

Chapter 13

Vector-Valued Functions and Motion in Space.

13.1 Curves in Space and Their Tangents:

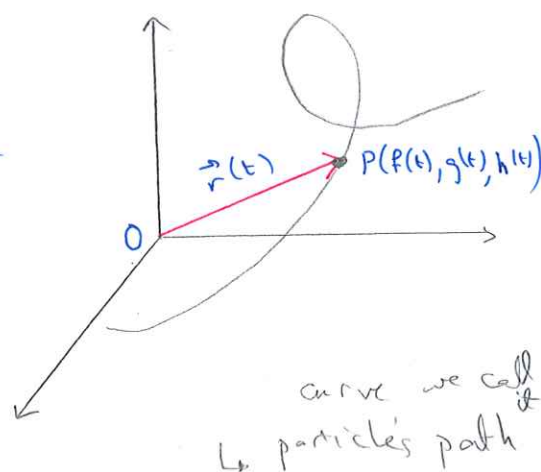
Assume a particle moves in space along a Curve C during a time Interval I

- The position of the particle at any time is determined by the position ^{vector function} vector:

function of time

$$\vec{r}(t) := f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

$\vec{OP}$



- The parametrization of the Curve C is:
 $x = f(t)$, $y = g(t)$, $z = h(t)$, $t \in I$.
- $f(t)$, $g(t)$, $h(t)$ are called Component functions.

Parametrization:

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1. $x = t$, $y = t$, $z = t$

$$0 \leq t \leq 1$$

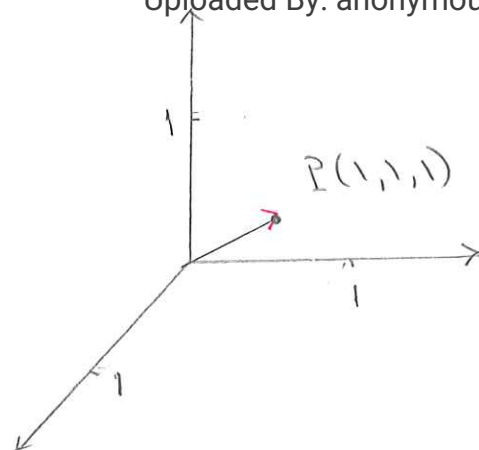
Initial point at $t = 0$

Terminal pt at $t = 1$

2. $x = t - 1$, $y = t - 1$, $z = t - 1$

$$1 \leq t \leq 2$$

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Note: The domain of the position vector $\vec{r}(t)$ is the Intersection of the domains of $f(t)$, $g(t)$ & $h(t)$.

Note: $\vec{r}(t)$ is called a vector-valued function; that assigns a vector in space to each element in the domain D .

Def: The vector $\vec{C} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$, where c_1, c_2, c_3 are constants is called Constant vector.

Limits and Continuity:

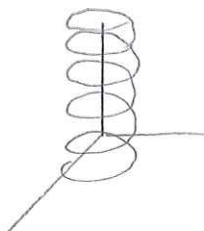
Def: Let $\vec{r}(t) = f(t) \vec{i} + g(t) \vec{j} + h(t) \vec{k}$ be a vector function with domain D , and $\vec{L}$ a vector, then:

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \lim_{t \rightarrow t_0} f(t) \vec{i} + \lim_{t \rightarrow t_0} g(t) \vec{j} + \lim_{t \rightarrow t_0} h(t) \vec{k} = \vec{L}$$

Example: Let $\vec{r}(t) = (\cos t) \vec{i} + (\sin t) \vec{j} + 2t \vec{k}$. STUDENTS HUB.com Uploaded By: anonymous

Find $\lim_{t \rightarrow \frac{\pi}{2}} \vec{r}(t)$.

$$\begin{aligned} \lim_{t \rightarrow \frac{\pi}{2}} \vec{r}(t) &= \lim_{t \rightarrow \frac{\pi}{2}} (\cos t) \vec{i} + \lim_{t \rightarrow \frac{\pi}{2}} (\sin t) \vec{j} + \lim_{t \rightarrow \frac{\pi}{2}} 2t \vec{k} \\ &= 0 \vec{i} + 1 \vec{j} + \pi \vec{k} = \vec{j} + \pi \vec{k} \end{aligned}$$



Example: $\vec{r}(t) = (\cos t) \vec{i} + (\sin t) \vec{j} + \lfloor t \rfloor \vec{k}$ ← greatest Integer function.
or "floor fun."

