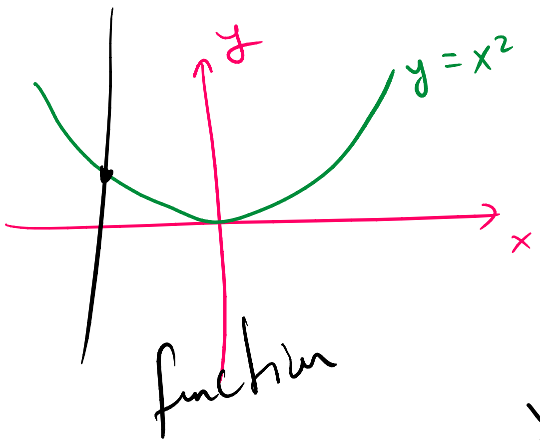
7.1

Wednesday, November 17, 2021 12:29 PM

# Inverse functions

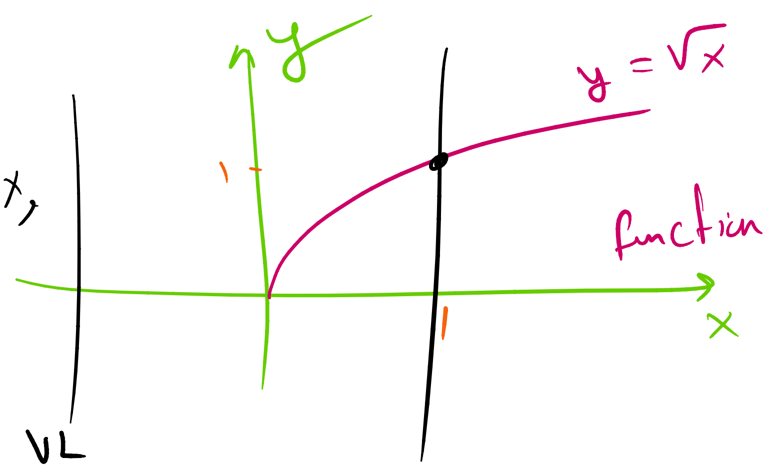


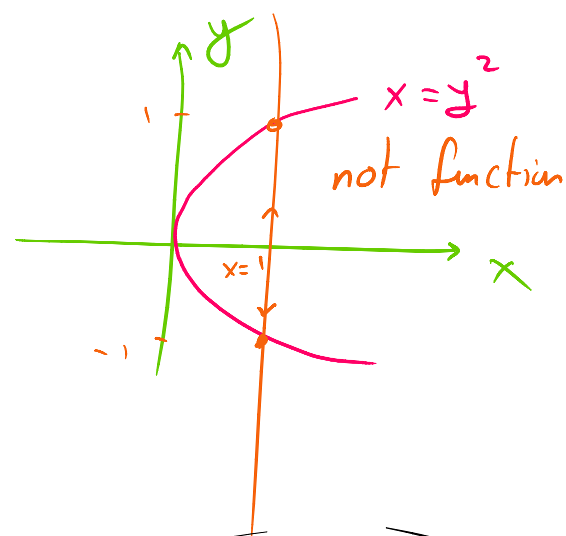

VLT vertical line

function: is relation "Rule" that assigns a unique value ($y \in \mathbb{R}$) to each element in domain ($x \in D$)

if any VL crosses f at most once the f is function

ex,



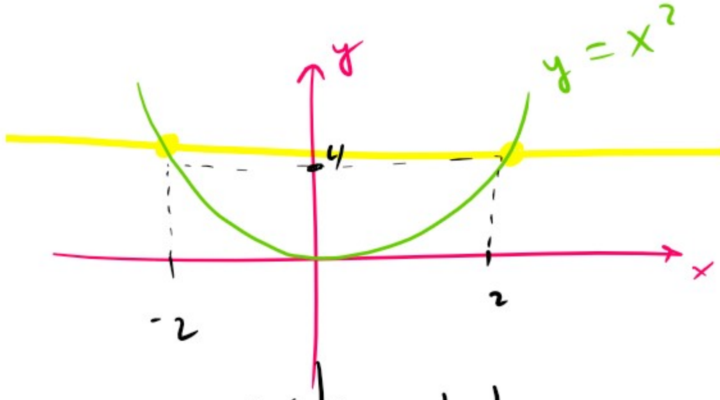




Q. what is 1-1 function?

one-to-one functions

A: f(x) is 1-1 function on domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2 \in D$

**Exp**




Is $y$ 1-1?

