# 6.3 Diagolization

**Def** • The matrix $A_{n\times n}$ is diagonalizable if there exists a nonsingular matrix $X$ and a diagonal matrix $D$ s.t $X^{-1}AX = D$. We say $X$ diagonalizes $A$.

• That is $A = XDX^{-1}$ we factorize $A$ into a product $XDX^{-1}$

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**Th 6.3.1** If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of $A_{n\times n}$ matrix with corresponding eigenvectors $x_1, x_2, \dots, x_k$, then $x_1, x_2, \dots, x_k$ are linearly independent.

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**Th 6.3.2** The matrix $A_{n\times n}$ is diagonalizable iff $A$ has $n$ linearly independent eigenvectors

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**Exp*** Factor the matrix $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$ into product $XDX^{-1}$.

• First: we find the eigenvalues $\Rightarrow |A - \lambda I| = 0 \Leftrightarrow$

$$\begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(-5-\lambda) + 6 = 0 \Leftrightarrow \lambda^2 + 3\lambda - 4 = 0$$

$$\Leftrightarrow (\lambda - 1)(\lambda + 4) = 0 \Leftrightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -4$$

• Second: we find the eigenvectors:

$\Rightarrow (A - \lambda_1 I)X = 0 \Leftrightarrow \left[\begin{array}{cc|} 1 & -3 \\ 2 & -6 \end{array}\right]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_1 = 3x_2$

$$\Leftrightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\Rightarrow (A - \lambda_2 I)X = 0 \Leftrightarrow \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = 2x_1$

$$\Leftrightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Take $\alpha = 1$

• $X = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow X^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

• $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} = X^{-1}AX$

• Hence, $A = XDX^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

## Remarks

[1] If $\underset{n\times n}{A}$ is diagonalizable (means $A = XDX^{-1}$), then

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \quad \text{and} \quad X = [x_1 \ x_2 \ \cdots \ x_n] \quad \text{where}$$

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ and
$x_1, x_2, \ldots, x_n$ are the corresponding eigenvectors.

[2] The diagolizing matrix $X$ is not unique. That is,
in Exp* if we take $\alpha = -1$, then $X = \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}$ and so
$X^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}$. Hence, $A = XDX^{-1}$ ✓ "check"

[3] If $A$ is diagonalizable, then $A$ can be factored into a product $XDX^{-1}$. That is $A = XDX^{-1}$

[4] If $A$ is diagonalizable, then the $k^{th}$ power of $A$ is

$$A^k = XD^kX^{-1} = X \begin{bmatrix} (\lambda_1)^k & & & 0 \\ & (\lambda_2)^k & & \\ & & \ddots & \\ 0 & & & (\lambda_n)^k \end{bmatrix} X^{-1}$$

since
$$\begin{aligned} A^2 = A\,A &= (XDX^{-1})(XDX^{-1}) \\ &= X\,DD\,X^{-1} \\ &= X\,D^2\,X^{-1} \end{aligned}$$

in Exp* $A^2 = XD^2X^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ -6 & 18 \end{bmatrix}$

$A^{-1} = XD^{-1}X^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{4} \end{bmatrix}\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{3}{4} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$\boxed{5}$. If $A_{n\times n}$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

- If the eigenvalues are not distinct, then $A$ may or may not diagonalizable. This depends on whether $A$ has $n$ linearly independent eigenvectors.

In Exp* $A_{2\times 2}$ has 2 distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -4$
so $A$ is diagonalizable.

Exp Is $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ diagonalizable ?

- Find eigenvalues $\Rightarrow |A - \lambda I| = 0 \iff \begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix} = 0$

$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1.$

- Find the eigenvectors:

$\rightarrow \lambda_1 = 0 \quad \Rightarrow \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ we find $N(A)$

$$\left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_3 = \alpha \\ x_2 = \alpha \\ x_1 = \alpha \end{array}$$

$$N(A) = \left\{ x \in \mathbb{R}^3 : x = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\rightarrow \lambda_2 = \lambda_3 = 1 \quad \Rightarrow \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_3 = \beta \\ x_2 = \alpha \\ x_1 = \frac{\alpha}{2} + \beta \end{array}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} + \beta \\ \alpha \\ \beta \end{pmatrix} = \frac{\alpha}{2}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

- Hence, $X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ so $A$ is diagonalizable since $A$ has 3 linearly independent eigenvectors.

- Thus, $A = XDX^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$

 

**Def.** The matrix $\underset{n\times n}{A}$ is defective if $A$ has fewer than $n$ linearly independent eigenvectors.

- Hence, defective matrix is not diagonalizable.

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**Exp** The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is defective since $\lambda_1 = \lambda_2 = 1$ and the corresponding eigenvector is multiple of $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only. That is $[A - \lambda I]x = 0$ is $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = \alpha$ and $x_2 = 0$

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**Exp** Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}$

→ $A$ and $B$ have the same eigenvalues $\lambda_1 = 4$ and $\lambda_2 = \lambda_3 = 2$

→ $A$ has eigenspace spanned by $e_2$ corresponding to $\boxed{\lambda_1 = 4}$
$A$ " " " " $e_3$ " " $\boxed{\lambda = 2}$

Hence, $A$ is defective since it has only two linearly independent eigenvectors $e_2$ and $e_3$.

→ $B$ has eigenspace spanned by $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ corresponding to $\boxed{\lambda_1 = 4}$
" " " " " $e_3$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ " " $\boxed{\lambda = 2}$

Hence, $B$ is not defective (means $B$ is diagolizable) since it has three linearly independent eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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