$\vec{r}$ is discontinuous at every Integer.

Recall: $\lfloor 6.2 \rfloor = 6$, $\lfloor 4 \rfloor = 4$, $\lfloor -6.7 \rfloor = -7$.

$$\lim_{t \rightarrow \pi/2} \vec{r}(t) = 0\vec{i} + 1\vec{j} + \lfloor \frac{\pi}{2} \rfloor \vec{k} = \vec{j} + \vec{k}$$

Derivatives

Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a position vector
the difference between the particle's position at t & $t + \Delta t$ is

$$\Delta \vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$$

$$\begin{aligned} \frac{\Delta \vec{r}}{\Delta t} &= [f(t + \Delta t)\vec{i} + g(t + \Delta t)\vec{j} + h(t + \Delta t)\vec{k}] - [f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}] \\ &= \left[\frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \vec{i} + \left[\frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \vec{j} + \left[\frac{h(t + \Delta t) - h(t)}{\Delta t} \right] \vec{k} \end{aligned}$$

$$\text{Now } \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \left[\frac{df}{dt} \right] \vec{i} + \left[\frac{dg}{dt} \right] \vec{j} + \left[\frac{dh}{dt} \right] \vec{k}$$

$$\Rightarrow \underline{\underline{\vec{v}(t)}} = \vec{r}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

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Remarks: $\square$ The position vector $\vec{r}(t)$ is called smooth

if $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$.

that is, if f, g , and h have continuous derivatives that are not all zero.

[2] The Velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt}$

($\vec{v}$ is tangent on the Curve)

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

[3] Speed is the magnitude of the Velocity $S = |\vec{v}(t)|$.

[4] Direction of the motion is a unit vector given by :

$$\text{Direction} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Example: $\vec{v}_1 = (\cos t)\vec{i} + (\sin t)\vec{j}$.

$$S = |\vec{v}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

So $\vec{v}_1$ is a Unit vector.

Example: Find Direction of particle moves along a Curve

with position vector $\vec{r}(t) = 3t\vec{i} + 4t\vec{j}$.

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$$\vec{v}(t) = 3\vec{i} + 4\vec{j}$$

$$|\vec{v}(t)| = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\therefore \text{Direction} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

Note: $|\text{Direction}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$.

Note: Velocity = (Speed) . (Direction)

$$= |\vec{v}| \cdot \frac{\vec{v}}{|\vec{v}|}$$

Differentiation Rules:

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t .

$\vec{C}$ a constant vector, c any scalar and f any differentiable scalar function.

1. Constant function Rule: $\frac{d}{dt} \vec{C} = \vec{0}$.

2. Scalar multiple Rule: $\frac{d}{dt} [c \vec{u}(t)] = c \vec{u}'(t)$.

3. Sum Rule: $\frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$.
Difference =

4. Scalar multiple Rule: $\frac{d}{dt} [f(t) \vec{u}(t)] = f'(t) \vec{u}(t) + f(t) \vec{u}'(t)$

5. Dot product Rule: $\frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
 $\frac{du}{dt} \cdot v(t) + u(t) \cdot \frac{dv}{dt}$

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6. Cross product Rule: $\frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

7. Chain Rule: $\frac{d}{dt} [\vec{u}(f(t))] = f'(t) \vec{u}'(f(t))$

Example: $\vec{w}_1 = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{w}_2 = -\vec{i} - 7\vec{k}$

$$\vec{w}_1 \cdot \vec{w}_2 = -2 + 7 = +5 \Rightarrow (w_1, w_2)' = 0$$

Recall: ② $\vec{w}_1 \cdot \vec{w}_2 = |\vec{w}_1| |\vec{w}_2| \cos \theta$ ① $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Note: ③ If $\vec{u}_1 \cdot \vec{u}_2 = 0 \Rightarrow \vec{u}_1 \perp \vec{u}_2$ ④ $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$

