CALCULUS (2) CHAPTER 8(8.2&8.3)

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8.2) Trigonometric Integrals

Case1) $\int \sin x^{m} \cos x^{n}$

Odd=2k+1

نستخدم المتطابقة $\sin^2 x = 1 - \cos^2 x$

$\sin x^{m} = \sin x^{2k+1} = (\sin^2 x)^k \sin x$=$(1 - \cos^2 x)^k \sin x$

Use supposition let u= cos x than solve it

Case2) $\int \sin x^{m} \cos x^{n}$ → Odd=2k+1

Even

نستخدم المتطابقة $1 - \sin^2 x = \cos^2 x$

$\cos x^{m} = \cos x^{2k+1} = (\cos^2 x)^k \cos x$=$(1 - \sin^2 x)^k \cos x$

Use supposition let u= sin x than solve it

Even

Case3) $\int \sin x^{m} \cos x^{n}$

نستخدم المتطابقة $\sin^2 x = \frac{1-\cos 2x}{2}$ & $\cos x^2 = \frac{1+\cos 2x}{2}$

Examples

- 1)$\int \sin^3 x \cos^3 x \, \mathrm{d}x = \int \sin^2 x \cos^3 x \sin x$DX   Case (1)
- =$\int (1 - \cos^2 x) \cos^3 x \sin x$ DX
- Let u=cosx⇒du=-sinx dx
- -$\int (1 - u^2) u^3 \, \mathrm{d}u = -\int u^3 - u^5$du=$\frac{u^6}{6} - \frac{u^4}{4} + c$
- =$\frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + c$

- 2)$\int (\sin x)^2 (\cos x)^3\, dx$ Case(2)
- = $\int \sin^2 x \cos^2 x \cos x$DX = $\int (1-\sin^2 x)\cos x \sin^2 x$ DX
- Let u=sinx⇒du=cosx dx
- =$\int u^2(1-u^2)\,du = \int u^2 - u^4$du
- = $\frac{u^3}{3} - \frac{u^5}{5} + c = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

- 3)$\int 16 \sin^2 x \cos^2$x=16$\int 1(\frac{1-\cos 2x}{2})(\frac{1+\cos 2x}{2})$DX Case (3)
- =4$\int 1-\cos^2 2x$ dx $= 4$x $- 4\int \cos^2 2x$
- =4x-4$\int \frac{1+\cos 4x}{2}$dx=4x-$\frac{4}{2}$x-2$\int \cos$4x $= 4x - 2x - \frac{1}{2}$sin4x=2x-$\frac{1}{2}$sin4x+c

PRODUCT OF SIN AND COS

- $\int \sin mx \sin nx \quad dx$
- $\int \sin mx \cos nx$ dx
- $\int \cos mx \cos ns$ dx
- $\sin mx \sin nx$ $=\frac{1}{2}$(cos(m-n)x-cos(m+n)x)
- $\sin mx \cos nx$ $=\frac{1}{2}$(sin(m-n)x+sin(m+n)x)
- $\cos mx \cos ns =$ $\frac{1}{2}$(cos(m-n)x+cos(m+n)x)

EXAMPLES

- 4) $\int \cos 3x \cos 4x \, dx = \int \frac{1}{2}$ (cos(−x) +cos(7x)) DX
- $= \frac{1}{2} \int \cos x + \cos 7x \, dx = \frac{1}{2} \sin x + \frac{1}{14} \sin 7x + c$
- 5) $\int_0^{\pi} \sqrt{1 - \sin^2 x} \, dx = \int_0^{\pi} \sqrt{\cos^2 x} \, dx = \int_0^{\pi} |\cos x| dx$
- $= \int_0^{\frac{\pi}{2}} cosx \, dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$
- $= \sin x|_0^{\frac{\pi}{2}} - \sin|_{\frac{\pi}{2}}^{\pi} = \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 2$

