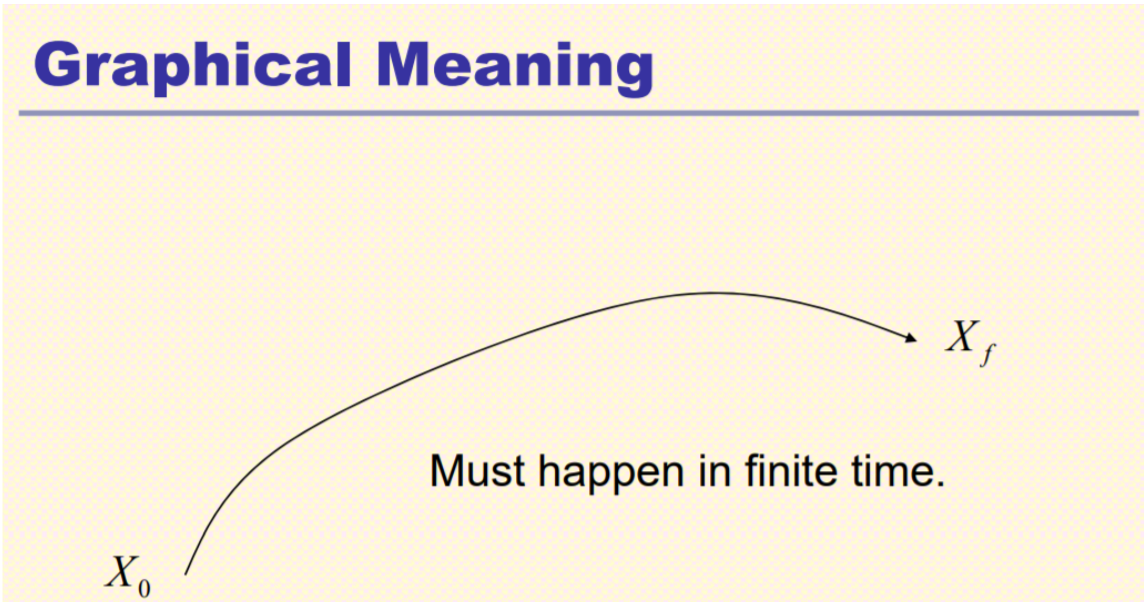
Controllability:

- A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state x_0 to any other state x_f in a **finite interval of time**.



- Controllability depends upon the system matrix A and the control influence matrix B.
- Study the controllability is the first step to design any type of controllers.
- To test the controllability there are two methods.

<u>Method 1:</u>

Let us define the state space representation for LTI system:

$$\dot{x} = Ax + Bu \\ y = cx + Du \tag{1}$$

The controllability can be studied by calculating the controllability matrix and check its rank.

- The controllability matrix is defined by:

$$M = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B] \in R^{n*n}.$$

- The system given by Equation (1) is completely state controllable if and only if the column vectors of matrix M are linearly independent, or the rank of the controllability matrix (M) is (n) then the system is completely state controllable.

- To check the rank of the M matrix calculate $|M|$.

  -if $|M|$=0 the system not fully state controllable. While if $|M| \neq$0 then the system is fully state controllable.

- In Matlab, to check the controllability use the following commands:

  -M= ctrb(A,B)

  -rank(M)

Example 1:

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$\mathbf{M} = \left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 2 & -2 \\ 1 & -2 \end{bmatrix}$$

$rank(\mathbf{M}) = 2 \quad \therefore$ The system is controllable.

$$|M| = -4 + 2 = -2$$

Example 2:

Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Since

$$[\mathbf{B} \;\vdots\; \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{singular}$$

the system is not completely state controllable.

$$|M| = 0$$

<u>Method 2:</u> By using the diagonalized conniconal form:

1) Find the eigenvalues, the eigenvectors, and the similarity transformation matrix Q.

2) Find the diagonalize conniconal form for the system as shown in previous lecture.