Example: $\vec{w}_1 = t\vec{i} + 3t\vec{j}$, $\vec{w}_2 = 3\vec{i} + 7t\vec{j}$

$$\vec{w}_1 \cdot \vec{w}_2 = 3t + 21t^2$$

$$(\vec{w}_1 \cdot \vec{w}_2)' = 3 + 42t$$

Using Rule 5: $(w_1, w_2)' = (\underbrace{\vec{i} + 3\vec{j}}_{\vec{w}_1}) \cdot (\underbrace{3\vec{i} + 7t\vec{j}}_{\vec{w}_2}) + (\underbrace{t\vec{i} + 3t\vec{j}}_{\vec{w}_1}) \cdot (\underbrace{7\vec{j}}_{\vec{w}_2})$

$$= 3 + 21t + 21t = 3 + 42t$$

Example: $\vec{w}_1 \times \vec{w}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 0 & -7 \end{vmatrix} = -21\vec{i} + 15\vec{j} + 3\vec{k}$

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Recall • $\vec{w}_1 \times \vec{w}_2 = -\vec{w}_2 \times \vec{w}_1$

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• while $\vec{w}_1 \cdot \vec{w}_2 = \vec{w}_2 \cdot \vec{w}_1$

• $\vec{w}_1 \times \vec{w}_2 = (|\vec{w}_1| |\vec{w}_2| \sin \theta) \vec{n}$

where $\vec{n}$: normal unit vector (RHR)

Right-hand Rule.

$$|\vec{w}_1 \times \vec{w}_2| = |\vec{w}_1| |\vec{w}_2| \sin \theta$$

Corollary:

Remark: Assume $\vec{r}(t)$ is differentiable vector function with constant length. then: $\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$.

proof: $|\vec{r}(t)| = C$

$$|\vec{r}(t)|^2 = C^2$$

$$\text{but } \vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = C^2.$$

$$\text{Now } \frac{d}{dt} (\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}(t) \cdot \frac{d\vec{r}}{dt} + \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$$

$$\Rightarrow 2 \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$$

$$\Rightarrow \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0.$$

Result: $\vec{r}(t) \perp \frac{d\vec{r}}{dt}$ iff $\vec{r}(t)$ has constant length.

Example: $\vec{r}(t) = (\cos t) \vec{i} + (\sin t) \vec{j}$.

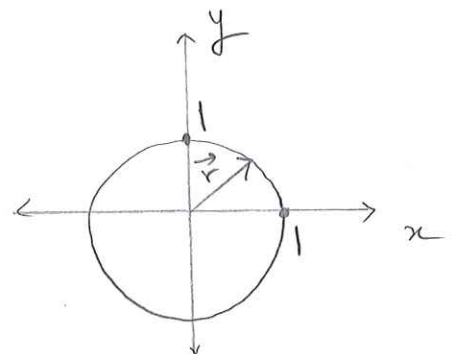
$$|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \quad (\text{constant length})$$

Cartesian equation: $\begin{cases} x = \cos t \\ y = \sin t \\ x^2 + y^2 = 1 \end{cases}$

parametric equations

$$r^2 = x^2 + y^2 = 1$$

$$\tan \theta = \frac{y}{x}$$

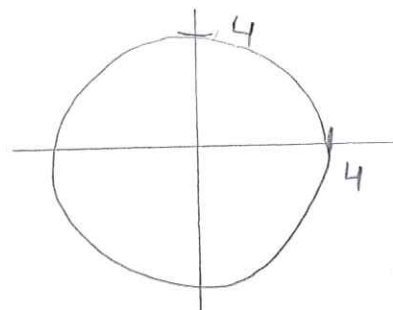


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Example: Given a position vector of a particle moves along the path $\vec{r}(t) = (4 \cos \frac{t}{2})\vec{i} + (4 \sin \frac{t}{2})\vec{j}$

