# 8.1 Integration by Parts

$\boxed{6}$ Evaluate the follwing integral using integration by parts. $\int_1^e x^3 \ln x \, dx$

Let $u = \ln x \qquad dv = x^3\, dx$

$du = \frac{dx}{x} \qquad v = \frac{x^4}{4}$

$$\int_1^e x^3 \ln x \, dx = \frac{x^4}{4} \ln x \Big|_1^e - \int_1^e \frac{x^4}{4} \cdot \frac{dx}{x}$$

$$= \frac{x^4}{4} \ln x \Big|_1^e - \int_1^e \frac{x^3}{4} dx$$

$$= \frac{x^4}{4} \ln x \Big|_1^e - \frac{1}{4}\left(\frac{x^4}{4}\right) \Big|_1^e$$

$$= \left[\frac{x^4}{4} \ln x - \frac{x^4}{16}\right] \Big|_1^e$$

$$= \frac{e^4}{4} \ln e - \frac{e^4}{16} - \left[\frac{1}{4} \ln 1 - \frac{1}{16}\right]$$

$$= \frac{e^4}{4}(1) - \frac{e^4}{16} - \left[\frac{1}{4}(0) - \frac{1}{16}\right]$$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4}{16} + \frac{1}{16}$$

**11** Evaluate the following integral using integration by parts :- $\int \tan^{-1} y \, dy$

Let $u = \tan^{-1} y \qquad dv = dy$

$du = \dfrac{dy}{1+y^2} \qquad v = y$

$$\int \tan^{-1} y \, dy = y \tan^{-1} y - \int \frac{y}{1+y^2} \, dy$$

$$= y \tan^{-1} y - \frac{1}{2} \int \frac{2y}{1+y^2} \, dy$$

$$= y \tan^{-1} y - \frac{1}{2} \ln |1+y^2| + c$$

$$= y \tan^{-1} y - \ln \sqrt{1+y^2} + c$$

$\boxed{16}$ Evaluate the integrat using integration by parts

$$\int p^4 e^{-p}\, dp$$

let

| | | |
|---|---|---|
| $u = p^4$ | $\xrightarrow{+}$ | $dV = e^{-p}$ |
| $4p^3$ | $\xrightarrow{-}$ | $-e^{-p}$ |
| $12p^2$ | $\xrightarrow{+}$ | $e^{-p}$ |
| $24p$ | $\xrightarrow{-}$ | $-e^{-p}$ |
| $24$ | $\xrightarrow{+}$ | $e^{-p}$ |
| $0$ | | $-e^{-p}$ |

$$\int p^4 e^{-p}\, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24p e^{-p} - 24 e^{-p} + c$$

$\boxed{22}$ Evaluate $\int e^{-y} \cos y \, dy$

$u = e^{-y}$ $\quad dv = \cos y \, dy$

$du = -e^{-y}$ $\quad v = \sin y$

$$\int e^{-y} \cos y \, dy = \sin y \, e^{-y} + \int e^{-y} \sin y \, dy$$

$\int e^{-y} \sin y \, dy$ :- $\quad u = e^{-y} \quad dv = \sin y \, dy$

$du = -e^{-y} dy \quad v = -\cos y$

$$\int e^{-y} \sin y \, dy = -e^{-y} \cos y - \int e^{-y} \cos y \, dy$$

$$\therefore \int e^{-y} \cos y \, dy = \sin y \, e^{-y} + \left(-e^{-y} \cos y - \int e^{-y} \cos y \right. + c$$

$$2 \int e^{-y} \cos y \, dy = \sin y \, e^{-y} - e^{-y} \cos y + c$$

$$\int e^{-y} \cos y \, dy = \frac{e^{-y} \sin y}{2} - \frac{e^{-y} \cos y}{2} + c$$

$\boxed{30}$ Evaluate $\int z(\ln z)^2 dz$

Let $u = \ln z \rightarrow du = \frac{dz}{z} \rightarrow dz = z\,du$
$= e^u\,du$

$$\int z(\ln z)^2 dz = \int e^u \cdot u^2 \cdot e^u du = \int e^{2u} u^2 du$$

$$\begin{array}{ccc} u^2 & & e^{2u} \\ & + & \\ 2u & & \frac{e^{2u}}{2} \\ & - & \\ 2 & & \frac{e^{2u}}{4} \\ & + & \\ 0 & & \frac{e^{2u}}{8} \end{array}$$

$$\therefore \int e^{2u} u^2 du = u^2 \cdot \frac{e^{2u}}{2} - 2u \cdot \frac{e^{2u}}{4} + 2 \cdot \frac{e^{2u}}{8} + c$$

$$\int e^{2u} u^2 du = \frac{u^2 e^{2u}}{2} - \frac{u e^{2u}}{2} + \frac{e^{2u}}{4} + c$$

$$= \frac{(\ln z)^2 \; z^2}{2} - \frac{\ln(z) \cdot z^2}{2} + \frac{z^2}{4} + c$$

