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MATH1321

Calculus 2

Chapter 10.6

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10.6 Alternating Series (AS)

The AS has the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} U_n = U_1 - U_2 + U_3 - U_4 + \ldots$

$a_n = (-1)^{n+1} U_n$ and $U_n = |a_n|$

When the AS Converge?

Theorem: AS Test

The AS conv. if:

1) $U_n > 0 \ \forall n$.

2) U_n decreases for large n. ($U_{n+1} \leqslant U_n$)

3) $\lim_{n \to \infty} U_n = 0$.

But if $\lim_{n \to \infty} U_n \neq 0$ then AS div. by n^{th} term test.

Ex. Check conv. or div. of ?

1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (Alternating Harmonic Series)

Apply AST: $U_n = \frac{1}{n} > 0 \ \forall n$ and $\downarrow$ for large n.

$\lim_{n \to \infty} \frac{1}{n} = 0$

Hence $\sum_{n=1}^{\infty} a_n$ conv. by AST.

2) $\sum_{n=1}^{\infty} (-1)^n (0.2)^n$ (conv. geometric series to $-\frac{1}{6}$)

Apply AST: $U_n = (0.2)^n > 0 \ \forall n$ and $\downarrow$ for large n.

$\lim_{n \to \infty} (0.2)^n = 0$.

Hence $\sum_{n=1}^{\infty} a_n$ conv. by AST.

3) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{4}{3}\right)^n$

$\lim_{n \to \infty} \left(\frac{4}{3}\right)^n = \infty \neq 0$

Hence $\sum_{n=1}^{\infty} a_n$ div. by n^{th} term test.

4) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1+n}{2n+5}\right)$

$\lim_{n \to \infty} \left(\frac{1+n}{2n+5}\right) = \frac{1}{2} \neq 0$

Hence $\sum_{n=1}^{\infty} a_n$ div. by n^{th} term test.

Remark:

$\sum_{n=1}^{\infty} \frac{1}{n}$ div. Harmonic Series.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv. Alternating HS.

1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ conv. AHS.

2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv. AHS.

$\to$ Rule: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ conv. by AST if $0 < P < \infty$.

Absolute Convergence:

$\sum a_n$ conv. abs. if $\sum |a_n|$ conv.

Ex. Check if the following series are conv. abs. ?

1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ a_n conv. by AST since $P>0$.

$|a_n|$ conv. by P-series Test.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ conv. abs.

2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ a_n conv. by AST since $P>0$.

$|a_n|$ div. Harmonic Series.

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv. but not conv. abs.

Note

Any infinite series conv. but not Abs. is called Conv. Conditionally.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^P}$ conv. conditionally if $0 < P \leq 1$.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^P}$ conv. abs. if $P > 1$.

Remark: If $\sum |a_n|$ then $\sum a_n$ conv. (works for Ex. 1)

That is if the infinite series conv. Abs. then it is conv.

CONVESE IS NOT TRUE

Ex. Check conv. or div. of ?

1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}$ Apply AST

$\lim_{n\to\infty} U_n = 3 \neq 0$

Hence $\sum_{n=1}^{\infty} a_n$ div. by n^{th} term test.

2) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$ Apply AST

$\lim_{n\to\infty} U_n = 0$ $\frac{n}{n^3+1}$ is positive $\forall n$ and $\downarrow$ for large n.

Hence $\sum_{n=1}^{\infty} a_n$ conv. by AST. also conv. abs. (by LCT or DCT $\frac{1}{n^2}$)

Theorem: Alternating Estimation Theorem AET

Assume $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} U_n = U_1 - U_2 + U_3 - U_4 + \dots = L$

If we estimate L by finding the first sum of n term

$$S_n = U_1 - U_2 + U_3 - U_4 + \dots + (-1)^{n+1} U_n \quad (S_n \approx L)$$

Then: 1) The remainder $L - S_n$ has same sign as a_{n+1}.

2) The error $= |L - S_n| < U_{n+1} = |a_{n+1}|$

3) $\text{Min}\{S_n, S_{n+1}\} < L < \text{Max}\{S_n, S_{n+1}\}$

Ex. Given $\sum_{n=1}^{\infty} (-1)^{n+1} (\frac{2}{3})^n$.

1) Find L ?

$\sum_{n=1}^{\infty} a_n = \frac{2}{3} - (\frac{2}{3})^2 + (\frac{2}{3})^3 + \dots$ geometric

$= \frac{a}{1-r} = \frac{2}{5} = 0.4.$

2) S_3 ? $S_3 = \sum_{n=1}^{3} a_n = \frac{2}{3} - (\frac{2}{3})^2 + (\frac{2}{3})^3 = 0.519.$

3) Remainder when n=3 ?

$L - S_3 = 0.4 - 0.519 = -0.119$

4) a_4 ? $= (-1)^5 (\frac{2}{3})^4 = -(\frac{2}{3})^4.$

Note that $L - S_3$ has same sign as a_4.

5) Error ? $= |L - S_3| = 0.119.$ E_3

6) U_4 ? $U_4 = |a_4| = (\frac{2}{3})^4.$

7) Upper bound for the error ?

$E_3 < U_4 \longrightarrow 0.119 < 0.198$

8) S_4 ? $= S_3 - (\frac{2}{3})^4 = 0.321.$

9) Check $\min\{S_3, S_4\} < L < \max\{S_3, S_4\}$?

$\min\{0.519, 0.321\} < 0.4 < \max\{0.519, 0.321\}$

$0.321 < 0.4 < 0.519.$

Ex. Estimate $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$ with error of magnitude less than 5×10^{-6} ?

$\frac{1}{(2n)!} < 5 \times 10^{-6}$ (Find n)

$(2n)! > \frac{1}{5 \times 10^{-6}} \longrightarrow (2n)! > 200000$ (try)

$n = 3 \quad 6! > 200000 \quad \times$

$n = 4 \quad 8! > 200000 \quad \times$

$n = 5 \quad 10! > 200000 \quad \checkmark \longrightarrow n \geq 5$

$S_5 = \sum_{n=0}^{4} (-1)^n \frac{1}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!}$

by Calculator $S_5 \approx 0.54.$

To check error $\longrightarrow$ Error $= |L - S_5| = |L - 0.54| < U_6$

$U_6 = |a_6| = \left| (-1)^5 \frac{1}{10!} \right| = \frac{1}{10!} = 0.275 \times 10^{-6}.$

Therefore $0.275 \times 10^{-6} < 5 \times 10^{-6}$