1. Find a Cartesian equation in x - y plane. Draw it.

$$x = 4 \cos \frac{t}{2}, \quad y = 4 \sin \frac{t}{2}$$



$$r^2 = x^2 + y^2 = 16 \cos^2 \frac{t}{2} + 16 \sin^2 \frac{t}{2} = 16$$

2. Find velocity and acceleration at $t = \pi, \frac{3\pi}{2}$.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-2 \sin \frac{t}{2})\vec{i} + (2 \cos \frac{t}{2})\vec{j}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (-\cos \frac{t}{2})\vec{i} - (\sin \frac{t}{2})\vec{j}$$

$$\vec{v}(\pi) = -2\vec{i}$$

$$\vec{v}(\frac{3\pi}{2}) = -\sqrt{2}\vec{i} - \sqrt{2}\vec{j}$$

$$\vec{a}(\pi) = -\vec{j}$$

$$\vec{a}(\frac{3\pi}{2}) = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

3. Find the speed & the Direction of the motion at $t = 2\pi$.

$$|\vec{v}(t)| = \sqrt{2^2 \sin^2 \frac{t}{2} + 2^2 \cos^2 \frac{t}{2}} = \sqrt{4} = 2$$

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$$D|_{2\pi} = \frac{\vec{v}(2\pi)}{|\vec{v}(2\pi)|} = \frac{-2\vec{j}}{2}$$

(13.1) Q(2) $\vec{r}(t) = \frac{t}{t+1} \vec{i} + \frac{1}{t} \vec{j}$, $t = -\frac{1}{2}$

$x = \frac{t}{t+1}$ and $y = \frac{1}{t}$.

$\Rightarrow x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y} \Leftrightarrow y = \frac{1}{x} - 1$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{1}{(t+1)^2} \vec{i} - \frac{1}{t^2} \vec{j}$

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{-2}{(t+1)^3} \vec{i} + \frac{2}{t^3} \vec{j}$

$\vec{v}(-\frac{1}{2}) = 4\vec{i} - 4\vec{j}$ & $\vec{a} = -16\vec{i} - 16\vec{j}$.

for Q(20)

Recall: The tangent line to a smooth curve

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ at $t = t_0$

is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$

parallel to $\frac{d\vec{r}}{dt}(t_0) = \vec{v}(t_0)$

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Recall: A parametrization of the line through

a point $a = (f(t_0), g(t_0), h(t_0))$ & parallel to $\vec{v} = \frac{d\vec{r}}{dt}(t_0)$

is $\vec{I} = a + t\vec{v}$

13.1 (17) $\vec{r}(t) = (\ln(t^2+1))\vec{i} + (\tan^{-1}t)\vec{j} + \sqrt{t^2+1}\vec{k}$

Find the angle between $\vec{v}$ & $\vec{a}$ when $t=0$.

$$\vec{v}(t) = \frac{2t}{t^2+1}\vec{i} + \frac{1}{t^2+1}\vec{j} + \frac{t}{\sqrt{t^2+1}}\vec{k}$$

$$\vec{a}(t) = \left[\frac{-2t^2+2}{(t^2+1)^2} \right]\vec{i} - \left[\frac{2t}{(t^2+1)^2} \right]\vec{j} + \left[\frac{1}{(t^2+1)^{3/2}} \right]\vec{k}$$

$$\vec{v}(0) = \vec{j} \quad \& \quad \vec{a}(0) = 2\vec{i} + \vec{k}$$

$$|\vec{v}||\vec{a}|\cos\theta = \vec{v}(0) \cdot \vec{a}(0)$$

$$\vec{v}(0) \cdot \vec{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

(20) Find parametric equations for the line that is tangent to the given curve at $t=t_0$.

$$\vec{r}(t) = t^2\vec{i} + (2t-1)\vec{j} + t^3\vec{k}, \quad t_0 = 2$$

To find the parametric equations ^{for the line}, we need

1) a point (x_0, y_0, z_0) on the line

2) vector that is // to the line $[\vec{v} = \langle a, b, c \rangle]$

• We know that $\vec{v}(t_0) //$ line. ^{tangent line.}

$$\Rightarrow \vec{v}(t) = 2t\vec{i} + 2\vec{j} + 3t^2\vec{k}$$

$$\therefore \vec{v}(2) = 4\vec{i} + 2\vec{j} + 12\vec{k} \Rightarrow \vec{v} = \langle 4, 2, 12 \rangle$$

$$\& \quad \vec{r}(2) = 4\vec{i} + 3\vec{j} + 8\vec{k} \Rightarrow (4, 3, 8)$$

$$\Rightarrow \begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

$$\begin{aligned} x &= 4 + 4t \\ y &= 3 + 2t \\ z &= 8 + 12t \end{aligned}$$

(10)

