Pole Placement Control Design

Assumptions:

- The system is completely state controllable.
- The state variables are measurable and are available for feedback.
- Control input is unconstrained.

Pole Placement Control Design

Objective:
The closed loop poles should lie at $\mu_1, \ldots, \mu_n,$ which are their 'desired locations'.

Difference from classical approach:
Not only the "dominant poles", but "all poles" are forced to lie at specific desired locations.

Necessary and sufficient condition:
The system is completely state controllable.

Closed Loop System Dynamics

$$\dot{X} = AX + BU$$

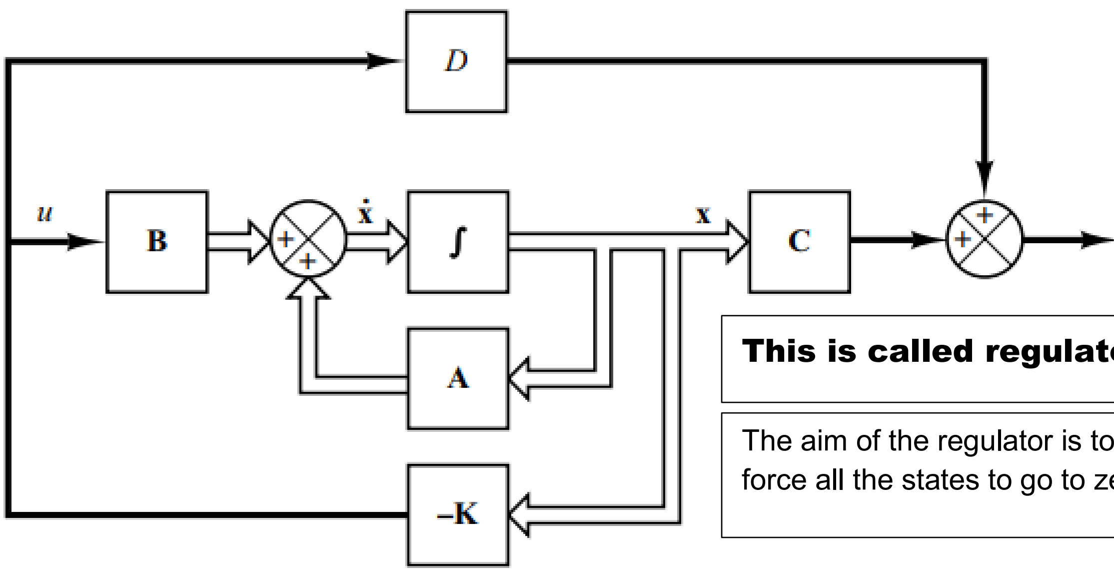
The control vector U is designed in the following state feedback form

$$U = -KX$$

This leads to the following closed loop system

$$\dot{X} = (A - BK)X = A_{CL}X$$

where $A_{CL} \triangleq (A - BK)$



This is called regulator

The aim of the regulator is to force all the states to go to zero

Philosophy of Pole Placement Control Design

The gain matrix K is designed in such a way that

$$\left|sI-(A-BK)\right|=(s-\mu_1)(s-\mu_2)\quad\cdots\quad(s-\mu_n)$$

where $\mu_1,\cdots,\mu_n$ are the desired pole locations.

Pole Placement Design Steps: Method 1 (low order systems, n $\leq$ 3)

- Check controllability
- Define $K=\begin{bmatrix}k_1 & k_2 & k_3\end{bmatrix}$
- Substitute this gain in the desired characteristic polynomial equation

  $$\left|sI-A+BK\right|=(s-\mu_1)\cdots(s-\mu_n)$$
- Solve for k_1, k_2, k_3 by equating the like powers on both sides

Matlab:

P=[-lamda1 -lamda2 …….. –lamdan]   % desired eigenvalues

K=place(A,B, P)

Example 1: Consider the following system

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

1-Check the controllability:

$$M = [B\ AB] = \begin{bmatrix} 0 & 2 \\ 1 & 10 \end{bmatrix}$$

$|M| = -2$ Therefore the system is fully state controllable

2-Define: K= $[k_1 \quad k_2]$]

3-Design a regulator with the following performance $T_s = 1$ second and ζ=0.8

$$T_s = \frac{4}{\zeta\omega_n} = 1 \rightarrow \qquad \omega_n = 5 \text{ rad/s}$$