$x_1 = 2$ $\quad x_1 \neq x_2$

$x_2 = -2$

$f(x_1) = f(x_2) = 4$

not 1-1 function

$D = (-\infty, \infty)$

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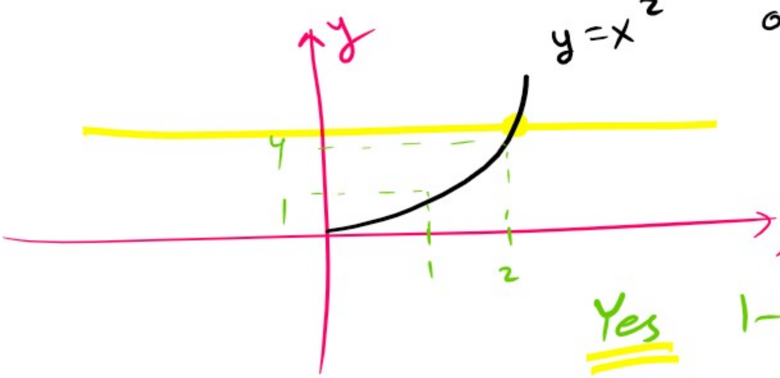
**Exp**




$y = x^2$ on $x \geq 0$

Is $y$ 1-1?

Yes 1-1 fuction

$D = [0, \infty)$

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We use HLT to check if $f$ is 1-1 or not

HLT: Horizontal Line Test

If any HL crosses $f$ at most once, then $f$ is 1-1 ✓

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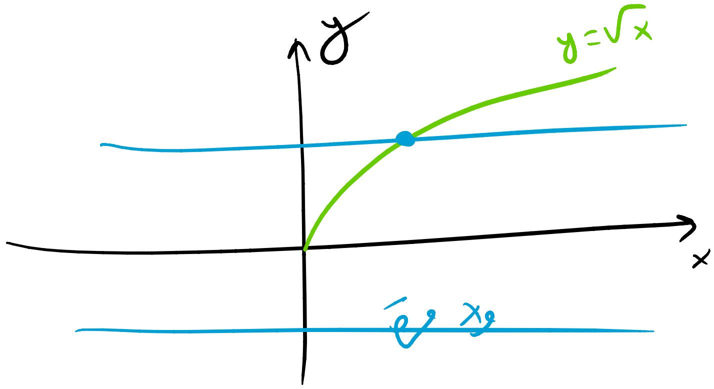
Is $y$ 1-1?

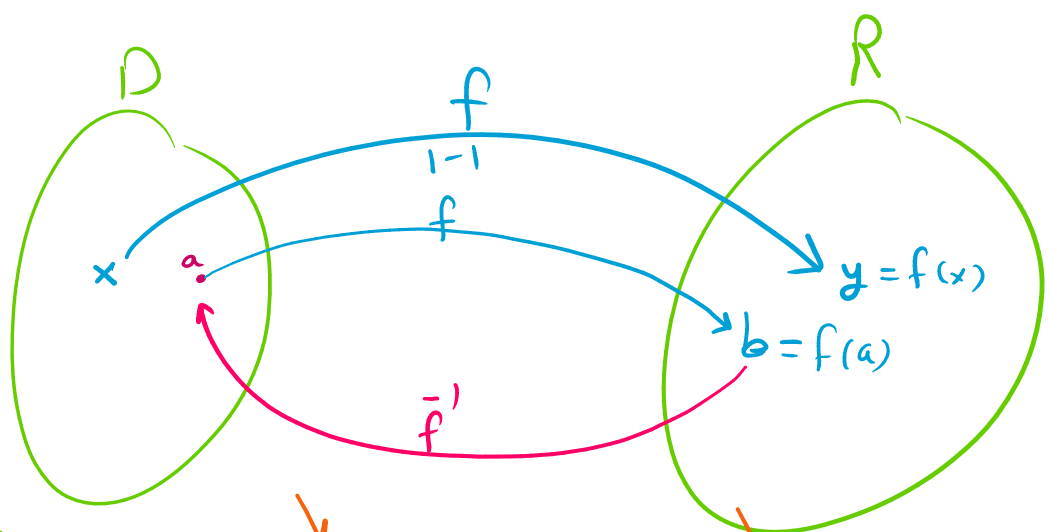
Exp $y = \sqrt{x}$ Is $y$ 1-1?