13.2 Integrals of Vector functions

Def: If $\vec{R}(t)$ is any antiderivative of $\vec{r}(t)$, then the indefinite Integral of $\vec{r}$ is:

$$\int \vec{r}(t) dt = \vec{R}(t) + \vec{C}$$

Def: If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a, b]$, then so is $\vec{r}$, and the definite Integral of $\vec{r}$ is:

$$\int_a^b \vec{r}(t) dt = \left(\int_a^b f(t) dt \right) \vec{i} + \left(\int_a^b g(t) dt \right) \vec{j} + \left(\int_a^b h(t) dt \right) \vec{k}$$

Example: Let $\vec{r}(t) = (\cos t)\vec{i} + 3\vec{j} - 2t\vec{k}$.

Find (1) $\int \vec{r}(t) dt$

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$$\begin{aligned} \int \vec{r}(t) dt &= (\sin t + c_1)\vec{i} + (3t + c_2)\vec{j} - \left(\frac{2t^2}{2} + c_3\right)\vec{k} \\ &= \sin t \vec{i} + 3t \vec{j} - t^2 \vec{k} + \underbrace{c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}}_{\vec{C}} \end{aligned}$$

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$$\begin{aligned} (2) \int_0^\pi \vec{r}(t) dt &= \sin t \Big|_0^\pi \vec{i} + 3t \Big|_0^\pi \vec{j} - t^2 \Big|_0^\pi \vec{k} \\ &= 0 \vec{i} + 3\pi \vec{j} - \pi^2 \vec{k} = 3\pi \vec{j} - \pi^2 \vec{k}. \end{aligned}$$

Example: Let $\vec{a}(t) = (-3 \cos t)\vec{i} - (3 \sin t)\vec{j} + 2\vec{k}$

be the acceleration of a particle. Assume that at time $t=0$, the particle departed from the point $(3, 0, 0)$ with velocity $\vec{v}(0) = 3\vec{j}$.

Find the position of the particle at time $t=\pi$.

$$\vec{v}(t) = \int \vec{a}(t) dt = (-3 \sin t)\vec{i} + (3 \cos t)\vec{j} + 2t\vec{k} + \vec{C}$$

$$\vec{v}(0) = 3\vec{j} + \vec{C} = 3\vec{j} \Rightarrow \boxed{\vec{C} = 0}$$

$$\therefore \vec{v}(t) = (-3 \sin t)\vec{i} + (3 \cos t)\vec{j} + 2t\vec{k}$$

$$\text{Now } \vec{r}(t) = \int \vec{v}(t) dt = (3 \cos t)\vec{i} + (3 \sin t)\vec{j} + t^2\vec{k} + \vec{C}_2$$

$$\vec{r}(0) = 3\vec{i} + \vec{C}_2 = 3\vec{i} + 0\vec{j} + 0\vec{k} \quad (\text{from } (3, 0, 0))$$

$$\therefore \vec{r}(t) = (3 \cos t)\vec{i} + (3 \sin t)\vec{j} + t^2\vec{k}$$

$$\Rightarrow \vec{r}(\pi) = -3\vec{i} + \pi^2\vec{k}$$

13.2 (Q10) Evaluate $\int_0^{\frac{\pi}{4}} [\sec t \vec{i} + \tan^2 t \vec{j} - t \sin t \vec{k}] dt$

$$= \int_0^{\frac{\pi}{4}} [\sec t \vec{i} + (\sec^2 t - 1) \vec{j} - \underbrace{(t \sin t)}_{\text{by parts}} \vec{k}] dt$$

$$= \left[\ln(\sec t + \tan t) \right]_0^{\frac{\pi}{4}} \vec{i} + \left[\tan t - t \right]_0^{\frac{\pi}{4}} \vec{j} + \left[t \cos t - \sin t \right]_0^{\frac{\pi}{4}} \vec{k}$$

$$= \ln(1 + \sqrt{2}) \vec{i} + \left(1 - \frac{\pi}{4}\right) \vec{j} + \left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \vec{k}$$