Based on the desired eigenvalues are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -4 \pm 3j$$

The desired characteristic equation is: (s-s1)*(s-s2)

$$(s+4-3j)^*\,(s+4+3j) = s^2 + 8\,s + 25$$

4-Caculate the following equation:

$$[sI - A + BK] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} s-1 & -2 \\ -5+k_1 & s-10+k_2 \end{bmatrix}$$

$$|sI - A + BK| = s^2 + (k_2 - 11)s + (2k_1 - k_2)$$

5-Solve for k_1, k_2 by equating the like powers on both sides:

$$s^2 + 8\,s + 25 = s^2 + (k_2 - 11)s + (2k_1 - k_2)$$

$k_2 - 11 = 8$ ……… …. $k_2 = 19$

$2k_1 - k_2 = 25$ ……………. $k_1 = 22$

Consider the regulator system shown in Figure 10–2. The plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

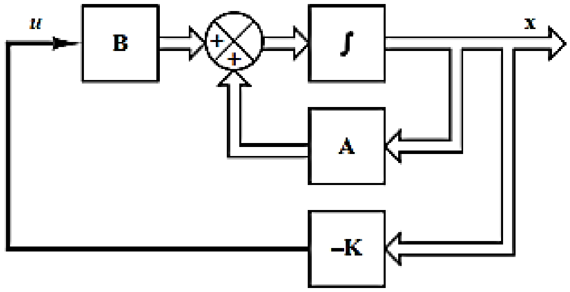
where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system uses the state feedback control $\mathbf{u} = -\mathbf{K}\mathbf{x}$. Let us choose the desired closed-loop poles at

$$s = -2 + j4, \qquad s = -2 - j4, \qquad s = -10$$

(We make such a choice because we know from experience that such a set of closed-loop poles will result in a reasonable or acceptable transient response.) Determine the state feedback gain matrix **K**.



First, we need to check the controllability matrix of the system. Since the controllability matrix **M** is given by

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & \vdots & \mathbf{AB} & \vdots & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that $|\mathbf{M}| = -1$, and therefore, rank $\mathbf{M} = 3$. Thus, the system is completely state controllable and arbitrary pole placement is possible.

The desired characteristic equation is

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$
$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

Hence,

$$\alpha_1 = 14, \qquad \alpha_2 = 60, \qquad \alpha_3 = 200$$

Method 2: By defining the desired state feedback gain matrix **K** as

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

and equating $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$ with the desired characteristic equation, we obtain

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix}$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

Thus,

$$6 + k_3 = 14, \qquad 5 + k_2 = 60, \qquad 1 + k_1 = 200$$

from which we obtain

$$k_1 = 199, \qquad k_2 = 55, \qquad k_3 = 8$$

or

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

Method two (Bass-Gura Approach): there are two cases:

Case 1: the system is in the first companion form

Pole Placement Control Design: Method – 2

$$\dot{X} = AX + Bu$$

$$u = -KX, \qquad K = \left[k_1 \; k_2 \cdots k_n\right]$$

Let the system be in first companion (controllable canonical) form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

After applying the control, the closed loop system dynamics is given by

$$\dot{X} = (A - BK)X = A_{CL}X$$

$$A_{CL} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ k_1 & k_2 & k_3 & \cdots & k_n \end{bmatrix}$$

$$A_{CL} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \\ (-a_n - k_1) & (-a_{n-1} - k_2) & \cdots & \cdots & \cdots & (-a_1 - k_n) \end{bmatrix} \cdots\cdots\cdots(1)$$

Pole Placement Control Design: Method – 2

If $\mu_1, \cdots, \mu_n$ are the desired poles. Then the desired characteristic polynomial is given by,

$$(s - \mu_1)\cdots(s - \mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

This characteristic polynomial, will lead to the closed loop system matrix as

$$A_{CL} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & & \\ \vdots & & \ddots & & 1 \\ 0 & 0 & & \cdots & \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} \cdots\cdots(2)$$

State space form

Pole Placement Control Design: Method – 2

Comparing Equation (1) and (2), we arrive at:

$$\begin{bmatrix} a_n + k_1 = \alpha_n \\ a_{n-1} + k_2 = \alpha_{n-1} \\ \vdots \\ a_1 + k_n = \alpha_1 \end{bmatrix} \Rightarrow \begin{bmatrix} k_1 = (\alpha_n - a_n) \\ k_2 = (\alpha_{n-1} - a_{n-1}) \\ \vdots \\ k_n = (\alpha_1 - a_1) \end{bmatrix}$$

$$K = (\alpha - a) \qquad (\text{Row vector form})$$

Case 2: the system is not in the first companion form:

What if the system is not given in the first companion form?