POWER OF TANX AND SECX

6) $\int 4\tan^3 x$ dx=4 $\int tanx \tan^2 x$ dx

= 4$\int tanx\,(\sec^2 x - 1)dx = 4\int \tan x \sec^2 x\, dx - 4\int tanx\, dx$

- =2 $\tan^2$x-4 $\int \frac{\sin x}{\cos x}$dx= 2 $\tan^2$x+4ln|cosx| +c
- = 2 $\tan^2$x-2ln| $\sec^2 x$ | +c = 2 $\tan^2$x-2ln| $tan^2 x + 1$| +c

- 7)$\int \sec^4 x \, dx = \int \sec^2 x \sec^2 x$ dx=$\int (1+\tan^2 x) \sec^2 x \, dx$
- Let u=tan x $\Rightarrow$ du=$\sec^2 x$ dx
- $= \int 1 + u^2 \, du = u + \frac{u^3}{3} + c$
- =tanx +$\frac{1}{3}\tan^3 x$+c =tanx + $\frac{1}{3}(\sec^2 x$-1)tanx +c = tanx-$\frac{1}{3} tanx$ + $\frac{1}{3}(\sec^2 x$-1)tanx +c
- =$\frac{2}{3}$tanx + $\frac{1}{3}\sec^2 x \tan x + c$

- OUTLINE SOLUTION

EVALUATE THE INTEGRALS

5. $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int \left(1 - \cos^2 x\right) \sin x\,dx = \int \sin x\,dx - \int \cos^2 x \sin x\,dx = -\cos x + \frac{1}{3}\cos^3 x + C$

11. $\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x \cos x dx = \int \sin^3 x\,(1 - \sin^2 x)\cos x\,dx = \int \sin^3 x \cos x\,dx - \int \sin^5 x \cos x\,dx$
$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

18. $\int 8\cos^4 2\pi x\,dx = 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx = 2\int (1 + 2\cos 4\pi x + \cos^2 4\pi x)dx = 2\int dx + 4\int \cos 4\pi x\,dx + 2\int \frac{1+\cos 8\pi x}{2}\,d$
$= 3\int dx + 4\int \cos 4\pi x\,dx + \int \cos 8\pi x\,dx = 3x + \frac{1}{\pi}\sin 4\pi\,x + \frac{1}{8\pi}\sin 8\pi\,x + C$

20. $\int_0^\pi 8\,\sin^4 y \cos^2 y\,dy = 8\int_0^\pi \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right) dy = \int_0^\pi dy - \int_0^\pi \cos 2y\,dy - \int_0^\pi \cos^2 2y\,dy + \int_0^\pi \cos^3 2y\,dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^\pi - \int_0^\pi \left(\frac{1+\cos 4y}{2}\right) dy + \int_0^\pi (1 - \sin^2 2y)\cos 2y\,dy = \pi - \frac{1}{2}\int_0^\pi dy - \frac{1}{2}\int_0^\pi \cos 4y\,dy + \int_0^\pi \cos 2y\,dy$
$- \int_0^\pi \sin^2 2y \cos 2y\,dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^\pi = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\,d\theta = \int_0^{\pi/2} \sin^2 2\theta(1 - \sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta\,d\theta - \int_0^{\pi/2} \sin^4 2\theta \cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

28. $\int_0^{\pi/6} \sqrt{1+\sin x}\, dx = \int_0^{\pi/6} \frac{\sqrt{1+\sin x}}{1} \frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\, dx = \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\, dx = \int_0^{\pi/6} \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\, dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$

$= \left[-2(1-\sin x)^{1/2}\right]_0^{\pi/6} = -2\sqrt{1-\sin\left(\frac{\pi}{6}\right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2-\sqrt{2}$

33. $\int \sec^2 x \tan x\, dx = \int \tan x \sec^2 x\, dx = \frac{1}{2}\tan^2 x + C$

36. $\int \sec^3 x \tan^3 x\, dx = \int \sec^2 x \tan^2 x \sec x \tan x\, dx = \int \sec^2 x(\sec^2 x - 1)\sec x \tan x\, dx$

$= \int \sec^4 x \sec x \tan x\, dx - \int \sec^2 x \sec x \tan x\, dx = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

38. $\int \sec^4 x \tan^2 x\, dx = \int \sec^2 x \tan^2 x \sec^2 x\, dx = \int \left(\tan^2 x + 1\right)\tan^2 x \sec^2 x\, dx = \int \tan^4 x \sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx$