(Q15) Solve the following IVP:

$$\frac{d^2 \vec{r}}{dt^2} = -32 \vec{k}, \quad \vec{r}(0) = 100 \vec{k} \text{ and } \left. \frac{d\vec{r}}{dt} \right|_{t=0} = 8\vec{i} + 8\vec{j}.$$

$$\frac{d\vec{r}}{dt} = \int -32 \vec{k} dt = -32t \vec{k} + \vec{C}_1.$$

$$\left. \frac{d\vec{r}}{dt} \right|_{t=0} = 8\vec{i} + 8\vec{j} = -32(0) \vec{k} + \vec{C}_1 \Rightarrow \vec{C}_1 = 8\vec{i} + 8\vec{j}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 8\vec{i} + 8\vec{j} - 32t \vec{k}.$$

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$$\text{Therefore } \vec{r}(t) = \int (8\vec{i} + 8\vec{j} - 32t \vec{k}) dt$$

$$\vec{r}(t) = 8t \vec{i} + 8t \vec{j} - 16t^2 \vec{k} + \vec{C}_2$$

$$\text{Using } \vec{r}(0) = 100 \vec{k} \Rightarrow \vec{C}_2 = 100 \vec{k}$$

$$\Rightarrow \vec{r}(t) = 8t \vec{i} + 8t \vec{j} + (100 - 16t^2) \vec{k}. \quad (13)$$

13.2 Q18 A particle traveling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at $t=0$. The particle moves toward the point $(3, 0, 3)$ with constant acceleration $2\vec{i} + \vec{j} + \vec{k}$. Find $\vec{r}(t)$?

sol:

$$\vec{v}(t) = \int (2\vec{i} + \vec{j} + \vec{k}) dt = 2t\vec{i} + t\vec{j} + t\vec{k} + \vec{C}_1 \quad \text{Eq. (1)}$$

We know that

$$\left. \vec{v}(t) \right|_{t=0} = \left(\begin{matrix} \text{Unit} \\ \text{Direction} \\ \text{vector} \end{matrix} \right) (\text{speed})$$

$$\Rightarrow \vec{v}(0) = \frac{\vec{PS}}{|\vec{PS}|} \cdot (2) = \frac{(3-1)\vec{i} + (0-(-1))\vec{j} + (3-2)\vec{k}}{|\vec{PS}|} (2).$$

$$\vec{v}(0) = \frac{1}{\sqrt{6}} (4\vec{i} + 2\vec{j} + 2\vec{k}) = \frac{4}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{2}{\sqrt{6}}\vec{k}$$

Substitute $t=0$ in Eq. (1) $\Rightarrow \vec{C}_1 = \frac{4}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{2}{\sqrt{6}}\vec{k}$

$$\vec{v}(t) = \left(2t + \frac{4}{\sqrt{6}} \right) \vec{i} + \left(t + \frac{2}{\sqrt{6}} \right) \vec{j} + \left(t + \frac{2}{\sqrt{6}} \right) \vec{k}$$

Now $\vec{r}(t) = \int \vec{v}(t) dt = \left(\frac{t^2}{2} + \frac{4t}{\sqrt{6}} \right) \vec{i} + \left(\frac{t^2}{2} + \frac{2t}{\sqrt{6}} \right) \vec{j} + \left(\frac{t^2}{2} + \frac{2t}{\sqrt{6}} \right) \vec{k} + \vec{D}$

But $\vec{r}(0) = (1)\vec{i} + (-1)\vec{j} + (2)\vec{k} = \vec{D}$

$$\Rightarrow \vec{r}(t) = \left(\frac{t^2}{2} + \frac{4t}{\sqrt{6}} + 1 \right) \vec{i} + \left(\frac{t^2}{2} + \frac{2t}{\sqrt{6}} - 1 \right) \vec{j} + \left(\frac{t^2}{2} + \frac{2t}{\sqrt{6}} + 2 \right) \vec{k} \quad (14)$$