Define a transformation $X = T\hat{X}$

$$\dot{\hat{X}} = T^{-1}\dot{X}$$

$$\dot{\hat{X}} = T^{-1}(AX + Bu)$$

$$\dot{\hat{X}} = (T^{-1}AT)\hat{X} + (T^{-1}B)u$$

Design a T such that $T^{-1}AT$ will be in first companion form.

Select $T = MW$

where $M \triangleq \begin{bmatrix} B & AB \cdots\cdots A^{n-1}B \end{bmatrix}$ is the controllability matrix

Pole Placement Control Design: Method – 2

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & & \iddots & \iddots & 0 \\ & \iddots & \iddots & \cdots & \vdots \\ a_1 & 1 & \cdots & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Next, design a controller for the transformed system (using the technique for systems in first companion form).

$$u = -\hat{K}\hat{X} = -(\hat{K}T^{-1})X = -KX$$

Note: *Because of its role in control design as well as the use of M (Controllability Matrix) in the process, the '<u>first companion form</u>' is also known as 'Controllable Canonical form'.*

Pole Placement Design Steps: Method 2: Bass-Gura Approach

- Check the controllability condition
- Form the characteristic polynomial for *A*

  $$|sI - A| = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$$

  find a_i's
- Find the Transformation matrix T
- Write the desired characteristic polynomial

  $$(s-\mu_1)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

  and determine the α_i's
- The required state feedback gain matrix is

  $$K = [(\alpha_n - a_n) \quad (\alpha_{n-1} - a_{n-1}) \quad \cdots \quad (\alpha_1 - a_1)]\, T^{-1}$$

Choice of closed loop poles : Guidelines

- Do not choose the closed loop poles far away from the open loop poles, otherwise it will demand high control effort

- Do not choose the closed loop poles very negative, otherwise the system will be fast reacting (i.e. it will have a small time constant)
  - In frequency domain it leads to large bandwidth, and hence noise gets amplified

Consider the regulator system shown in Figure 10–2. The plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

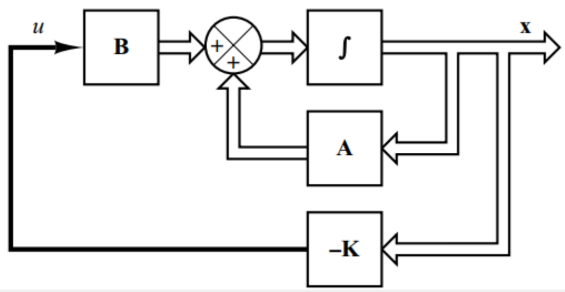
where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system uses the state feedback control $\mathbf{u} = -\mathbf{K}\mathbf{x}$. Let us choose the desired closed-loop poles at

$$s = -2 + j4, \qquad s = -2 - j4, \qquad s = -10$$

(We make such a choice because we know from experience that such a set of closed-loop poles will result in a reasonable or acceptable transient response.) Determine the state feedback gain matrix **K**.



First, we need to check the controllability matrix of the system. Since the controllability matrix **M** is given by

$$\mathbf{M} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

we find that $|\mathbf{M}| = -1$, and therefore, rank $\mathbf{M} = 3$. Thus, the system is completely state controllable and arbitrary pole placement is possible.

Next, we shall solve this problem. We shall demonstrate each of the three methods presented in this chapter.

