2.4: Independent Random variable.

Recall :

Def : A, B independent events $\Leftrightarrow P(A|B) = P(A)$, $P(B) \neq 0$

$P(B|A) = P(B)$, $P(A) \neq 0$

Prop. : A, B independent $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$, $P(A), P(B) \neq 0$.

Def 2 : let X_1, X_2 be Random variables , we say that X_1, X_2 are independent if $f(x_1, x_2) = f_1(x_1) f_2(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$.

Remark : X_1, X_2 independent Random variables , then

1. $f_{1|2}(x_1|x_2) = f_1(x_1) \quad \forall x_1 \in \mathbb{R}$, $f_2(x_2) \neq 0$

2. $f_{2|1}(x_2|x_1) = f_2(x_2) \quad \forall x_2 \in \mathbb{R}$, $f_1(x_1) \neq 0$.

Thm : let X_1, X_2 be Random variables with joint p.d.f $f(x_1, x_2)$

X_1, X_2 independent $\Leftrightarrow f(x_1, x_2) = h(x_1) \cdot g(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$

where $h(x_1) \geq 0$, $\forall x_1 \in \mathbb{R}$

$g(x_2) \geq 0$, $\forall x_2 \in \mathbb{R}$.

Thm 2: X_1, X_2 independent Random variable, then
$Pr(a < X_1 < b, c < X_2 < d) = Pr(a < X_1 < b) \cdot Pr(c < X_2 < d)$ for
$a < b$, $a, b \in \mathbb{R}$, $c < d$, $c, d \in \mathbb{R}$.

Thm 3: X_1, X_2 independent Random variables, then
$E(u(X_1) \cdot u(X_2)) = E(u(X_1)) \cdot E(u(X_2))$.

Thm 4: X_1, X_2 independent Random variable $\Leftrightarrow$
$M(t_1, t_2) = M_1(t_1) \cdot M_2(t_2)$, $\forall (t_1, t_2) \in \mathbb{R}^2$.

exp 1: $f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1 < 1, \ 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$

are X_1, X_2 independent ?

$f_1(x_1) = \begin{cases} x_1 + \frac{1}{2}, & 0 < x_1 < 1 \\ 0, & \text{elsewhere} \end{cases}$ $f_2(x_2) = \begin{cases} x_2 + \frac{1}{2}, & 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$

$\rightsquigarrow f_1(x_1) \cdot f_2(x_2) = \begin{cases} (x_1 + \frac{1}{2})(x_2 + \frac{1}{2}), & 0 < x_1 < 1, \ 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$

$\Rightarrow f(x_1, x_2) \neq f_1(x_1) \cdot f_2(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$

$\Rightarrow X_1, X_2$ are Not independent. " dependent ".

exp 2 : $f(x_1, x_2) = \begin{cases} 8x_1x_2 & , 0 < x_1 < x_2 < 1 \\ 0 & , \text{elsewhere} \end{cases}$

are x_1, x_2 independent ?

$f_1(x_1) = \begin{cases} \frac{8x_1(1-x_1^2)}{2} & , 0 < x_1 < 1 \\ 0 & , \text{elsewhere} \end{cases}$

$f_2(x_2) = \begin{cases} 4x_2^3 & , 0 < x_2 < 1 \\ 0 & , \text{elsewhere} \end{cases}$

$f_1(x_1) \cdot f_2(x_2) \neq f(x_1, x_2)$

$\Rightarrow$ x_1, x_2 dependent .

$\Rightarrow$ another solution of exp 1,2 :

we can't find $h(x_1) \geq 0 \quad \forall x_1 \in \mathbb{R}$ and $g(x_2) \geq 0 \quad \forall x_2 \in \mathbb{R}$

such that $f(x_1, x_2) = h(x_1)\, g(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$

Hence, x_1, x_2 dependent .

(check exp 3) .

exp3: In exp1, X_1 and X_2 were found to be dependent.

$$Pr(a<X_1<b,\ c<X_2<d) \neq Pr(a<X_1<b)\ Pr(c<X_2<d).$$

$$\Rightarrow Pr(0<X_1<\tfrac{1}{2},\ 0<X_2<\tfrac{1}{2}) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} (x_1+x_2)\,dx_1\,dx_2$$

$$= \int_0^{\frac{1}{2}} \frac{x_1^2}{2} + x_2 x_1 \Big|_0^{\frac{1}{2}} dx_2$$

$$= \int_0^{\frac{1}{2}} \frac{1}{8} + \frac{1}{2}x_2 \; dx_2$$

$$= \frac{1}{8}x_2 + \frac{x_2^2}{4}\Big|_0^{\frac{1}{2}} = \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \frac{1}{8}$$

But

$$Pr(0<X_1<\tfrac{1}{2}) = \int_0^{\frac{1}{2}} (x_1+\tfrac{1}{2})\,dx_1 = \frac{x_1^2}{2} + \frac{1}{2}x_1\Big|_0^{\frac{1}{2}} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

$$Pr(0<X_2<\tfrac{1}{2}) = \int_0^{\frac{1}{2}} (\tfrac{1}{2}+x_2)\,dx_2 = \frac{1}{2}x_2 + \frac{x_2^2}{2}\Big|_0^{\frac{1}{2}} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}.$$

$$\rightarrow Pr(0<X_1<\tfrac{1}{2},\ 0<X_2<\tfrac{1}{2}) \neq Pr(0<X_1<\tfrac{1}{2})\ Pr(0<X_2<\tfrac{1}{2}).$$

exp4: X, Y independent Random variable $\Rightarrow \rho = 0$ assuming $\sigma_x, \sigma_y > 0$

since $$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_x \sigma_y} = \frac{E(X)E(Y) - E(X)E(Y)}{\sigma_x \sigma_y} = 0$$