

10.4 Comparison Test

The comparison test: (D.C.T)

Let $\sum a_n$, $\sum c_n$ and $\sum b_n$ be series with nonnegative terms. Suppose that for some integer N .

$$d_n \leq a_n \leq c_n \text{ for all } n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges

b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges

The limit comparison test (L.C.T)

Suppose that $a_n > 0$, $b_n > 0$ for $n \geq N$

1) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both

converges or both diverges.

2) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$

converges

3) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$

diverges.

~~Remember~~ Remember that harmonic series $\sum \frac{1}{n}$ diverges

and p-series $\sum \frac{1}{n^p}$ converges if $p > 1$ and

diverges for $p \leq 1$

-1-

Questions: 8, 15, 18, 27, 28, 32, 40, 43, 52

Pages: 603 - 604

8 use D.C.T to determine if the following sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}} \quad \text{Compare with} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

(Note that $\frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}}$ and $\frac{1}{\sqrt{n}}$ nonnegative for $n \geq 1$)

(also note that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series, $p = \frac{1}{2}$ which is divergent series)

$$\text{for } n \geq 1, \sqrt{n} \geq 1, 2\sqrt{n} \geq 2$$

$$2\sqrt{n} + 1 \geq 3 \quad \text{multiply both sides by } n$$

$$2n\sqrt{n} + n \geq 3n \geq 3 \quad \text{add } n^2 \text{ for both sides}$$

$$n^2 + 2n\sqrt{n} + n \geq n^2 + 3$$

$$n(n + 2\sqrt{n} + 1) \geq n^2 + 3$$

$$\frac{n(\sqrt{n} + 1)^2}{n^2 + 3} \geq 1$$

$$\frac{(\sqrt{n} + 1)^2}{n^2 + 3} \geq \frac{1}{n} \quad \text{take the square root of both sides}$$

$$\frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}} \geq \frac{1}{\sqrt{n}} \quad \text{so } \frac{\sqrt{n} + 1}{\sqrt{n^2 + 3}} \text{ diverges by L.C.T}$$

15 use L.C.T. to determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}, \text{ compare with } \sum_{n=2}^{\infty} \frac{1}{n}$$

Note that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n}} = \lim_{n \rightarrow \infty} n = \infty$$

So by L.C.T $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges

18 use any method to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}, \text{ use L.C.T with } \sum_{n=1}^{\infty} \frac{1}{n}$$

(Note that $\frac{3}{n+\sqrt{n}}$, $\frac{1}{n}$ are positive for $n \geq 1$)

$$\lim_{n \rightarrow \infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{3n}{n+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{3}{1+\frac{1}{\sqrt{n}}} = 3$$

So both converge or both diverge.

but $\sum \frac{1}{n}$ diverges (harmonic series)

So by L.C.T $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges

27

$$\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$$

(use any method)

Compare with $\sum_{n=3}^{\infty} \frac{1}{n}$

(Note that $\frac{1}{n}$, $\frac{1}{\ln(\ln n)}$ are positive for $n \geq 3$)

$n > \ln n$ (take $\ln$ for both sides)

$$\ln n > \ln(\ln n)$$

$$\text{so } n > \ln n > \ln(\ln n)$$

$$\therefore \frac{1}{n} < \frac{1}{\ln(\ln n)}$$

$$\text{so } \sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$$

but $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series)

so by D.C.T $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges.

28

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$$

use L.C.T, Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note: $\frac{(\ln n)^2}{n^3}$, $\frac{1}{n^2}$ are positive for $n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges (P-series with $p > 1$)

$$\lim_{n \rightarrow \infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n^3} \cdot n^2 = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} = \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{2(\ln n)(\frac{1}{n})}{1} = \lim_{n \rightarrow \infty} \frac{2 \ln n}{n} = \lim_{n \rightarrow \infty} \frac{2(\frac{\infty}{\infty})}{1} = 0$$

so by L.C.T $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ Converges.

32

$$\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$$

use L.C.T compare with $\sum_{n=2}^{\infty} \frac{1}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{\frac{\ln(n+1)}{n+1}}{\frac{1}{n+1}} = \lim_{n \rightarrow \infty} \ln(n+1) = \infty$$

note that $\sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges by the integral test

$$\left(\int_2^{\infty} \frac{dx}{x+1} = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_2^b \right) = \lim_{b \rightarrow \infty} [\ln|b+1| - \ln|3|] = \infty$$

so by L.C.T $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

Note that you can use the integral test to show that $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

40

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$$

Compare with $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

Note that $\frac{2^n + 3^n}{3^n + 4^n}$ and $\frac{2^n + 3^n}{4^n}$ are positive for $n \geq 1$

$$\begin{aligned} \text{Note that } \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} &= \sum_{n=1}^{\infty} \left(\frac{2}{4} \right)^n + \left(\frac{3}{4} \right)^n \\ &= \sum_{n=1}^{\infty} \left(\frac{2}{4} \right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n = \frac{\frac{2}{4}}{1 - \frac{2}{4}} + \frac{\frac{3}{4}}{1 - \frac{3}{4}} \\ &= \frac{\frac{2}{4}}{\frac{2}{4}} + \frac{\frac{3}{4}}{\frac{1}{4}} = 1 + 3 = 4 \end{aligned}$$

So $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

by D.C.T since $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n} \leq \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

~~So~~ $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$ Converges.

We can use L.C.T $\left(\lim_{n \rightarrow \infty} \frac{\frac{2^n + 3^n}{3^n + 4^n}}{\frac{2^n + 3^n}{4^n}} = 1 \right)$

We can use L.C.T $\left(\lim_{n \rightarrow \infty} \frac{\frac{2^n + 3^n}{3^n + 4^n}}{\frac{3^n}{4^n}} = 1 \right)$

43

$$\sum_{n=2}^{\infty} \frac{1}{n!}, \text{ Compare with } \sum_{n=2}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

$$\leq \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $|r| = \left|\frac{1}{2}\right| < 1$, so it converges.

by D.C.T $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges.

also we can compare with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$
for $n \geq 2$, $(n-2)! \geq 1$

$$\text{So } n(n-1)(n-2)! \geq n(n-1)$$

$$n! \geq n(n-1)$$

$$\frac{1}{n!} \leq \frac{1}{n(n-1)}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

$$\text{Note that } \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n} \right]$$

which converges by n^{th} partial sum.

52

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$$

,

compare with

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

($\frac{\sqrt[n]{n}}{n^2}$, $\frac{1}{n^2}$ are positive for $n \geq 1$)

$$\lim_{n \rightarrow \infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{n^2} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$