# 3.3 Linear Independence

**Def** A minimal spanning set is a spanning set with no unnecessary elements. That is, all elements in the set are needed in order to span the vector space.

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**Exp** Let $S$ be the subset of $\mathbb{R}^3$ spanned by the vectors

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}^T, \quad \vec{x}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}^T, \quad \vec{x}_3 = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}^T$$

→ In fact every vector in $S$ can be represented in terms of $x_1$ and $x_2$ since $\vec{x}_3 \in \text{span}(\vec{x}_1, \vec{x}_2)$. That is $\boxed{\vec{x}_3 = 3\vec{x}_1 + 2\vec{x}_2}$ *

- This means that any linear combination of $\vec{x}_1, \vec{x}_2, \vec{x}_3$ can be reduced to a linear combination of $x_1$ and $x_2$ since

$$\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 + \alpha_3 \vec{x}_3 = \alpha \vec{x}_1 + \alpha_2 \vec{x}_2 + \alpha_3 (3\vec{x}_1 + 2\vec{x}_2)$$
$$= (\alpha_1 + 3\alpha_3)\vec{x}_1 + (\alpha_2 + 2\alpha_3)\vec{x}_2$$

- Thus, $S = \text{span}(x_1, x_2, x_3) = \text{span}(x_1, x_2)$

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→ From * $\Rightarrow$ $3\vec{x}_1 + 2\vec{x}_2 - 1\vec{x}_3 = \vec{0}$.

$\Rightarrow$ Since the three coefficients are non zero

- $\vec{x}_1 = \frac{-2}{3}\vec{x}_2 + \frac{1}{3}\vec{x}_3 \quad \Rightarrow \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \text{span}(\vec{x}_2, \vec{x}_3)$
- $\vec{x}_2 = \frac{-3}{2}\vec{x}_1 + \frac{1}{2}\vec{x}_3 \quad \Rightarrow \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \text{span}(\vec{x}_1, \vec{x}_3)$
- $\vec{x}_3 = 3\vec{x}_1 + 2\vec{x}_2 \quad \Rightarrow \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \text{span}(\vec{x}_1, \vec{x}_2)$




$$S = \text{span}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \text{span}(\vec{x}_2, \vec{x}_3) = \text{span}(\vec{x}_1, \vec{x}_3) = \text{span}(\vec{x}_1, \vec{x}_2)$$

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→ $\vec{x}_1$ is not multiple of $\vec{x}_2$
$\vec{x}_2$ 〃 〃 〃 〃 $\vec{x}_3$
$\vec{x}_3$ 〃 〃 〃 〃 $\vec{x}_1$

$\Rightarrow$ • $\text{span}(\vec{x}_1)$, $\text{span}(\vec{x}_2)$ are proper subspaces of $\text{span}(\vec{x}_1, \vec{x}_2)$

• ...

**Th**** $\boxed{1}$ If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span a vector space $V$ and one of these vectors can be written as linear combination of the other $n-1$ vectors, then those $n-1$ vectors span $V$.

**Proof** • let $\vec{v} \in V$ be any element

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_{n-1} \vec{v}_{n-1} + \alpha_n \vec{v}_n \quad \dots (1)$$

• Assume $\vec{v}_n$ is the vector that can be written as

$$\vec{v}_n = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \dots + \beta_{n-1} \vec{v}_{n-1}$$

• Hence (1) becomes

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_{n-1} \vec{v}_{n-1} + \alpha_n \left( \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \dots + \beta_{n-1} \vec{v}_{n-1} \right)$$

$$= (\alpha_1 + \alpha_n \beta_1) \vec{v}_1 + (\alpha_2 + \alpha_n \beta_2) \vec{v}_2 + \dots + (\alpha_{n-1} + \alpha_n \beta_{n-1}) \vec{v}_{n-1}$$

• Thus, $\vec{v}$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$. Hence, these vectors span $V$.