1-1

Q: why we need 1-1 functions?

A:




$f: D \to R$

$a \to f(a)=b$

$f^{-1}: R \to D$

$b \to f^{-1}(b)=a$

$\cancel{f^{-1}}\left(\cancel{f}(a)\right)=f^{-1}(b)$

$a = f^{-1}(b)$

Domain $f$ is Range of $f^{-1}$

Range $f$ is Domain of $f^{-1}$

Inverse of $f$ is denoted by $f^{-1}(x)$

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

$$\left[f(x)\right]^{-1} = \frac{1}{f(x)} \checkmark$$

$$\left(f^{-1}\circ f\right)(x) = \cancel{f^{-1}}\left(\cancel{f}(x)\right) = x \quad \forall\, x \in D(f)$$

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x \quad \forall\, x \in \underbrace{D(f^{-1})}_{R(f)}$$

Q2: Given function $f(x)$

Q : Given 1-1 function $f(x)$
How to find $f^{-1}(x)$ ?

A : $\boxed{1}$ Replace $f(x)$ by $y$

$\boxed{2}$ Solve for $x$ ضع x موضوع القانون

$\boxed{3}$ Replace $x$ by $y$
Replace $y$ by $x$

$\boxed{3}$ Replace $y$ by $f^{-1}(x)$

Exp Given $f(x) = x^2$ , $x \geq 0$

① Find $f^{-1}(x)$

$\boxed{1}$ $y = x^2$

$\boxed{2}$ $\sqrt{y} = \sqrt{x^2} \Rightarrow \sqrt{y} = |x| \Rightarrow x = \sqrt{y}$

