9.2 Continuous Functions and Limits at Infinity

Saturday, May 21, 2022 12:46 PM

Def The function f is continuous (cont.) at $x=c$ متصل

if ① $f(c)$ exists and

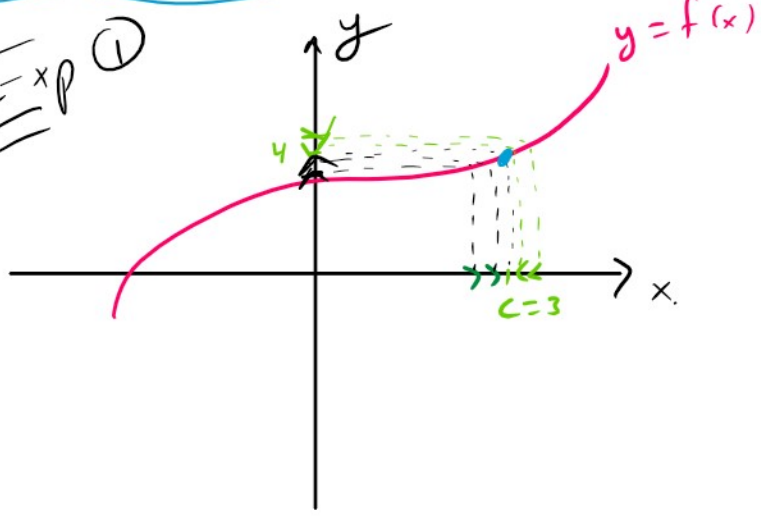
② $\lim_{x\to c} f(x)$ exists and $\rightarrow \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$

③ $\lim_{x\to c} f(x) = f(c)$

Remark: if one or more of the conditions ①,②,③ then

f is discont. at $x=c$ غير متصل (متقطع)

Exp ①



f is cont. at $x=3$

① $f(3) = 4$

② $\lim_{x\to 3^+} f(x) = 4 = \lim_{x\to 3^-} f(x)$

$\lim_{x\to 3} f(x) = 4$

③ $\lim_{x\to 3} f(x) = f(3)$

③ $\lim_{x \to 3} f(x) = f(3)$

$4 = 4$ ✓

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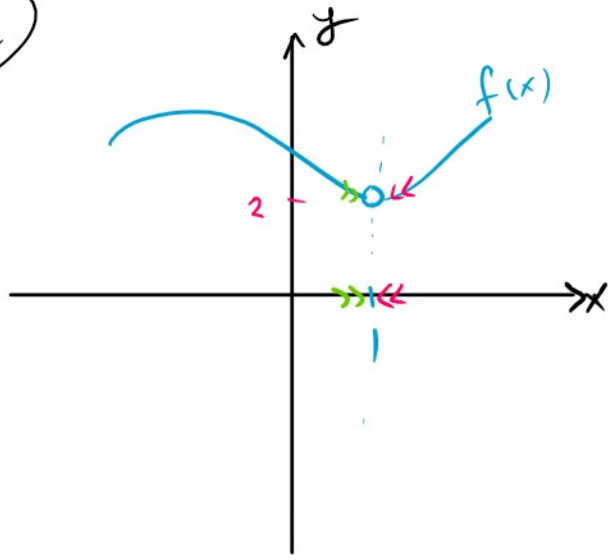
② 

① $f(1)$ undefined $\Rightarrow$

f is not cont. at $x=1$

(f discont at $x=1$)

② $\lim_{x \to 1^+} f(x) = 2 = \lim_{x \to 1^-} f(x)$

$\lim_{x \to 1} f(x) = 2$

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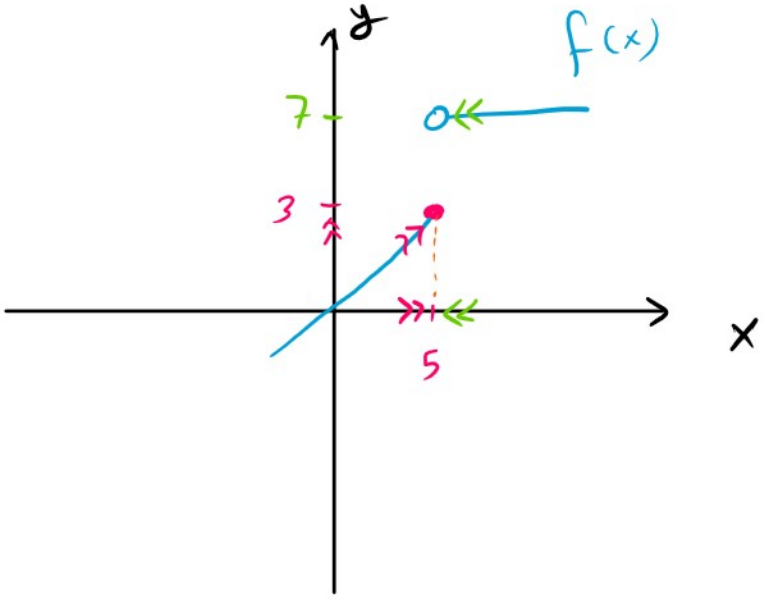
③ 

