2. $a < z < b$	Two	Yes
3. $ z > b$	Right	No

$$\begin{array}{ccccccc}
 y(n) & = & x(n) & * & h(n) \\
 \uparrow & & \uparrow & & \downarrow & & \updownarrow \\
 Y(z) & = & X(z) & \cdot & H(z)
 \end{array}$$

$H(z) \triangleq$ system function or Transfer function.

$h(n)$ is absolutely summable $\Rightarrow$ has F.T.

✓ Stable $\longleftrightarrow$ ROC includes Unit Circle.

✓ Causal $\longleftrightarrow h(n) = 0$ for $n < 0 \Rightarrow$ Right-sided
 $\Rightarrow$ ROC is outside outermost pole.

Example: $y(n) - \frac{1}{2}y(n-1] = x(n)$

To get system function, we use a useful property

$$\begin{array}{l}
 y(n) \xrightarrow{Z} Y(z) \\
 y(n+n_0) \xrightarrow{Z} z^{n_0} Y(z)
 \end{array}$$

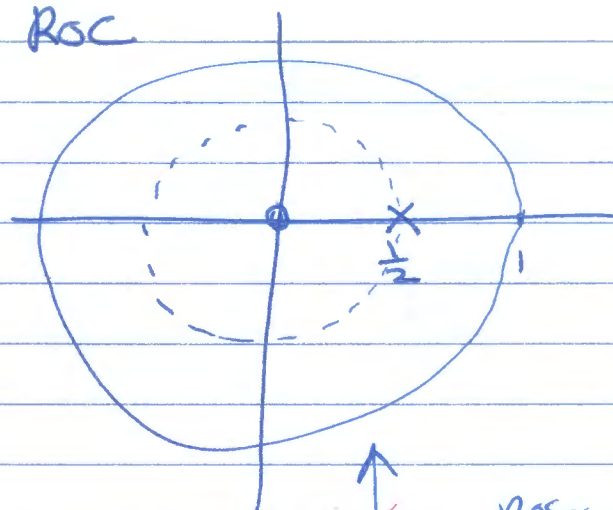
$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \text{ROC?}$$

zeros: at $z=0$

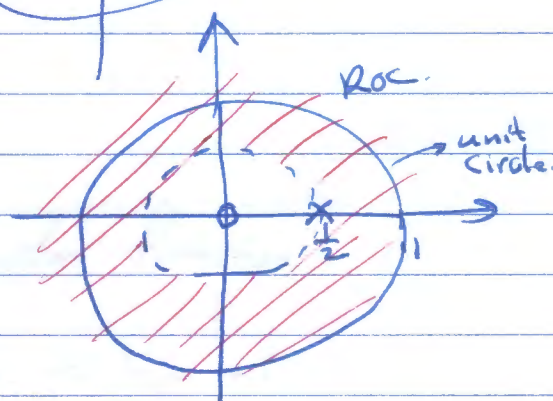
poles: at $z = \frac{1}{2}$

* There are two choices for ROC



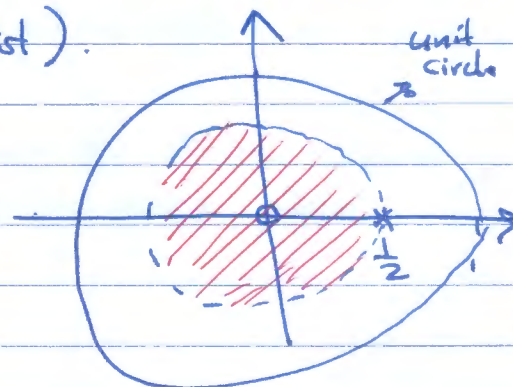
1 $|z| > \frac{1}{2}$

- Stable system (F.T. exist).
- Causal (Right-sided)
- $h(n) = (\frac{1}{2})^n u(n)$



2 $|z| < \frac{1}{2}$

- Not Stable. (F.T. doesn't exist).
- Non-Causal
- $h(n) = -(\frac{1}{2})^n u(-n-1)$