③ $y = \sqrt{x}$

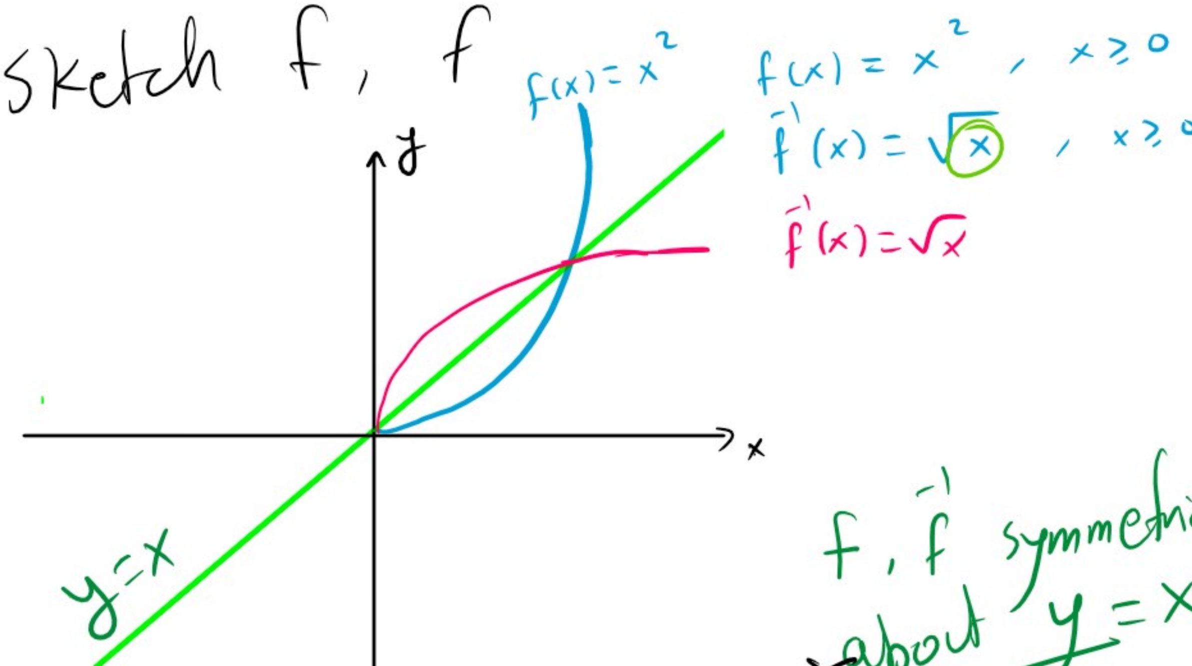
④ $f^{-1}(x) = \sqrt{x}$

② Find $D(f)$, $D(f^{-1})$, $R(f)$, $R(f^{-1})$

$D(f) = [0, \infty) = R(f^{-1})$

$D(f^{-1}) = [0, \infty) = R(f)$

③ Sketch $f$, $f^{-1}$




④ What symmetry we see? $\Rightarrow$ $f$, $f^{-1}$ symmetric about $y = x$

⑤ find $(f^{-1} \circ f)(x)$, $(f \circ f^{-1})(x)$

(5) find $(f^{-1} \circ f)(x)$, $(f \circ f^{-1})(x)$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x|$$
$$= x$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

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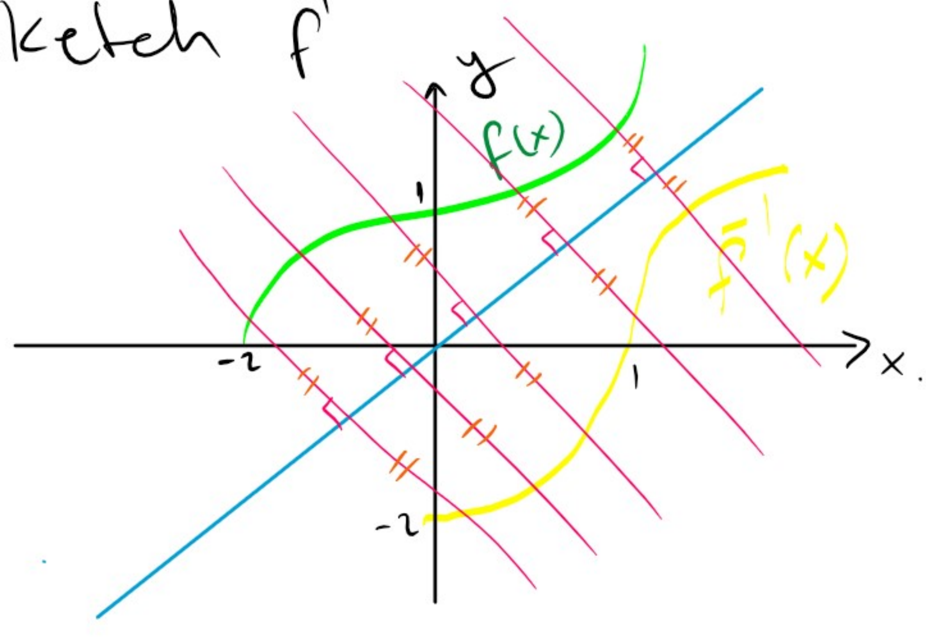
**Exp** Give $f$ in this graph

Sketch $f^{-1}$

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**Exp** Find $f^{-1}$ if $f(x) = x^2 - 2x$, $\underline{x \le 1}$

$D(f) = (-\infty, 1] = R(f^{-1})$

(1) $y = x^2 - 2x$

?   ✓

 

(1) $y = x^2 - 2x$

(2) $y = \underbrace{x^2 - 2x + 1}_{} - 1 = (x-1)^2 - \cancel{1}$

$+1 \qquad\qquad +1$

$$\sqrt{(x-1)^2} = \sqrt{y+1}$$

$$|x-1| = \sqrt{y+1}$$

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ \boxed{1-x \quad \text{if } x \leq 1} \end{cases}$$

$$1 - x = \sqrt{y+1}$$

$-1 \qquad -1$

$$-x = -1 + \sqrt{y+1}$$

$$\boxed{x = 1 - \sqrt{y+1}}$$

(3) $y = 1 - \sqrt{x+1}$

(4) $f^{-1}(x) = 1 - \sqrt{x+1}$

$x + 1 \geq 0$

$x \geq -1$

$$D(f^{-1}) = [-1, \infty) = R(f)$$

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**Q.** How can we derive $f^{-1}$

**A.** See next result

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$\underline{\underline{Th}}$ $f : D \longrightarrow R$ is 1-1 and $f'$ exists and never zero on $D$