$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

42. $\int 3\sec^4(3x)\, dx = \int (1+\tan^2(3x))\sec^2(3x)3dx = \int \sec^2(3x)3dx + \int \tan^2(3x)\sec^2(3x)3dx = \tan(3x) + \frac{1}{3}\tan^3(3x) + C$

45. $\int 4\tan^3 x\,dx = 4\int (\sec^2 x - 1)\tan x\,dx = 4\int \sec^2 x \tan x\,dx - 4\int \tan x\,dx = 4\frac{\tan^2 x}{2} - 4\ln|\sec x| + C$

$= 2\tan^2 x - 4\ln|\sec x| + C = 2\tan^2 x - 2\ln|\sec^2 x| + C = 2\tan^2 x - 2\ln(1 + \tan^2 x) + C$

47. $\int \tan^5 x\,dx = \int \tan^4 x \tan x\,dx = \int \left(\sec^2 x - 1\right)^2 \tan x\,dx = \int \left(\sec^4 x - 2\sec^2 x + 1\right)\tan x\,dx$

$= \int \sec^4 x \tan x\,dx - 2\int \sec^2 x \tan x\,dx + \int \tan x\,dx = \int \sec^3 x \sec x \tan x\,dx - 2\int \sec x \sec x \tan x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C = \frac{1}{4}\left(\tan^2 x + 1\right)^2 - \left(\tan^2 x + 1\right) + \ln|\sec x| + C = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C$

51. $\int \sin 3x \cos 2x\,dx = \frac{1}{2}\int (\sin x + \sin 5x)\,dx = -\frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C$

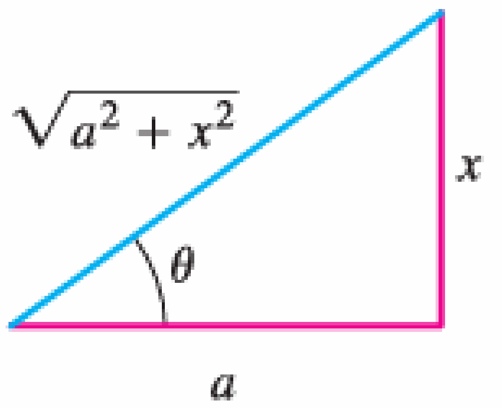
64. $\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}\,dx = \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\,dx = \int \frac{\sin x}{\cos^4 x}\,dx - \int \frac{\cos^2 x \sin x}{\cos^4 x}\,dx = \int \sec^3 x \tan x\,dx - \int \sec x \tan x\,dx$

$= \int \sec^2 x \sec x \tan x\,dx - \int \sec x \tan x\,dx = \frac{1}{3}\sec^3 x - \sec x + C$

67. $\int x\sin^2 x\,dx = \int x\,\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx \left[u = x,\ du = dx,\ dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x\right]$

$= \frac{1}{4}x^2 - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right] = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