$\boxed{25}$ Evalutate by using a substitution prior to integration by parts. $\int e^{\sqrt{3s+9}}\, ds$

① Substitution:- Let $x = \sqrt{3s+9}$

$$dx = \frac{3}{2\sqrt{3s+9}}\, ds$$

$$\frac{2}{3}\sqrt{3s+9}\, dx = ds$$

$$\frac{2}{3}\, x\, dx = ds$$

$$\int e^{\sqrt{3s+9}}\, ds = \int e^{x} \cdot \frac{2}{3} x\, dx = \frac{2}{3}\int x e^{x} dx$$

② By Parts:-

let $u = x$ $\quad dv = e^{x}\, dx$

$du = dx$ $\quad v = e^{x}$

$$\frac{2}{3}\int x\, e^{x} dx = \frac{2}{3}\left[\, x\, e^{x} - \int e^{x} dx \right]$$

$$= \frac{2}{3}\left[\, x\, e^{x} - e^{x} + c \right]$$

$$= \frac{2}{3}\left[\sqrt{3s+9}\; e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right] + c$$

$\boxed{36}$ Evaluate $\int \frac{(\ln x)^3}{x}\, dx$

by substitution:-

let $u = \ln x \rightarrow du = \frac{dx}{x}$

$$\int \frac{(\ln x)^3}{x}\, dx = \int u^3\, du = \frac{u^4}{4} + c$$

$$= \frac{(\ln x)^4}{4} + c$$

$\boxed{39}$ Evaluate $\int x^3 \sqrt{x^2+1}\, dx$

$$= \int x^2 \cdot x\sqrt{x^2+1}\, dx$$

let $u = x^2$ $\qquad dv = x\sqrt{x^2+1}\, dx$

$du = 2x\,dx$ $\qquad v = \int x\sqrt{x^2+1}\, dx = \frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2}$

$$v = \frac{1}{3}(x^2+1)^{3/2}$$

$$\int x^2 \cdot x\sqrt{x^2+1}\, dx = x^2 \cdot \frac{1}{3}(x^2+1)^{3/2} - \int 2x \cdot \frac{1}{3}(x^2+1)^{3/2} dx$$

$$= \frac{x^2(x^2+1)^{3/2}}{3} - \frac{1}{3}\frac{(x^2+1)^{5/2}}{5/2} + C$$

$$= \frac{x^2(x^2+1)^{3/2}}{3} - \frac{2}{15}(x^2+1)^{5/2} + C$$

$$\int x\sqrt{x^2+1}\, dx$$

let $Z = x^2+1 \rightarrow dZ = 2x\,dx$

$$\int x\sqrt{x^2+1}\, dx = \int x\sqrt{Z}\,\frac{dZ}{2x} = \int Z^{\frac{1}{2}}\, dZ$$

$$= \frac{Z^{3/2}}{3/2} = \frac{2}{3}(x^2+1)^{3/2}$$

$\boxed{46}$ Evaluate $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

① Substitution:- let $S = \sqrt{x} \rightarrow dS = \frac{1}{2\sqrt{x}} dx$

$$\int \sqrt{x}\, e^{\sqrt{x}}\, dx = \int \sqrt{x}\, e^{S} \cdot 2\sqrt{x}\, dS$$

$$= \int 2S^2 e^{S}\, dS$$

② By parts:

| | |
|---|---|
| $2S^2$ | $e^S$ |
| $4S$ | $e^S$ |
| $4$ | $e^S$ |
| $0$ | $e^S$ |

$$\int 2S^2 e^S\, dS = 2S^2 e^S - 4Se^S + 4e^S + C$$

$$= 2x\, e^{\sqrt{x}} - 4\sqrt{x}\, e^{\sqrt{x}} + 4 e^{\sqrt{x}} + C$$

$\boxed{50}$ Evaluate $\int_0^{\frac{1}{\sqrt{2}}} 2x \sin^{-1}(x^2)\, dx$

By parts: let $u = \sin^{-1}(x^2)$ $\quad dv = 2x\, dx$

$du = \frac{2x\, dx}{\sqrt{1-x^4}} \qquad v = x^2$

$$\int_0^{\frac{1}{\sqrt{2}}} 2x \sin^{-1}(x^2)\, dx = x^2 \sin^{-1}(x^2) \Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}}\, dx$$