$a \longrightarrow \underline{\underline{b = f(a)}}$

then $f^{-1} : R \longrightarrow D$ is diff on $R$ with

$b \longrightarrow a$

$$\left.\frac{d f^{-1}}{dx}\right|_{x=b} = \frac{1}{f'\left(f^{-1}(b)\right)}$$

$f^{-1}(b) \to a$

$x = b \to f(a)$

$b = f(a)$

$f^{-1}(b) = \cancel{f^{-1}}\left(\cancel{f}(a)\right)$

$f^{-1}(b) = a$

$f : D \longrightarrow R$

$a \longrightarrow b = f(a)$

$f' : D \longrightarrow R$

$f'' : D \longrightarrow R$

$\Rightarrow$ $f(a)$, $f'(a)$, $f''(a)$ $\Big\}$ $a \in D(f)$

$f^{-1} : R \longrightarrow D$

$b \longrightarrow a = f^{-1}(b)$

$(f^{-1})' : R \longrightarrow D$

$f^{-1}(b)$, $(f^{-1})'(b)$ $\Big\}$ $\Rightarrow b \in D(f^{-1}) = R$

$(f^{-1})' : R \longrightarrow D$

$(f^{-1})'' : R \longrightarrow D$

$(f^{-1})'(b)$

$(f^{-1})''(b)$

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Exp $y = x^2$ , $x \geq 0$

Find $(f^{-1})'(y)$

a ? x

b ?

$\boxed{b = 4}$

$f(a) = 4 \rightarrow \boxed{f(a) = b}$

$a^2 = 4$

$a = \pm 2 \Rightarrow \boxed{a = 2}$

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$f(x) = x^2$

$f'(x) = 2x$

S1 $(f^{-1})'(4) = \frac{1}{f'(a)} = \frac{1}{f'(2)}$

b

$\boxed{a = f^{-1}(b)}$

$= \frac{1}{2(2)}$

$= \frac{1}{4}$

S2 $f^{-1}(x) = \sqrt{x}$

$(f^{-1})' = \frac{1}{2\sqrt{x}}$

$(f^{-1})'(4) = \frac{1}{\ldots} = \frac{1}{\ldots} = \frac{1}{4}$

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}$$

**Exp** Let $f(x) = 3x^2$

Find $\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})}$

$f(a) = b$

$f(a) = f(\sqrt{2})$

$\boxed{a = \sqrt{2}}$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})} = \frac{1}{f'(a)} = \frac{1}{f'(\sqrt{2})} = \frac{1}{3(\sqrt{2})^2}$$

$$= \frac{1}{3(2)} = \frac{1}{6}$$

$\to b$

**Exp** $f(x) = 5x + e^{2x}$

Find $(f^{-1})'(1)$

$f(a) = b$

$f(a) = 1$

$\boxed{5a + e^{2a} = 1}$

$a = ??$ $0$

$$\left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{f'(a)} = \frac{1}{f'(0)}$$