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linearly Dependent Definition

$\boxed{2}$ Given $n$ vectors: $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. It is possible to write one of these vectors as a linear combination of the other $n-1$ vectors iff $\exists$ scalars $c_1, c_2, \dots, c_n$ not all zero s.t

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = \vec{0}$$

**Proof** $\Rightarrow$ Suppose $\vec{v}_n = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_{n-1} \vec{v}_{n-1} \Rightarrow$

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_{n-1} \vec{v}_{n-1} - \vec{v}_n = \vec{0}$$

 Let $c_i = \alpha_i$, $i = 1, \dots, n-1$ and $c_n = -1 \Rightarrow$ 

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = \vec{0}$$

$\Leftarrow$ If $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n = \vec{0}$ and at least one of $c_i \neq 0$, say $c_n$, then

$$\vec{v}_n = -\frac{c_1}{c_n} \vec{v}_1 + \frac{-c_2}{c_n} \vec{v}_2 + \dots + \frac{-c_{n-1}}{c_n} v_{n-1}$$

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**Def** The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in a vector space $V$ are **linearly independent** if $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ implies that $c_1 = c_2 = \dots = c_n = 0$.

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Note that From Th*

- If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a minimal spanning set, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent.
- If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent and span $V$, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a minimal spanning set.

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* A minimal spanning set is called a **basis** (next section).

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**Exp** ① The vectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\end{pmatrix}$ are linearly independent $\begin{vmatrix}1&1\\1&2\end{vmatrix} \neq 0$

since if $c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ then $\begin{matrix}c_1 + c_2 = 0\\ c_1 + 2c_2 = 0\end{matrix}$

$$\left[\begin{array}{cc|c}1&1&0\\1&2&0\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c}1&1&0\\0&1&0\end{array}\right] \Rightarrow \boxed{c_2 = 0} \Rightarrow \boxed{c_1 = 0}$$

is the only solution.

② The vectors $\begin{pmatrix}1\\1\\3\end{pmatrix}$ and $\begin{pmatrix}0\\2\\1\end{pmatrix}$ are linearly independent

since if $c_1\begin{pmatrix}1\\1\\3\end{pmatrix} + c_2\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ then $\boxed{c_1 = 0}$

$c_1 + c_2(2) = 0$

$\boxed{c_2 = 0}$

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**Def** The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in a vector space $V$ are **linearly dependent** if $\exists$ scalars $c_1, c_2, \dots, c_n$ not all zero s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$





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**Exp** ① The vectors $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}$ are linearly dependent

- since $(2)\begin{pmatrix}1\\1\end{pmatrix} + (-1)\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ and $\boxed{c_1 = 2}$, $\boxed{c_2 = -1}$
- That is the system $\left[\begin{array}{cc|c}1&2&0\\1&2&0\end{array}\right]$ has non trivial solution
- That is $\begin{vmatrix}1&2\\1&2\end{vmatrix} = 0$

② The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{x} = (1,2,3)$ are linearly dependent

since $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{x} = \vec{0}$ implies that

$c_1 = 1, \quad c_2 = 2, \quad c_3 = 3, \quad c_4 = -1$

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③ $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}$ are linearly dependent

since if $c_1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + c_3\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ then

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ c_1 + 2c_3 &= 0 \\ c_2 + 3c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0\\ 1 & 0 & 2 & 0\\ 0 & 1 & 3 & 0\end{array}\right] R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 3 & 0\end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0\\ 0 & 1 & 3 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

has nontrivial solution $c_1 = 2, \ c_2 = 3, \ c_3 = -1$

Note that $\begin{vmatrix} 1 & 0 & 2\\ 1 & 0 & 2\\ 0 & 1 & 3\end{vmatrix} = 0$

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**Th** Let $x_1, x_2, \cdots, x_n$ be $n$ vectors in $\mathbb{R}^n$. Let $X = [x_1, x_2, \cdots, x_n]$.

The vectors $x_1, x_2, \cdots, x_n$ are linearly dependent iff $X$ is singular.