$\int \frac{2x^3}{\sqrt{1-x^4}}\, dx$ : let $s = 1 - x^4 \to ds = -4x^3\, dx$

$$\int \frac{2x^3}{\sqrt{1-x^4}}\, dx = \int \frac{2\cancel{x^3}}{\sqrt{s}} \cdot \frac{ds}{-4\cancel{x^3}} = \frac{-1}{2} \int s^{-\frac{1}{2}}\, ds$$

$$= \frac{-1}{2}\left(\frac{s^{\frac{1}{2}}}{\frac{1}{2}}\right) = -\sqrt{s} = -\sqrt{1-x^4}$$

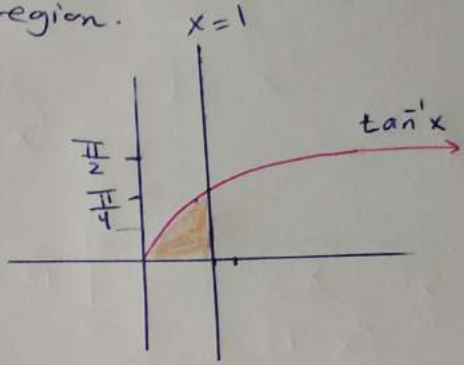
$$= x^2 \sin^{-1}(x^2) \Big|_0^{\frac{1}{\sqrt{2}}} + \sqrt{1-x^4} \Big|_0^{\frac{1}{\sqrt{2}}}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \sin^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right) + \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^4} - \left(0 + \sqrt{1-0}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right) + \sqrt{\frac{3}{4}} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

58 Consider the region bounded by the graphs of $y = \tan^{-1}x$ , $y = 0$ and $x = 1$

a) Find the area of the region.




$\tan^{-1}x = y$

$\tan^{-1}(1) = \frac{\pi}{4} \rightarrow (1, \frac{\pi}{4})$

$$A = \int_0^1 \tan^{-1}x \, dx$$

by parts : let $u = \tan^{-1}x$ $\quad dv = dx$

$du = \frac{dx}{1+x^2}$ $\quad v = x$

$$A = \int_0^1 \tan^{-1}x \, dx = x \tan^{-1}x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$

$$= x \tan^{-1}x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2} \, dx$$

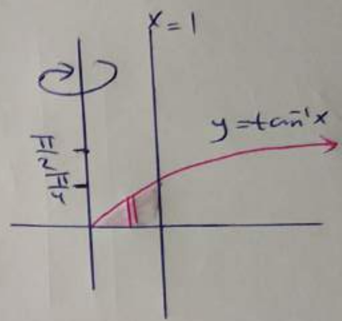
$$= \left[ x \tan^{-1}x - \frac{1}{2}\ln|1+x^2| \right] \Big|_0^1$$

$$= 1(\tan^{-1}(1)) - \frac{1}{2}\ln|2| - \left[ 0 - \frac{1}{2}\ln(1) \right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$

b) Find the volume of the solid formed by revolving this region about the y-axis

Shell method:-




$$V = \int_a^b 2\pi \, R \, H \, dx$$

$$= \int_0^1 2\pi (x)(\tan^{-1}x)\, dx$$

$$= 2\pi \int_0^1 x \tan^{-1} x \, dx$$

by parts let $u = \tan^{-1}x$ $\quad dv = x\,dx$

$du = \dfrac{dx}{1+x^2}$ $\quad v = \dfrac{x^2}{2}$

$$2\pi \int_0^1 x \tan^{-1} x\, dx = 2\pi \left[ \frac{x^2}{2} \tan^{-1}x \Big|_0^1 - \int_0^1 \frac{x^2}{2(1+x^2)}\, dx \right]$$

$$= 2\pi \left[ \frac{x^2}{2}\tan^{-1}x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1}{1+x^2}\, dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\, dx \right]$$

$$= 2\pi \left[ \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x \right] \Big|_0^1$$

$$= 2\pi \left[ \frac{1}{2}(\tan^{-1}1) - \frac{1}{2}(1) + \frac{1}{2}\tan^{-1}(1) - (0) \right]$$

$$= 2\pi \left[ \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} \right] = 2\pi\left( \frac{\pi}{4} - \frac{1}{2} \right)$$