Case I: the eigenvalues for the system are distinct or complex. Therefore, compute $(\bar{A}, \bar{B})$

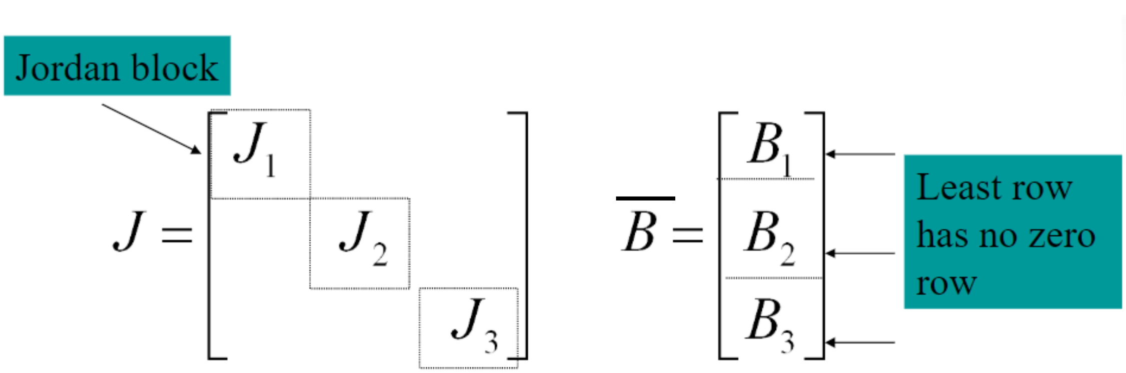
$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \bar{B} = Q^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

If there is no zero rows, then the system is fully state controllable. Otherwise, the system is not fully state controllable

-By using this method we can determine which state exactly is uncontrollable if there is.

-Also this method is preferred if the number of states is greater than four.

Case II: some of the eigenvalues for the system are repeated. Therefore, compute $(\bar{J}, \bar{B})$



The system is completely state controllable if and only if

(1) No two Jordan blocks in **J** are associated with the same eigenvalues.

(2) The elements of any row of $\bar{B}$ that correspond to the last row of each Jordan block are not all zero.

(3) The elements of each row of that correspond to distinct eigenvalues are not all zero.

The following systems are completely state controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc:cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hdashline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The following systems are not completely state controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc:cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hdashline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} u$$

Observability:

- A system is said to be observable at time t_0 with the system in state x_0, it is possible to determine this state from the observation of the output over a **finite interval of time.**

- Observability depends upon the system matrix A and the output matrix C.
- Study the observability is the first step to design any type of observers.
- To test the observability there are two methods.

<u>Method 1:</u>

Let us define the state space representation for LTI system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= cx + Du \end{aligned} \tag{1}$$

The observability can be studied by calculating the observability matrix and check its rank.

- The observability matrix is defined by:

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in R^{np*n}.$$

- The system given by Equation (1) is completely state observable if and only if the row vectors of matrix O are linearly independent, or the rank of the observability matrix (O) is (n) then the system is completely state observable.

- To check the rank of the O matrix, calculate $|O|$ if it is a square matrix.

  -if $|O|$=0 the system not fully state observable. While if $|O|$ ≠0 then the system is fully state observable.

- In Matlab, to check the observability use the following commands:

  -O= obsv(A,C)

  -rank(O)

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$O_B = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$rank(O_B) = 1 \neq 2 \quad \therefore$ The system is NOT observable.

$|O|$=0

<u>Method 2:</u> By using the diagonalized conniconal form:

3) Find the eigenvalues, the eigenvectors, and the similarity transformation matrix Q.
4) Find the diagonalize conniconal form for the system as shown in previous lecture.

Case I: the eigenvalues for the system are distinct or complex. Therefore, compute $(\bar{A}, \bar{C})$

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \bar{C} = CQ = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_n \end{bmatrix}$$

If there is no zero columns, then the system is fully state Observable. Otherwise, the system is not fully state Observable.

-By using this method we can determine which state exactly is unobservable if there is.

-Also this method is preferred if the number of states is greater than four.

Case II: some of the eigenvalues for the system are repeated. Therefore, compute $(\bar{J}, \bar{C})$

Jordan block

$$J = \begin{bmatrix} \boxed{J_1} & & \\ & \boxed{J_2} & \\ & & \boxed{J_3} \end{bmatrix}$$

$$\overline{C} = \left[C_1 \;\middle|\; C_2 \;\middle|\; C_3 \right]$$

First column has no zero column

The system is completely state controllable if and only if :

(1) No two Jordan blocks in **J** are associated with the same eigenvalues.

(2) No columns of $\bar{C}$ that correspond to the first row of each Jordan block consist of zero elements.

(3) No columns of $\bar{C}$ that correspond to distinct eigenvalues consist of zero elements.

The following systems are completely observable.

a)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$
$$y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix},$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

The following systems are completely observable.

a)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$
$$y = \begin{bmatrix}0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

b)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0 & -3 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

c)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\end{bmatrix} = \left[\begin{array}{ccc|cc}2 & 1 & 0 & & 0\\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1\\ 0 & & & 0 & -3\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix},$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$$