**Proof** • Note that $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ is equivalent to

$$Xc = 0$$

• This system has non trivial solution iff $X$ is singular.

• Thus, $x_1, x_2, \cdots, x_n$ are linearly dependent iff $X$ is singular.

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Exp ① The vectors $\begin{pmatrix}4\\2\\3\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}2\\-5\\3\end{pmatrix}$ are linearly dependent

since $\begin{vmatrix} 4 & 2 & 2\\ 2 & 3 & -5\\ 3 & 1 & 3\end{vmatrix} = 0 \Leftrightarrow$ The solution set $= \{(-2\alpha, 3\alpha, \alpha) : \alpha \in \mathbb{R}\}$

② The vectors $x_1 = \begin{pmatrix}1\\-1\\2\\3\end{pmatrix}, x_2 = \begin{pmatrix}-2\\3\\1\\-2\end{pmatrix}, x_3 = \begin{pmatrix}1\\0\\7\\7\end{pmatrix}$ are linearly dep.

since $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0\\ -1 & 3 & 0 & 0\\ 2 & 1 & 7 & 0\\ 3 & -2 & 7 & 0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$ This system has non trivial solutions. $c_3$ is free.

**Th** Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be vectors in a vector space $V$. The vector $\vec{v} \in \text{Span}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$ can be written uniquely as a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ iff $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent.

**Proof** If $\vec{v} \in \text{Span}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n)$, then

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n \quad \text{---- (1)}$$

$\Leftarrow$ (By Contradiction) Assume $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent but $\vec{v}$ can be written not uniquely as linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$. Hence,

$$\vec{v} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n \quad \text{---- (2)}$$

- $(1)-(2) \Rightarrow \vec{0} = (\alpha_1 - \beta_1)\vec{v}_1 + (\alpha_2 - \beta_2)\vec{v}_2 + \cdots + (\alpha_n - \beta_n)\vec{v}_n$
  - since $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent $\Rightarrow$

    $$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_n - \beta_n = 0$$

  - Hence, $\alpha_i = \beta_i$ for all $i = 1, 2, \cdots, n$ $\times$
- This means if $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent, then (1) is the unique representation of the vector $\vec{v}$.

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$\Rightarrow$ (By Contradiction) Assume $\vec{v}$ is uniquely written as a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ but $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly dependent.

- Hence, there exists $c_1, c_2, \cdots, c_n$ not all zero s.t

$$\vec{0} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n \quad \text{---- (3)}$$

- $(1) + (3) \Rightarrow$ Let $\beta_i = c_i + \alpha_i$

$$\vec{v} = (c_1 + \alpha_1)\vec{v}_1 + (c_2 + \alpha_2)\vec{v}_2 + \cdots + (c_n + \alpha_n)\vec{v}_n$$
$$= \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n$$

- Since $c_i$ are not all zero $\Rightarrow$ at least one $\beta_i \neq \alpha_i$ for some $i$ $\Rightarrow \vec{v}$ can be written not uniquely as a linear combination of $\vec{v}_1, \cdots, \vec{v}_n$ $\times$

Exp (The Vector space $P_n$) show that $P_1(x) = x^2 - 2x + 3$, $P_2(x) = 2x^2 + x + 8$, $P_3(x) = x^2 + 8x + 7$ are linearly dependent.

Let $c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) = 0 = 0x^2 + 0x + 0$

$$c_1[x^2-2x+3) + c_2(2x^2+x+8) + c_3(x^2+8x+7) = 0$$

$$(c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0$$

$$c_1 + 2c_2 + c_3 = 0$$
$$-2c_1 + c_2 + 8c_3 = 0$$
$$3c_1 + 8c_2 + 7c_3 = 0$$

$\Rightarrow$ The coefficient matrix is singular since $\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = 0$. Hence, $P_1, P_2, P_3$ are linearly dependent. since nontrivial solution exists.