Inverse Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow x(n) = \frac{1}{2\pi j} \oint_C X(z) dz$$

1. Inspection Method (Informal)

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u(-n-1) \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

E.g. $X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, |z| > \frac{1}{2}$

by Inspection (from table) $\Rightarrow x(n) = (\frac{1}{2})^n u(n)$

if ROC $|z| < \frac{1}{2} \Rightarrow x(n) = -(\frac{1}{2})^n u(-n-1)$

2. Partial Fraction Expansion

$$X(z) = \frac{P(z^{-1})}{Q(z^{-1})} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

$$X(z) = \frac{z^N \sum_{k=0}^M b_k z^{M-k}}{z^N \sum_{k=0}^N a_k z^{N-k}}$$

$\Rightarrow$ M zeros and N poles at nonzero locations in z-plane.

In addition, There will be either $M-N$ poles at $z=0$ if $M < N$ or $N-M$ zeros at $z=0$ if $N < M$. i.e. $X(z)$ has the same number of zeros and poles and there are no poles or zeros at $z=\infty$.

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

c_k 's are nonzero zero

d_k 's " " poles

* IF $M < N$ and poles are all first-order, then $X(z)$ can be expressed as:

$$X(z) = \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}$$

Multiplying both sides by $(1 - d_k \bar{z}^{-1})$ and evaluating for $z = d_k$ shows that the coefficient A_k can be found by

(Residues) $A_k = (1 - d_k \bar{z}^{-1}) X(z) \Big|_{z=d_k}$

Example: $X(z) = \frac{1}{(1 - \frac{1}{4}\bar{z}^{-1})(1 - \frac{1}{2}\bar{z}^{-1})}, |z| > \frac{1}{2}$

$$X(z) = \frac{A_1}{1 - \frac{1}{4}\bar{z}^{-1}} + \frac{A_2}{1 - \frac{1}{2}\bar{z}^{-1}}$$

$$A_1 = (1 - \frac{1}{4}\bar{z}^{-1}) X(z) \Big|_{z=\frac{1}{4}} = -1$$

$$A_2 = (1 - \frac{1}{2}\bar{z}^{-1}) X(z) \Big|_{z=\frac{1}{2}} = 2$$

So, $X(z) = \frac{2}{1 - \frac{1}{2}\bar{z}^{-1}} + \frac{-1}{1 - \frac{1}{4}\bar{z}^{-1}}$

$$\Rightarrow h(n) = 2 \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n). \quad \left(\text{Right-sided since } \text{Roc } |z| > \frac{1}{2} \right)$$

* IF $M \geq N$:

we divide $P(\bar{z}^{-1})$ by $Q(\bar{z}^{-1})$ by long division and we use partial fraction expansion for the remainder.

Example:

$$X(z) = \frac{1 + 2\bar{z}^{-1} + \bar{z}^{-2}}{1 - \frac{3}{2}\bar{z}^{-1} + \frac{1}{2}\bar{z}^{-2}}, |z| > 1$$

$$X(z) = \frac{(1 + \bar{z}^{-1})^2}{(1 - \frac{1}{2}\bar{z}^{-1})(1 - \bar{z}^{-1})}$$

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}, \quad B_0 = 2 \text{ found by long division}$$

$$A_1 = \left(1 - \frac{1}{2}z^{-1}\right)X(z) \Big|_{z=\frac{1}{2}} = -9$$

$$A_2 = (1 - z^{-1})X(z) \Big|_{z=1} = 8$$

$$\begin{array}{r} 2 \\ \hline \frac{1}{2}z^2 - \frac{3}{2}z^1 + 1 \overline{) z^2 + 2z^1 + 1} \\ \underline{z^2 - 3z^1 + 2} \\ 5z^1 - 1 \end{array}$$

$$\text{So, } X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}$$

by inspection and using linearity of z-transform, and ROC $|z| > 1 \Rightarrow$ Right sided.

$$x(n) = 2\delta(n) - 9\left(\frac{1}{2}\right)^n u(n) + 8u(n).$$

3. Power Series Expansion:

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} = \dots + x(-2)z^2 + x(-1)z + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + \dots$$

Example: $X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$

Although $X(z)$ is obviously a rational function, its only poles are at $z=0$, so Partial Fraction Expansion is not appropriate.