① $f(5) = 3$ ✓

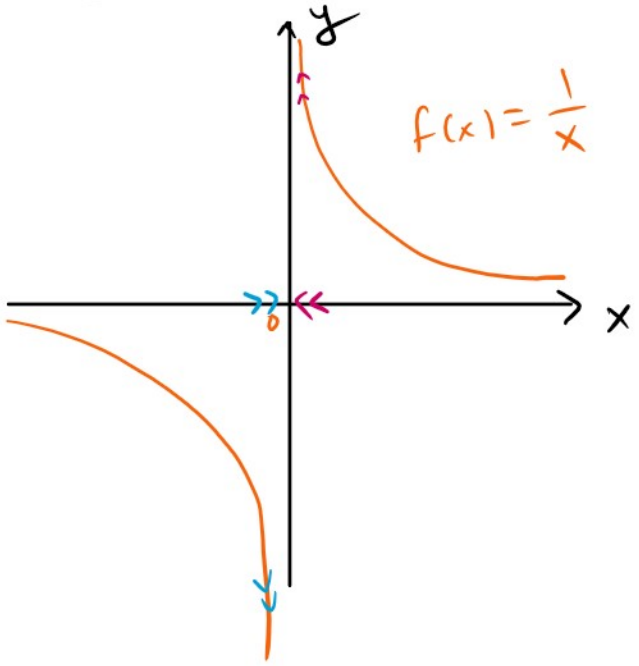
② $\lim_{x \to 5^+} f(x) = 7$

$\lim_{x \to 5^-} f(x) = 3$

$\lim_{x \to 5} f(x)$ DNE

So f is discont. at $x=5$

(4)



① $f(0)$ undefined

⇓

f is not cont. at $x=0$

② $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \frac{1}{\text{small}+} = \infty$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = \frac{1}{\text{small}-} = -\infty$

⇓ $\lim_{x\to 0} \frac{1}{x}$ DNE

Remark ① Any **Polynomial function** is cont. at every point (cont. on $\mathbb{R} = (-\infty, \infty)$)

Exp $f(x) = 2x^2 - 4x + 1$ كثيرة حدود من الدرجة 2

Poly. of degree 2 ⇐ مثال على جميع الدوال كثيرة الحدود

Take any # ⇒ $x = -1$

f cont. at $x=-1$

① $f(-1) = 2(-1)^2 - 4(-1) + 1 = 2 + 4 + 1 = 7$

② $\lim_{x\to -1} f(x) = \lim_{x\to -1} (2x^2 - 4x + 1) = 7$

③ $\lim_{x\to -1} f(x) = f(-1)$ ⇔ $7 = 7$

f cont. on $\mathbb{R} = (-\infty, \infty)$ since f is poly.

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② Any Rational function $R(x) = \frac{f(x)}{g(x)}$, f, g are Poly's. is cont. on $\mathbb{R} \setminus \{\text{zeros of } g\}$

Exp $R(x) = \frac{3x}{x^2 - 16}$ (3x: poly of degree 1; x^2-16: poly of degree 2) Find values of x where R(x) is

① cont.

② discont.

① R(x) is cont. on $\mathbb{R} \setminus \{\pm 4\}$

② R(x) is discont at $x = \pm 4$

$g(x) = 0$

$x^2 - 16 = 0$

$\sqrt{x^2} = \sqrt{16}$

$|x| = 4$

$x = 4$ or $x = -4$

Exp For what values of x, the function

① $f(x) = \frac{x^2 + 3x - 2}{x^2 - 5x}$ cont. ?

Rational function

Poly's

$x^2 - 5x = 0$

$x(x-5) = 0$

$x = 0$ or $x = 5$

f(x) cont on $\mathbb{R} \setminus \{0, 5\}$

- $f(x)$ cont on $\mathbb{R}\setminus\{0,5\}$ — $x=0$ or $x=5$

$f(x)$ is cont on $(-\infty,0)\cup(0,5)\cup(5,\infty)$

- f is not cont at $x=0$ and $x=5$

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② $h(x) = \dfrac{3-4x}{2x-8}$ ← Poly

↓ نسبي Rational

$h(x)$ is cont on $\mathbb{R}\setminus\{4\}$

$= (-\infty,4)\cup(4,\infty)$

$2x-8=0$

$2x=8$

$x=4$

$h(x)$ is discont. at $x=4$

Exp For what values of x the following Piecewise functions متعدد القاعدة are cont. ?