13.3 Arc length in Space.

Def: The length of a smooth curve:

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \leq t \leq b \text{ is}$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Remark: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$$\Rightarrow |\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Hence $L = \int_a^b |\vec{v}(t)| dt$

Example: Given $\vec{r}(t) = (2+t)\vec{i} - (1+t)\vec{j} + \sqrt{2}t\vec{k}$
for $0 \leq t \leq 3$. Find the arc length for the
curve parametrized by the position vector $\vec{r}(t)$.

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$$\vec{v}(t) = \vec{i} - \vec{j} + \sqrt{2}\vec{k}$$

$$|\vec{v}(t)| = \sqrt{1+1+(\sqrt{2})^2} = \sqrt{4} = 2.$$

$$\therefore L = \int_0^3 2 dt = 6 - 0 = 6.$$

Unit Vector
↓
Tangent.

$$\frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}.$$

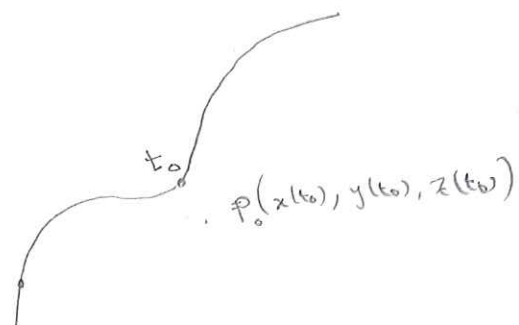
(15)

Def: Arc length parameter.

If we choose a basepoint $P(t_0)$ on a smooth curve C then the arc length parameter from the basepoint

$P(t_0)$ is defined by:

$$s(t) = \int_{t_0}^t |\vec{v}(u)| du \quad \text{--- (*)}$$



Remark: Note that $L = s(b) - s(a)$.

proof: $s(b) - s(a) = \int_{t_0}^b |\vec{v}(t)| dt - \int_{t_0}^a |\vec{v}(t)| dt$

$$= - \int_b^{t_0} |\vec{v}(t)| dt - \int_{t_0}^a |\vec{v}(t)| dt =$$

$$= - \left[\int_b^a |\vec{v}(t)| dt \right] = \int_a^b |\vec{v}(t)| dt = L.$$

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Speed on a smooth curve: $\frac{ds}{dt} = |\vec{v}(t)|$ FTOF from (*)

Unit Tangent vector: $\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$

Note: $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \vec{v} \cdot \frac{1}{|\vec{v}|} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{r}}{ds}$

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(16)

Example: Find the arc length parameter along the Curve

$$\vec{r}(t) = (\cos t + t \sin t) \vec{i} + (\sin t - t \cos t) \vec{j}$$

Consider the base point $P(1, 0, 0)$ and $\frac{\pi}{2} \leq t \leq \pi$.

So! we need to find t_0 such that!

$$\cos t_0 + t_0 \sin t_0 = 1 \quad \& \quad \sin t_0 - t_0 \cos t_0 = 0$$

clearly $t_0 = 0$.

$$\begin{aligned} \therefore \vec{v}(t) &= \frac{d\vec{r}}{dt} = (-\sin t + t \cos t + \sin t) \vec{i} + \\ &\quad (\cos t + t \sin t - \cos t) \vec{j} \\ &= (t \cos t) \vec{i} + (t \sin t) \vec{j} \end{aligned}$$

$$|\vec{v}(t)| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = |t| = t$$

$$\therefore s(t) = \int_0^t \tau d\tau = \frac{t^2}{2}$$

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[2] Find the arc length L

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$$L = S(\pi) - S\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}$$

$$\text{OR: } L = \int_{\frac{\pi}{2}}^{\pi} t dt = \frac{3\pi^2}{8}$$

13.3(16) Find the point on the Curve

$$\vec{r}(t) = (12 \sin t) \vec{i} - (12 \cos t) \vec{j} + 5t \vec{k}.$$

at a distance 13π units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.