Method 1: The first method is to use Equation (10–13). The characteristic equation for the system is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix}$$

$$= s^3 + 6s^2 + 5s + 1$$

$$= s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Hence,

$$a_1 = 6, \qquad a_2 = 5, \qquad a_3 = 1$$

The desired characteristic equation is

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

$$\mathbf{K} = \left[\alpha_3 - a_3 \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1\right]$$

where $\mathbf{T} = \mathbf{I}$ for this problem because the given state equation is in the controllable canonical form. Then we have

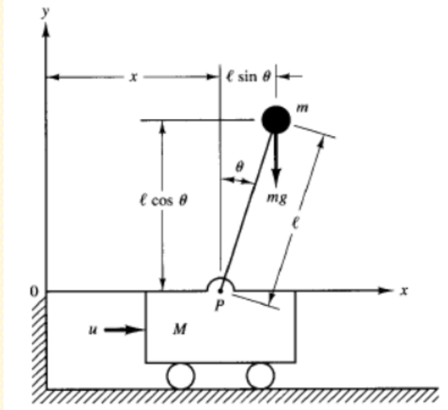
$$\mathbf{K} = [200 - 1 \;\vdots\; 60 - 5 \;\vdots\; 14 - 6]$$

$$= [199 \quad 55 \quad 8]$$

Example: Inverted Pendulum

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 1: Check controllability

$$\mathbf{M} = [\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B} \;\vdots\; \mathbf{A}^3\mathbf{B}] = \begin{bmatrix} 0 & -1 & 0 & -20.601 \\ -1 & 0 & -20.601 & 0 \\ 0 & 0.5 & 0 & 0.4905 \\ 0.5 & 0 & 0.4905 & 0 \end{bmatrix}$$

$$|M| \neq 0$$

Hence, the system is controllable.

Step 2: Form the characteristic equation and get a_i's

$$|s\mathbf{I} - \mathbf{A}| = \begin{bmatrix} s & -1 & 0 & 0 \\ -20.601 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 0.4905 & 0 & 0 & s \end{bmatrix}$$

$$= s^4 - 20.601s^2$$

$$= s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0$$

$$a_1 = 0, \qquad a_2 = -20.601, \qquad a_3 = 0, \qquad a_4 = 0$$

Step 3: Find Transformation $T = MW$ and its inverse

$$\mathbf{W} = \begin{bmatrix} a_3 & a_2 & a_1 & 1 \\ a_2 & a_1 & 1 & 0 \\ a_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -20.601 & 0 & 1 \\ -20.601 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{T} = \mathbf{MW} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -9.81 & 0 & 0.5 & 0 \\ 0 & -9.81 & 0 & 0.5 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} -\frac{0.5}{9.81} & 0 & -\frac{1}{9.81} & 0 \\ 0 & -\frac{0.5}{9.81} & 0 & -\frac{1}{9.81} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Step 4: Find ***α_i's from desired poles*** $\mu_1, \mu_2, \mu_3, \mu_4$

$$\mu_1 = -2 + j2\sqrt{3}, \quad \mu_2 = -2 - j2\sqrt{3}, \quad \mu_3 = -10, \quad \mu_4 = -10$$

$$\begin{aligned}(s-\mu_1)(s-\mu_2)(s-\mu_3)(s-\mu_4) &= (s + 2 - j2\sqrt{3})(s + 2 + j2\sqrt{3})(s+10)(s+10) \\ &= (s^2 + 4s + 16)(s^2 + 20s + 100) \\ &= s^4 + 24s^3 + 196s^2 + 720s + 1600 \\ &= s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 = 0\end{aligned}$$

$$\alpha_1 = 24, \quad \alpha_2 = 196, \quad \alpha_3 = 720, \quad \alpha_4 = 1600$$

Step 5: Find State Feed back matrix ***K*** and input ***u***

$$\begin{aligned}\mathbf{K} &= [\alpha_4 - a_4 \mid \alpha_3 - a_3 \mid \alpha_2 - a_2 \mid \alpha_1 - a_1]\,\mathbf{T}^{-1} \\ &= [1600 - 0 \mid 720 - 0 \mid 196 + 20.601 \mid 24 - 0]\,\mathbf{T}^{-1} \\ &= [1600 \quad 720 \quad 216.601 \quad 24]\begin{bmatrix} -\frac{0.5}{9.81} & 0 & -\frac{1}{9.81} & 0 \\ 0 & -\frac{0.5}{9.81} & 0 & -\frac{1}{9.81} \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\ &= [-298.1504 \quad -60.6972 \quad -163.0989 \quad -73.3945]\end{aligned}$$

$$u = -\mathbf{Kx} = 298.1504x_1 + 60.6972x_2 + 163.0989x_3 + 73.3945x_4$$