① $f(x) = \begin{cases} (3x+1)^2 - 5 & \text{if } x \geq 2 \\ 7 & \text{if } x < 2 \end{cases}$

poly of degree 2

$f \to 7^x$

Check $f(2) \overset{??}{=} \lim_{x\to 2} f(x)$ → ONE

poly of degree 0 → $7x^0$

Check $f(2) = \lim_{x\to 2} f(x)$ → DNE

$f(2) = (3\boxed{2}+1)^2 - 5 = (6+1)^2 - 5 = 7^2 - 5 = 49 - 5 = 44$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x+1)^2 - 5 = 44$

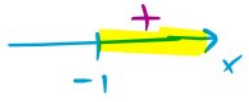
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 7 = 7$

$\lim_{x\to 2} f(x)$ DNE

So $f(2) \neq \lim_{x\to 2} f(x) \Rightarrow f$ is not cont. (dis cont) at $x=2$

f cont. on $\mathbb{R}\setminus\{2\} = (-\infty, 2)\cup(2, \infty)$

(2) $f(x) = \begin{cases} x+1 & \text{if } x > -1 \\ 1-x^2 & \text{if } x \leq -1 \end{cases}$



poly of degree 1

poly of degree 2

Check $f(-1) \overset{??}{=} \lim_{x\to -1} f(x)$

$0 \overset{??}{=} 0$

$f(-1) = 1 - \boxed{-1}^2 = 1 - 1 = 0$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x+1) = \boxed{-1} + 1 = 0$

$\lim_{x\to -1} f(x) = 0$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (\quad)$ $\lim_{x \to -1} f(x) =$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (1 - x^2) = 1 - (-1)^2 = 1 - 1 = 0$

$\Rightarrow$ f is cont. at $x = -1$

so f is cont. on $\mathbb{R} = (-\infty, \infty)$

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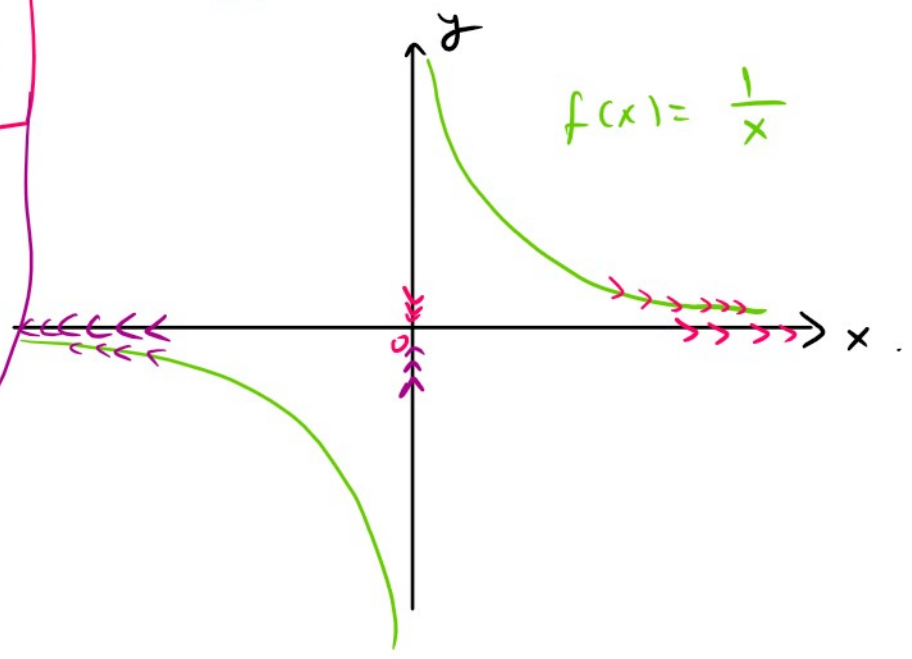
limits at infinity

$\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$



Remark (limits at infinity)