solution set $= \{(3\alpha, -2\alpha, \alpha) : \alpha \in \mathbb{R}\}$

How to determine whether $f_1, f_2, \ldots, f_n$ in $C^{(n-1)}[a,b]$ are linearly independent? (The Vector space $C^{(n-1)}[a,b]$)

* If $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly dependent, then $\exists$ scalars $c_1, c_2, \ldots, c_n$ not all zero s.t

$\forall x \in [a,b] \Rightarrow$

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{since } f^{(n-1)} \text{ exists}$$
$$c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$$
$$\vdots$$
$$c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \cdots + c_n f^{(n-1)}(x) = 0$$

Hence, the system

$$\underbrace{\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

will have non trivial solution





Thus, * Remark: If $f_1, f_2, \ldots, f_n$ are linearly dependent in $C^{(n-1)}[a,b]$, then $\forall x \in [a,b]$, the coefficient matrix $A$ is singular ($|A| = 0$).

**Def** Let $f_1, f_2, \dots, f_n$ be functions in $C^{(n-1)}[a,b]$. We define the Wronskian of $f_1, f_2, \dots, f_n$ on $[a,b]$ by

$$w(f_1, f_2, \dots, f_n)(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{vmatrix}$$

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Note that Remark* means that:

If $f_1, f_2, \dots, f_n$ are linearly dependent, then $w(f_1, f_2, \dots, f_n)(x) = 0$ for every $x \in [a,b]$

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**Th**3 let $f_1, f_2, \dots, f_n$ be elements of $C^{(n-1)}[a,b]$.

If there exist a point $x_0 \in [a,b]$ s.t $w(f_1, f_2, \dots, f_n)(x_0) \neq 0$, then $f_1, f_2, \dots, f_n$ are linearly independent.

**Proof** (by contradiction)

If $f_1, f_2, \dots, f_n$ are linearly dependent, then the coefficient matrix is singular $\forall x \in [a,b]$. That is, $w(f_1, f_2, \dots, f_n)(x) = 0 \quad \forall x \in [a,b]$. X since $\exists x_0 \in [a,b]$ s.t $w(f_1, f_2, \dots, f_n)(x_0) = 0$.

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**Exp** Are $e^{x}$ and $e^{-x}$ linearly independent in $C(-\infty, \infty)$

$$w(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2 \neq 0$$

Hence, they are linearly independent.

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**Note** If $f_1, f_2, \dots, f_n$ are linearly independent in $C^{(n-1)}[a,b]$, then $f_1, f_2, \dots, f_n$ " " " in $C[a,b]$

**Exp** Are $x^2$, $x|x|$ linearly independent in $C[-1,1]$ 68

- Both functions are in $C'[-1,1]$ since $x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

$(x|x|)' = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases} = 2|x|$

- Hence, $W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 0$

- No information can be concluded since $W=0$

So, we let $c_1 x^2 + c_2 x|x| = 0 \quad \forall x \in [-1,1]$

when $x=1 \Rightarrow c_1 + c_2 = 0$
$x=-1 \Rightarrow c_1 - c_2 = 0$
$\Rightarrow c_1 = c_2 = 0$ is the only solution.

- Hence, $x^2$ and $x|x|$ are linearly independent.

**Remark**** ① If $W(f_1, f_2, \dots, f_n) = 0$, then $f_1, f_2, \dots, f_n$ may be linearly independent or linearly dependent.

② If $W(f_1, f_2, \dots, f_n) \neq 0$, then $f_1, f_2, \dots, f_n$ are linearly indep.

**Exp** Show that $x+2$, $x^2-1$ are linearly independent

- $W(x+2, x^2-1) = \begin{vmatrix} x+2 & x^2-1 \\ 1 & 2x \end{vmatrix} = 2x^2+4x-x^2+1 = x^2+4x+1$

- since $x+2$ and $x^2-1$ are continuous at $x=0$ and it follows that $W(x+2, x^2-1)(0) = (0)^2 + 4(0) + 1 = 1 \neq 0$, $x+2$, $x^2-1$ are linearly independent by $\underline{Th}^3$