8.3) TRIGONOMETRIC SUBSTITUTIONS

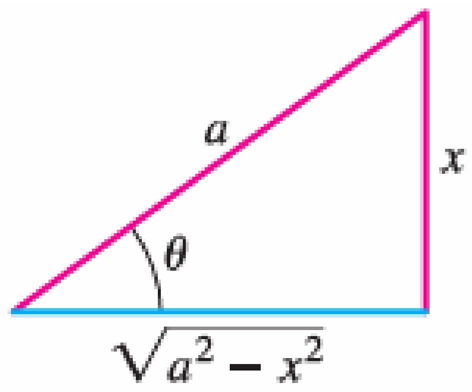


$$x = a \tan \theta$$

$$\sqrt{a^2 + x^2} = a|\sec \theta|$$

$$\theta = \tan^{-1} \frac{x}{a}$$

$$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$$

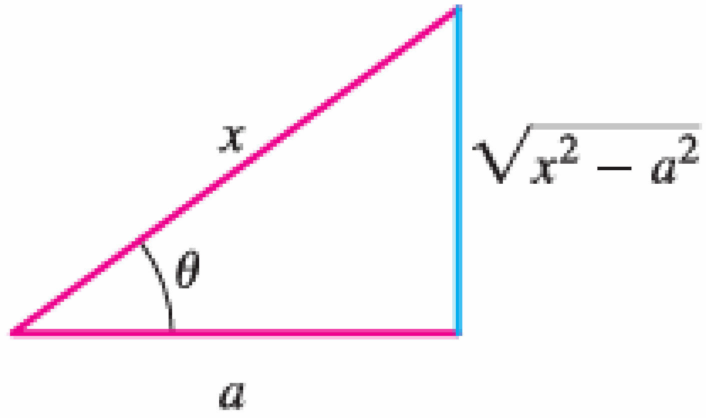


$$x = a \sin \theta$$

$$\sqrt{a^2 - x^2} = a|\cos \theta|$$

$$\theta = \sin^{-1} \frac{x}{a}$$

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$



$$x = a \sec \theta$$

$$\sqrt{x^2 - a^2} = a|\tan \theta|$$

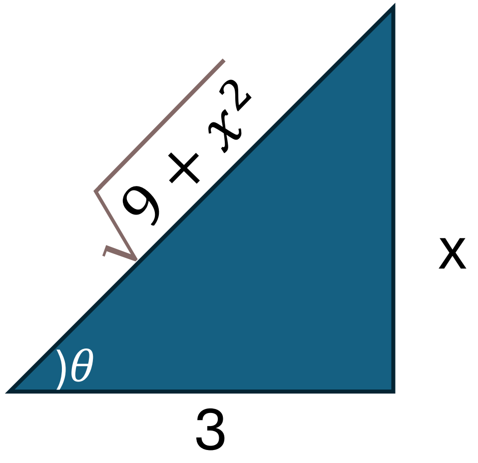
$$\theta = \sec^{-1} \frac{x}{a}$$

$$0 \leq \theta \leq \frac{\pi}{2} \text{ if } \frac{x}{a} \geq 1$$

$$\frac{\pi}{2} < \theta \leq \pi \text{ if } \frac{x}{a} \leq -1$$

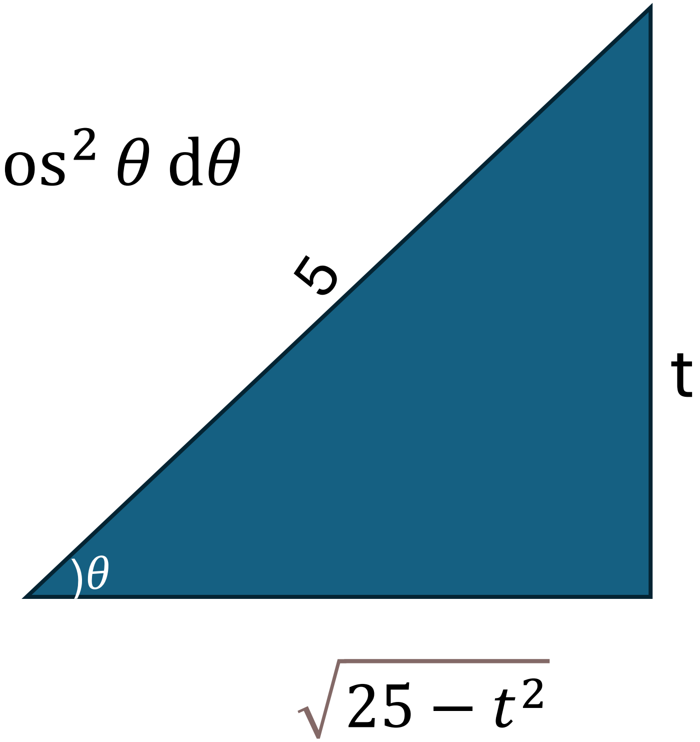
EXAMPLE(1)