(1) $\lim_{x \to \infty} k = k$ k constant

(1) $\lim_{x \to \infty} K = K$

Exp $\lim_{x \to \infty} \sqrt{5} = \sqrt{5}$

$\lim_{x \to \infty} -4 = -4$

$\lim_{x \to -\infty} \sqrt{5} = \sqrt{5}$

$\lim_{x \to -\infty} -4 = -4$

(2) $\lim_{x \to -\infty} K = K$

(3) $\lim_{x \to \infty} \frac{K}{x^p} = 0$ if $p > 0$

Exp $\lim_{x \to \infty} \frac{1}{x^1} = 0$ $\quad p = 1 > 0$

$\lim_{x \to \infty} \frac{7}{x^3} = 0$ ✓ $\quad p = 3 > 0$

$\lim_{x \to \infty} \frac{-10}{x^{15}} = 0$ ✓ $\quad p = 15 > 0$

$\lim_{x \to \infty} \frac{110}{x^{110}} = 0$ ✓ $\quad p = 110 > 0$

(4) $\lim_{x \to -\infty} \frac{K}{x^p} = 0$ if $p > 0$

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(5) If $R(x) = \frac{f(x)}{g(x)}$ ($f(x) \to$ poly, $g(x) \to$ poly) Rational function (اقتران نسبي)

(A) if $\deg g(x) > \deg f(x)$ then

$$\lim_{x \to \infty} R(x) = \lim_{x \to -\infty} R(x) = 0$$

Exp $\lim_{x \to \infty} \frac{4x - 1}{x^3 + 5} = 0$ (نأخذ 1 من x^3)

$$\lim_{x \to \infty} \frac{\frac{4x}{x^3} - \frac{1}{x^3}}{\frac{x^3}{x^3} + \frac{5}{x^3}} = \lim_{x \to \infty} \frac{\frac{4}{x^2} - \frac{1}{x^3}}{1 + \frac{5}{x^3}}$$

($\frac{4}{x^2} \to 0$, $p = 2 > 0$; $\frac{1}{x^3} \to 0$, $p = 3 > 0$; $\frac{5}{x^3} \to 0$, $p = 3 > 0$)

$$= \frac{0 - 0}{1 + 0} = \frac{0}{1} = 0$$

(B) • If degree of $f(x) >$ deg $g(x)$ then

$$\lim_{x \to \infty} R(x) = \infty \text{ or } -\infty$$

$$\lim_{x \to -\infty} R(x) = \infty \text{ or } -\infty$$

$x \to -\infty$

Exp $\lim_{x \to \infty} \dfrac{x^2 + 5}{2 + 3x}$ $\quad \frac{+}{+} = +\infty$

$\dfrac{x^2}{3x}$

$\dfrac{x}{3}$

$\dfrac{\infty}{3} = \infty$

$\lim_{x \to \infty} \dfrac{\frac{x^2+5}{x^2}}{\frac{2+3x}{x^2}} = \lim_{x \to \infty} \dfrac{1 + \frac{5}{x^2}}{\frac{2}{x^2} + \frac{3}{x}}$ $\quad p=2>0 \to 0$, $p=1>0 \to 0$

$= \dfrac{1+0}{0+0} = \dfrac{1}{0}$

$= \infty$

Exp $\lim_{x \to -\infty} \dfrac{1 - x^3}{x^2 + 5}$ $\quad \frac{\infty}{+} = -\frac{-}{+} = +\infty$ (∞ or $-\infty$)

$\dfrac{-x^3}{x^2}$

$-x$

$-(-\infty)$

$+\infty$

$-\dfrac{(-)}{(+)} = +\infty$

$\lim_{x \to -\infty} \dfrac{\frac{1-x^3}{x^2}}{\frac{x^2+5}{x^2}} = \lim_{x \to -\infty} \dfrac{\frac{1}{x^2} - x}{1 + \frac{5}{x^2}} = \dfrac{0 - x}{1+0}$ $\quad \frac{1}{x^2} \to 0$, $\frac{5}{x^2} \to 0$

$= \lim_{x \to -\infty} -x = +\infty$

Exp $\lim \dfrac{4x^3 - 5x^6}{8x^2 + 1}$ $\quad$ (∞ or $-\infty$)

$$\lim_{x \to \infty} \frac{-5x^6}{8x^2 + 1}$$

$$\frac{-5x^6}{8x^2} = \frac{-5}{8}x^4 = -\frac{5}{8}\infty = -\infty$$

$$\frac{-5}{8} = -\infty$$

[3] If $\deg f(x) = \deg g(x)$ then

$$\lim_{x \to \infty \text{ or } -\infty} R(x) = \frac{\text{معامل أكبر درجة في البسط}}{\text{معامل أكبر درجة في المقام}} = K$$

$y = K$ is Horizontal Asymptote

Exp $$\lim_{x \to \infty} \frac{3x^2 - 8}{6x^2 - 9} = \frac{3}{6} = \frac{1}{2}$$

$y = \frac{1}{2}$ is Hor. Asy.

Exp $$\lim_{x \to -\infty} \frac{1 - x^{(1)}}{1 + 7x^{(1)}} = \frac{-1}{7}$$

$y = -\frac{1}{7}$ is H. Asy.