* by Multiplying factors:

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

therefore, by inspection, $x(n) = \begin{cases} 1, & n=-2 \\ -\frac{1}{2}, & n=-1 \\ -1, & n=0 \\ \frac{1}{2}, & n=1 \\ 0, & \text{otherwise} \end{cases}$

Equivalently, $x(n) = \delta(n+2) - \frac{1}{2}\delta(n+1) - \delta(n) + \frac{1}{2}\delta(n-1)$

Example: $X(z) = \log(1 + a z^{-1})$, $|z| > |a|$

by using power series expansion for $\log(1+x)$ with $|x| < 1$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$$\boxed{\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}}$$

Therefore,

$$x(n) = \begin{cases} (-1)^{n+1} \frac{a^n}{n}, & n \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Example 1

$$X(z) = \frac{1}{1 - a z^{-1}}, \quad |z| > |a|$$

* since ROC is exterior of a circle $\Rightarrow$ Right-sided seq.

* since $X(z)$ approaches a finite constant as z approaches ∞
 $\Rightarrow$ seq. $x(n)$ is Causal.

* by using long division

$$X(z) = 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \dots$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow x(n) = a^n u(n).$$

$$\begin{array}{r} 1 - a z^{-1} \overline{) 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \dots} \\ \underline{1 - a z^{-1}} \phantom{+ a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \dots} \\ a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \dots \\ \underline{a z^{-1} - a^2 z^{-2}} \phantom{+ a^3 z^{-3} + a^4 z^{-4} + \dots} \\ a^2 z^{-2} + a^3 z^{-3} + a^4 z^{-4} + \dots \\ \underline{a^2 z^{-2} - a^3 z^{-3}} \phantom{+ a^4 z^{-4} + \dots} \\ a^3 z^{-3} + a^4 z^{-4} + \dots \\ \underline{a^3 z^{-3} - a^4 z^{-4}} \\ a^4 z^{-4} + \dots \\ \vdots \end{array}$$

4. Contour Integration (Formal method)

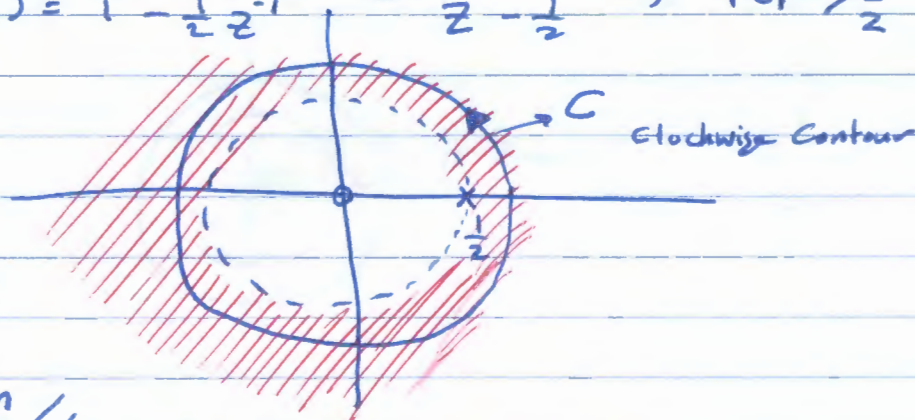
$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Complex integration

$$= \sum (\text{residues of } X(z) z^{n-1} \text{ at poles of } X(z) z^{n-1} \text{ inside } C)$$

Example:

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2}$$



$$X(z) z^{n-1} = \frac{z^n}{(z - \frac{1}{2})}$$

$n \geq 0 \Rightarrow$ 1 pole at $z = \frac{1}{2}$
Residue of $\frac{z^n}{z - \frac{1}{2}}$ at $z = \frac{1}{2}$

$$\left. \frac{z^n}{(z - \frac{1}{2})} \right|_{z = \frac{1}{2}} = \left(\frac{1}{2}\right)^n$$

$$n \geq 0 \Rightarrow x(n) = \left(\frac{1}{2}\right)^n$$

for $n < 0$ one pole at $z = \frac{1}{2}$
 n poles at $z = 0$

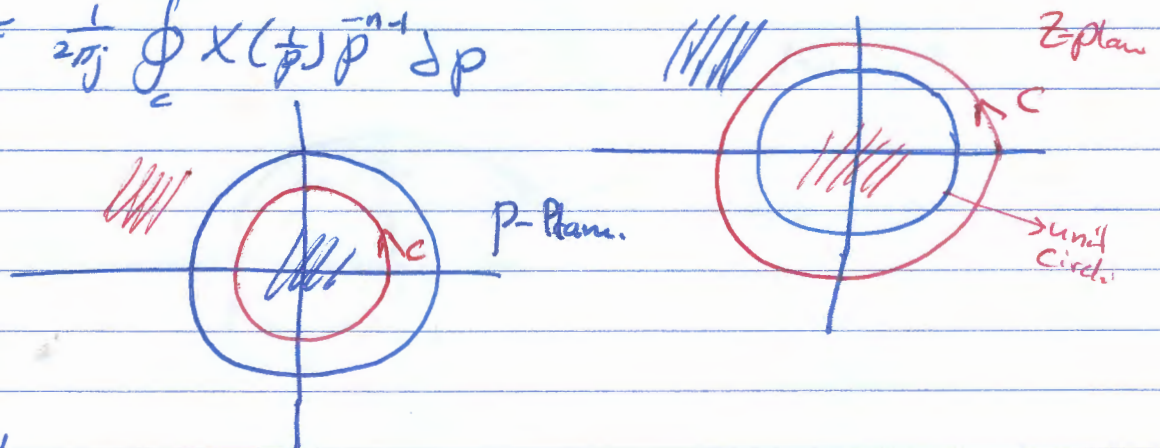
Easy way (we know $x(n) = 0$ for $n < 0$)

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$$\text{Let } z = -p^{-1} \Rightarrow z^{n-1} = -p^{-n+1}$$

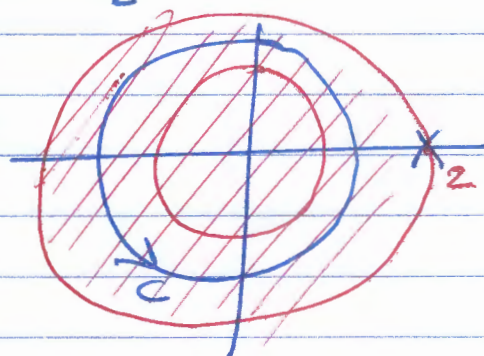
$$\text{If } z = re^{j\theta} \text{ then } p = \left(\frac{1}{r}\right)e^{-j\theta}$$

$$\begin{aligned} X(n) &= \frac{-1}{2\pi j} \oint_C X\left(\frac{1}{p}\right) p^{-n+1} dp \\ &= \frac{1}{2\pi j} \oint_C X\left(\frac{1}{p}\right) p^{-n+1} dp \end{aligned}$$



$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\begin{aligned} X\left(\frac{1}{p}\right) &= \frac{1}{1 - \frac{1}{2}p}, \quad |p| < \frac{1}{2} \\ &= \frac{-2}{p-2} \end{aligned}$$



$$X\left(\frac{1}{p}\right) p^{-n+1}$$

$n < 0 \Rightarrow 1 \text{ Pole at } p=2$

$\therefore x(n) = 0 \text{ for } n < 0$

$n \geq 0$: 1 pole at $p=2 \rightarrow$ this case is already handled.
(n+1) poles at $p=0$

$$\text{Finally } x(n) = \left(\frac{1}{2}\right)^n u(n)$$

Z-Transform Properties

$$x(n) \xrightarrow{Z} X(z), \text{ ROC} = R_x$$

$$x_1(n) \xrightarrow{Z} X_1(z), \text{ ROC} = R_{x_1}$$

$$x_2(n) \xrightarrow{Z} X_2(z), \text{ ROC} = R_{x_2}$$

① Linearity

$$ax_1(n) + bx_2(n) \xrightarrow{Z} aX_1(z) + bX_2(z)$$

$$\text{ROC} = R_{x_1} \cap R_{x_2}$$

② Time shifting

$$x(n-n_0) \xrightarrow{Z} z^{-n_0} X(z), \text{ ROC} = R_x \text{ (except for possible addition or deletion of } z=0 \text{ or } z=\infty)$$