- 1) $\int \frac{dx}{\sqrt{x^2+9}}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- let x =3tanθ ⇒dx=3sec$^2\ \theta\ d\theta$
- = $\int \frac{3\sec^2\theta\ d\theta}{\sqrt{9tan^2\ \theta+9}} = \int \frac{\sec^2\theta\ d\theta}{\underbrace{\sqrt{tan^2\ \theta+1}}_{\sec^2\theta}} = \int \frac{\sec^2\theta\ d\theta}{|sec\theta|} = \int \sec\theta\ d\theta = \ln|\sec\theta + \tan\theta|$+c
- =ln$\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C = \ln|\sqrt{9+x^2}$ +x|+c



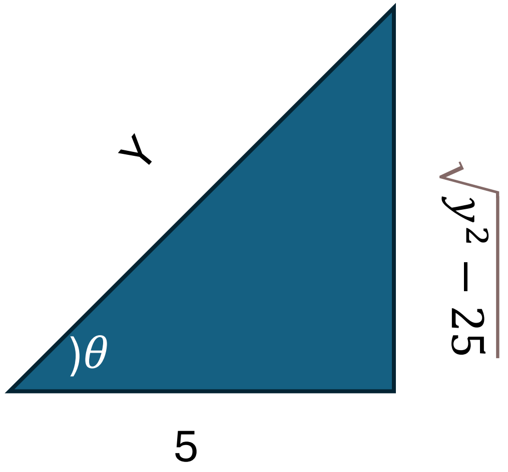
EXAMPLE (2)

- $\int \sqrt{25 - t^2}\, \mathrm{d}t$
- Let t=5sinθ $\Rightarrow$dt=5cosθ dθ , $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
- = $\int \sqrt{25 - 25\sin^2\theta}\, 5\cos\theta\, d\theta$
- =25 $\int \sqrt{1 - \sin^2\theta}\, \cos\theta\, d\theta$=25$\int \sqrt{\cos^2\theta}\, \cos\theta\, \mathrm{d}\theta$ =25$\int \cos^2\theta\, \mathrm{d}\theta$ $= \frac{25}{2}\int 1 + \cos 2\theta\, \mathrm{d}\theta$
- = $\frac{25}{2}\left(\left(\theta + \frac{\sin 2\theta}{2}\right)\right.$+c
- $\frac{25}{2}\left(\left(\theta + sin\theta cos\theta\right)\right.$+c
- = $\frac{25}{2}\left(\left(\sin^{-1}\left(\frac{t}{5}\right) + \frac{t}{5}\frac{\sqrt{25-t^2}}{5}\right)\right.$+c = $\frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right)$+ $\frac{t\sqrt{25-t^2}}{2}$



EXAMPLE 3

- $\int \sqrt{\frac{y^2-25}{y^3}}\, dy$
- Let y=5secθ ⇒dy=5sec θ tan θdθ , 0<θ<$\frac{\pi}{2}$
- = $\sqrt{\frac{25(\sec\theta)^2-25}{125(\sec\theta)^3}}$ 5sec θ tan θdθ =$\frac{1}{5}\int \frac{\sqrt{(\sec\theta)^2-1}}{\sec^2\theta}\tan\theta\, d\theta$=$\frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta} d\theta$
- $= \frac{1}{5}\int \sin^2\theta d\theta$ = $\frac{1}{10}\int 1-\cos 2\theta\, d\theta = \frac{1}{10}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$
- = $\frac{1}{10}(\theta - \sin\theta\cos\theta)$+c
- =$\frac{1}{10}\, sec^2\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y}(\frac{5}{y})$+c



EXAMPLE(4)

- $\int_{-2}^{2} \frac{dx}{4+x^2}$
- Let x=2tanθ ⇒dx=2sec$^2\theta$ dθ , $\frac{-\pi}{4} < 0 < \frac{\pi}{4}$
- When x=-2 ⇒ $\theta = \tan^{-1}(-1) = \frac{-\pi}{4}$
- x =2 ⇒ $\theta = \tan^{-1}(1) = \frac{\pi}{4}$
- = $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2\theta\, d\theta}{4+y\tan^2\theta} = \frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta} d\theta$=