Sol:

Let $P(t_0)$ denote the point. Then:

$$\vec{v}(t) = (12 \cos t) \vec{i} + (12 \sin t) \vec{j} + 5 \vec{k}.$$

It starts from the point $(0, -12, 0) \Rightarrow t = 0$, then:

$$\begin{aligned} \text{opposite } (-13\pi) &= \int_0^{t_0} \sqrt{(12 \cos t)^2 + (12 \sin t)^2 + 5^2} dt \quad (\text{Arc length}) \\ &= \int_0^{t_0} \sqrt{169} dt = \int_0^{t_0} 13 dt = 13 \Big|_0^{t_0} = 13t_0 \end{aligned}$$

$\Rightarrow$

$$t_0 = -\pi$$

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even or odd?

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$$\begin{aligned} \text{Then: } P(-\pi) &= (12 \sin(-\pi)) \vec{i} - 12 \cos(-\pi) \vec{j} + 5(-\pi) \vec{k} \\ &= -12 \vec{j} - 5\pi \vec{k}. \end{aligned}$$

13.3 (Q 15) Find the length of the Curve

$$\vec{r}(t) = (\sqrt{2}t)\vec{i} + (\sqrt{2}t)\vec{j} + (1-t^2)\vec{k}$$

from $(\underbrace{0, 0, 1}_{t=0})$ to $(\underbrace{\sqrt{2}, \sqrt{2}, 0}_{t=1})$

$$\left. \begin{array}{l} \sqrt{2}t = \sqrt{2} \\ \& \sqrt{2}t = \sqrt{2} \\ \& 1-t^2 = 0 \end{array} \right\} \Rightarrow t=1$$

So i: $L = \int_a^b |\vec{v}(t)| dt$

$$\vec{v}(t) = \sqrt{2}\vec{i} + \sqrt{2}\vec{j} - 2t\vec{k}$$

$$|\vec{v}(t)| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (-2t)^2} = \sqrt{4 + 4t^2} = 2\sqrt{1+t^2}$$

$$\therefore L = \int_0^1 2\sqrt{1+t^2} dt$$

Now (Calc. 1 section 8.3) Trigonometric substitution:

$$\text{Let } t = \tan \theta \Rightarrow dt = \sec^2 \theta d\theta$$

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$$\text{when } t = 0 \Rightarrow \theta = 0$$

$$t = 1 \Rightarrow \theta = \frac{\pi}{4}$$

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$$\Rightarrow L = \int_0^{\frac{\pi}{4}} 2\sqrt{1+\tan^2 \theta} (\sec^2 \theta) d\theta = \int_0^{\frac{\pi}{4}} 2\sqrt{\sec^2 \theta} \sec^2 \theta d\theta$$

$$\therefore L = \int_0^{\frac{\pi}{4}} (2) \sec^3 \theta \, d\theta.$$

Using Integration by parts:

$$\text{Let } u = \sec \theta, \quad dv = \sec^2 \theta \, d\theta$$

$$du = \sec \theta \tan \theta \, d\theta, \quad v = \tan \theta$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = \left[\sec \theta \tan \theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec \theta \tan^2 \theta \, d\theta \right]$$

$$= \sec \theta \tan \theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1) \sec \theta \, d\theta$$

$$= \sec \theta \tan \theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3 \theta \, d\theta + \int_0^{\frac{\pi}{4}} \sec \theta \, d\theta$$

to the first part $\leftarrow$

$$\Rightarrow \frac{2}{2} \int_0^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = \frac{\sec \theta \tan \theta}{2} \Big|_0^{\frac{\pi}{4}} + \frac{\ln(\sec \theta + \tan \theta)}{2} \Big|_0^{\frac{\pi}{4}}$$

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$$\text{Therefore: } L = 2 \left(\frac{\sec \theta \tan \theta}{2} \right) \Big|_0^{\frac{\pi}{4}} + 2 \left(\frac{\ln(\sec \theta + \tan \theta)}{2} \right) \Big|_0^{\frac{\pi}{4}}$$

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$$\therefore L = \sqrt{2} + \ln(\sqrt{2} + 1).$$