Proof:

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

Substitute $m = n - n_0$

$$Y(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x(m) z^{-m} = z^{-n_0} X(z)$$

Example:

$$X(z) = \frac{1}{z - \frac{1}{4}}, \quad |z| > \frac{1}{4}$$

from ROC $[|z| > \frac{1}{4}] \Rightarrow x(n)$ is right-sided sequence.

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}$$

This z-transform is of the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

with $M = N \Rightarrow$ by long division

$$\begin{array}{r} -\frac{1}{4}z^{-1} + 1 \overline{) z^{-1}} \\ \underline{-z^{-1} + \frac{1}{4}} \\ 4 \end{array}$$

$$\text{So, } \frac{z}{1 - \frac{1}{4}z^{-1}} = -4 + \frac{4}{1 - \frac{1}{4}z^{-1}}$$

$$\Rightarrow x(n) = -4\delta(n) + 4\left(\frac{1}{4}\right)^n u(n).$$

* Alternatively, $x(n]$ can be obtained directly by applying time-shifting property.

$$X(z) = z^{-1} \left(\frac{1}{1 - \frac{1}{4}z^{-1}} \right), |z| > \frac{1}{4}$$

$$\Rightarrow x(n) = \left(\frac{1}{4}\right)^{n-1} u(n-1).$$

3. Multiplying by exponential sequence:

$$z_0^n x(n) \xrightarrow{Z} X(z/z_0), \text{ ROC} = R_x |z_0|$$

Similar to Fourier Transform $e^{j\omega n} x(n) \xrightarrow{\text{F.T.}} X(e^{j(\omega - \omega_0)}).$

Example:

$$u(n) \xrightarrow{Z} \frac{1}{1 - z^{-1}}, |z| > 1$$

$$x(n) = r^n \cos(\omega_0 n) u(n)$$

First, $x(n)$ can be expressed as:

$$x(n) = \frac{1}{2} (r e^{j\omega_0})^n u(n) + \frac{1}{2} (r e^{-j\omega_0})^n u(n)$$

$$\Rightarrow \frac{1}{2} (r e^{j\omega_0})^n u(n) \xrightarrow{Z} \frac{\frac{1}{2}}{1 - r e^{j\omega_0} z^{-1}}, |z| > r$$

$$\frac{1}{2} (r e^{-j\omega_0})^n u(n) \xrightarrow{Z} \frac{\frac{1}{2}}{1 - r e^{-j\omega_0} z^{-1}}, |z| > r$$

$$X(z) = \frac{\frac{1}{2}}{1 - r e^{j\omega_0} z^{-1}} + \frac{\frac{1}{2}}{1 - r e^{-j\omega_0} z^{-1}}, |z| > r$$

$$= \frac{(1 - r \cos \omega_0 z^{-1})}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}}, |z| > r$$

4. Differentiation of $X(z)$

$$n x(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz}, \text{ ROC} = R_x$$

proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} -n x(n) z^{-n-1}$$

$$\frac{dX(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (n x(n)) z^{-n} = Z\{n x(n)\} \quad \therefore$$

Example:

$$X(z) = \log(1 + a z^{-1}), \quad |z| > |a|$$

$$\frac{dX(z)}{dz} = \frac{-a z^{-2}}{1 + a z^{-1}}$$

$$n x(n) \xleftrightarrow{z} -z \frac{dX(z)}{dz} = \frac{a z^{-1}}{1 + a z^{-1}}, \quad |z| > |a|$$

$$n x(n) = a (-a)^{n-1} u(n-1)$$

$$x(n) = (-1)^{n+1} \frac{a^n}{n} u(n-1) \xleftrightarrow{z} \log(1 + a z^{-1}), \quad |z| > |a|$$