$\frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{4}$

$$\int_{-2}^{2} \frac{dx}{4+x^2}$$

=$\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_{-2}^{2}$

= $\frac{1}{2}(\tan^{-1}1 - \tan^{-1}-1)$

= $\frac{1}{2}\left(\frac{\pi}{4} - \frac{-\pi}{4}\right) = \frac{\pi}{4}$

- OUTLINE SOLUTION

EVALUATE THE INTEGRALS

8. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\, d\theta,\ \sqrt{1-9t^2} = \cos\theta;$

$$\int \sqrt{1-9t^2}\, dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$$

10. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta,\ \sqrt{25x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$

$$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$$

12. $y = 5\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 5\sec\theta\tan\theta\, d\theta,\ \sqrt{y^2-25} = 5\tan\theta;$

$$\int \frac{\sqrt{y^2-25}}{y^3}\, dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)\, d\theta}{125\sec^3\theta} = \frac{1}{5}\int \tan^2\theta\cos^2\theta\, d\theta = \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{10}(\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + C = \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2-1} = \tan\theta;$

$$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\, d\theta = 2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \theta + \sin\theta\cos\theta + C$$

$$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$$

18. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\, d\theta,\ \sqrt{x^2+1} = \sec\theta;$

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta \sec\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$$

24. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\, d\theta,\ (4-x^2)^{3/2} = 8\cos^3\theta;$

$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$$

26. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2-1)^{5/2} = \tan^5\theta;$

$$\int \frac{x^2\, dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$$

29. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ (4x^2+1)^2 = \sec^4\theta;$

$$\int \frac{8\, dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right) d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$$

33. $v = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dv = \cos\theta\, d\theta,\ (1-v^2)^{5/2} = \cos^5\theta;$

$$\int \frac{v^2\, dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\, d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$$

USE AN APPROPRIATE SUBSTITUTION AND THEN A TRIGONOMETRIC SUBSTITUTION TO EVALUATE THE INTEGRALS

38. $y = e^{\tan\theta}, 0 \le \theta \le \frac{\pi}{4}$, $dy = e^{\tan\theta}\sec^2\theta\, d\theta$, $\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$;

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta = [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$$

45. Let $u = \sqrt{x} \Rightarrow x = u^2 \Rightarrow dx = 2u\,du \Rightarrow \int\sqrt{\frac{4-x}{x}}\,dx = \int\sqrt{\frac{4-u^2}{u^2}}\,2u\,du = 2\int\sqrt{4-u^2}\,du$;

$u = 2\sin\theta$, $du = 2\cos\theta\, d\theta$, $0 < \theta \le \frac{\pi}{2}$, $\sqrt{4-u^2} = 2\cos\theta$

$$2\int\sqrt{4-u^2}\,du = 2\int(2\cos\theta)(2\cos\theta)\,d\theta = 8\int\cos^2\theta\,d\theta = 8\int\frac{1+\cos 2\theta}{2}\,d\theta = 4\int d\theta + 4\int\cos 2\theta\,d\theta$$

$$= 4\theta + 2\sin 2\theta + C = 4\theta + 4\sin\theta\cos\theta + C = 4\sin^{-1}\left(\frac{u}{2}\right) + 4\left(\frac{u}{2}\right)\left(\frac{\sqrt{4-u^2}}{2}\right) + C = 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{x}\sqrt{4-x} + C$$

$$= 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sqrt{4x-x^2} + C$$

46. Let $u = x^{3/2} \Rightarrow x = u^{2/3} \Rightarrow dx = \frac{2}{3}u^{-1/3}du$

$$\int\sqrt{\frac{x}{1-x^3}}\,dx = \int\sqrt{\frac{u^{2/3}}{1-(u^{2/3})^3}}\left(\frac{2}{3}u^{-1/3}\right)du = \int\frac{u^{1/3}}{\sqrt{1-u^2}}\left(\frac{2}{3u^{1/3}}\right)du = \frac{2}{3}\int\frac{1}{\sqrt{1-u^2}}\,du = \frac{2}{3}\sin^{-1}u + C = \frac{2}{3}\sin^{-1}(x^{3/2}) + C$$

- "اللهم انفعني بما علمتني، وعلمني ما ينفعني، وارزقني علما تنفعني به"