Problem

Exercise refers to the Euler phi function, denoted ϕ, which is defined as follows: For each integer $n \geq 1$, $\phi(n)$ is the number of positive integers less than or equal to n that have no common factors with n except ±1. For example, $\phi(10) = 4$ because there are four positive integers less than or equal to 10 that have no common factors with 10 except ±1; namely, 1, 3, 7, and 9.

Exercise

Prove that if p is a prime number and n is an integer with $n > 1$, then $\phi(pn) = pn - pn-1$.

Step-by-step solution

Step 1 of 1

The objective is to prove that if p is a prime number and n is an integer with $n \geq 1$, then $\phi(p^n) = p^n - p^{n-1}$

The positive integers less than p^n and co-prime to p^n are different from the divisors of p^n

$$\begin{array}{l} 1, p, 2p, 3p, \ldots (p-1)p, \\ p^2, 2p^2, 3p^2, \ldots, (p-1)p^2, \\ p^3, 2p^3, 3p^3, \ldots, (p-1)p^3, \\ \vdots \quad \vdots \quad \vdots \quad \ldots \quad \vdots \\ p^{n-1}, 2p^{n-1}, 3p^{n-1}, \ldots, (p-1)p^{n-1}, p^n \end{array}$$

Are the divisors of p^n. These are p^{n-1} in number.

That is, there are p^{n-1} integers between 1 and p^n divisible by p,

Namely, $p, 2p, 3p, \ldots, (p^{n-1})p$

All the remaining positive integers less than p^n are co-prime to p^n

Thus, the set $\{1, 2, \ldots, p^n\}$ contains exactly $p^n - p^{n-1}$ integers that are relatively prime to p^n, and so $\phi(p^n) = p^n - p^{n-1}$.

Therefore, for $n \geq 1$, the number of positive integers not exceeding n that are relative prime to n.

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