Example: Find $Z\{n a^n u(n)\}$, $x(n) = n a^n u(n) = n(a^n u(n))$

$$X(z) = -z \frac{d}{dz} \left(\frac{1}{1 - a z^{-1}} \right), \quad |z| > |a|$$

$$= \frac{a z^{-1}}{(1 - a z^{-1})^2}, \quad |z| > |a|$$

Therefore,

$$n a^n u(n) \xleftrightarrow{z} \frac{a z^{-1}}{(1 - a z^{-1})^2}, \quad |z| > |a|$$

5 Conjugate of a complex sequence.

$$x^*(n) \xrightarrow{z} X^*(z^*), \text{ ROC} = R_x$$

6 Time Reversal

$$x^*(-n) \xrightarrow{z} X^*(1/z^*), \text{ ROC} = 1/R_x$$

i.e. ROC is inverted $\Rightarrow$ if $R_x \Rightarrow r_R < |z| < r_L \Rightarrow$ ROC becomes

$$\frac{1}{r_L} < |z| < \frac{1}{r_R}, \text{ thus if } z_0 \text{ is in ROC of } x(n), \text{ then}$$

$$\frac{1}{z_0^*} \text{ is in ROC of } z \{x^*(-n)\}.$$

* If $x(n)$ is real sequence $\Rightarrow$

$$x(-n) \xrightarrow{z} X\left(\frac{1}{z}\right), \text{ ROC} = R_x$$

Example:

$$x(n) = a^{-n} u(-n), \quad X(z) = ?$$

$x(n)$ is time-reversal of $a^n u(n)$.

$$X(z) = \frac{1}{1 - az} = \frac{-a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}, \quad |z| < |a^{-1}|$$

7 Convolution of sequences:

$$x_1(n) * x_2(n) \xrightarrow{z} X_1(z) X_2(z), \text{ ROC contains } R_{x1} \cap R_{x2}$$

Proof:

$$y(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] z^{-n}$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}$$

let $m = n - k$

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{m=-\infty}^{\infty} x_2(m) z^{-m} \right] z^{-k}$$

$$\Rightarrow Y(z) = X_1(z) X_2(z) \quad \#$$

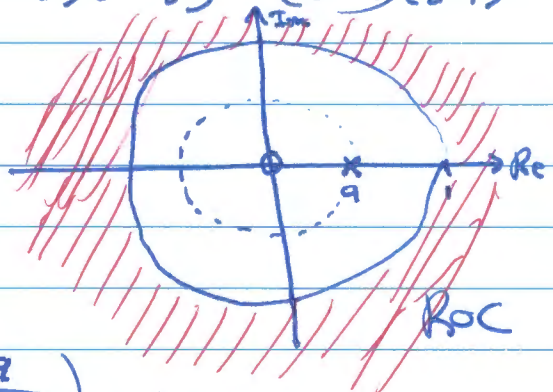
Example: $x_1(n) = a^n u(n)$ and $x_2(n) = u(n)$

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - a z^{-1}}, \quad |z| > |a|$$

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

if $|a| < 1$

$$\Rightarrow Y(z) = \frac{1}{(1 - a z^{-1})(1 - z^{-1})} = \frac{z^2}{(z - a)(z - 1)}, \quad |z| > 1$$



z-plane.

$$Y(z) = \frac{1}{1-a} \left(\frac{1}{1-z^{-1}} - \frac{a}{1-a z^{-1}} \right), \quad |z| > 1$$

$$\Rightarrow y(n) = \frac{1}{1-a} \left(u(n) - a^{n+1} u(n) \right)$$

See table 3.2 page 126 which summarizes z-transform properties.

Initial Value Theorem

if $x(n) = 0$ for $n < 0$ (i.e. $x(n)$ is causal), then

$$x(0) = \lim_{z \rightarrow \infty} X(z)$$

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(n) \lim_{z \rightarrow \infty} z^{-n} = x(0)$$