$\to b$

$f' = 5 + 2e^{2x}$

$$= \frac{1}{5+2} = \frac{1}{7}$$

$$= \frac{1}{5+2e^0} = \frac{1}{5+2(1)} = \frac{1}{5+2} = \frac{1}{7}$$

⑫ Exp

$$y = f(x) = 1 - \frac{1}{x} \quad , \; x > 0$$

① find $R(f^{-1}) = D(f) = (0, \infty)$

② Find $f^{-1}$

1. $y = 1 - \frac{1}{x}$

$$\frac{1}{x} = 1 - y$$

$$x = \frac{1}{1-y}$$

2. $y = \frac{1}{1-x}$

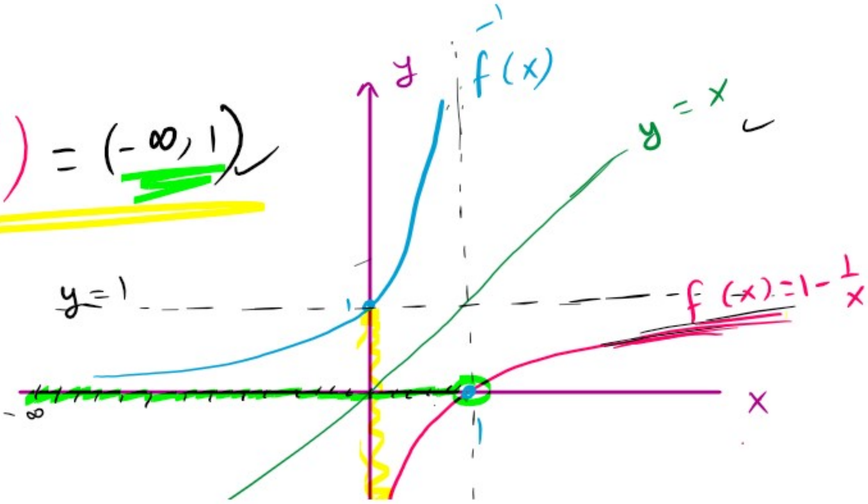
3. $f^{-1}(x) = \frac{1}{1-x}$ → قطعيش

$D(f^{-1}) = \mathbb{R} \setminus \{1\}$ x

3) Find $R(f)$

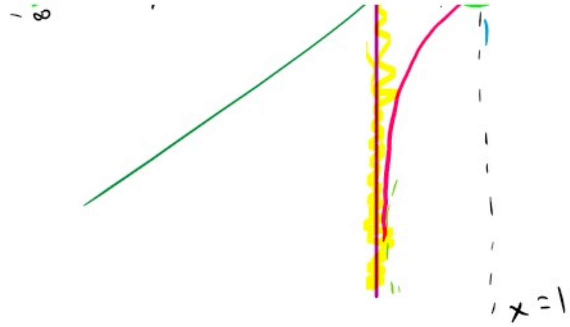
$R(f) = D(f^{-1}) = (-\infty, 1)$

$f(x) = 1 - \frac{1}{x} \; , \; x > 0$



$f(x) = $ ① $x$

$f^{-1}(x) = \frac{1}{1-x}$ ✓

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**Q31** $f(x) = \frac{x+3}{x-2}$ $\rightarrow$ $D(f) = \mathbb{R} \setminus \{2\}$

1-1

① Find $f^{-1}$

$\boxed{1}$ $y = \frac{x+3}{x-2}$

$\boxed{2}$ $y(x-2) = x+3$

$yx - 2y = x + 3$

$yx - x = 2y + 3$

$x(y-1) = 2y+3$

$x = \frac{2y+3}{y-1}$

$\boxed{3}$ $y = \frac{2x+3}{x-1}$

$\boxed{4}$ $f^{-1}(x) = \frac{2x+3}{x-1}$

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② $D(f^{-1}) = \mathbb{R} \setminus \{1\} = R(f)$

③ $R(f^{-1}) = D(f) = \mathbb{R} \setminus \{2\}$

④ Show that $f(f^{-1}(x)) = x$

$f = \dfrac{x+3}{x-2}$

$$f\left(\frac{2x+3}{x-1}\right) = \frac{\frac{2x+3}{x-1} + 3}{\frac{2x+3}{x-1} - 2} = \frac{\frac{2x+3+3(x-1)}{\cancel{x-1}}}{\frac{2x+3-2(x-1)}{\cancel{x-1}}}$$

$$= \frac{2x\cancel{+3}+3x\cancel{-3}}{\cancel{2x}+3\cancel{-2x}+2} = \frac{5x}{5} = x$$

(41) $f(x) = x^3 - 3x^2 - 1, \quad x \geq 2$

Find $\left.\dfrac{df^{-1}}{dx}\right|_{x=-1}$

$x = -1 = f(3)$

$-1 \to b$, $3 \to a$

$f' = 3x^2 - 6x$

$f'(3) = 3(3)^2 - 6(3)$

$= 3(9) - 18$

$= 27 - 18$

$= 9$

$\left.\dfrac{df^{-1}}{dx}\right| = \dfrac{1}{\phantom{x}} = \dfrac{1}{\phantom{x}}$

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$$\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{f'(a)} = \left.\frac{1}{f'(3)}\right| \quad = 9$$

$$= \frac{1}{9}$$

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$$f(x) = 1 - \frac{1}{x} = \frac{x}{x} - \frac{1}{x} = \frac{x-1}{x}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\infty} \frac{1x-1}{1x} = 1 \Rightarrow \boxed{y = 1 \quad \text{H. Asy}}$$

$\boxed{x = 0}$ V. Asy since $\lim_{x\to 0^+} \frac{x-1}{x} = \frac{0-1}{\text{small} +} = -\infty$

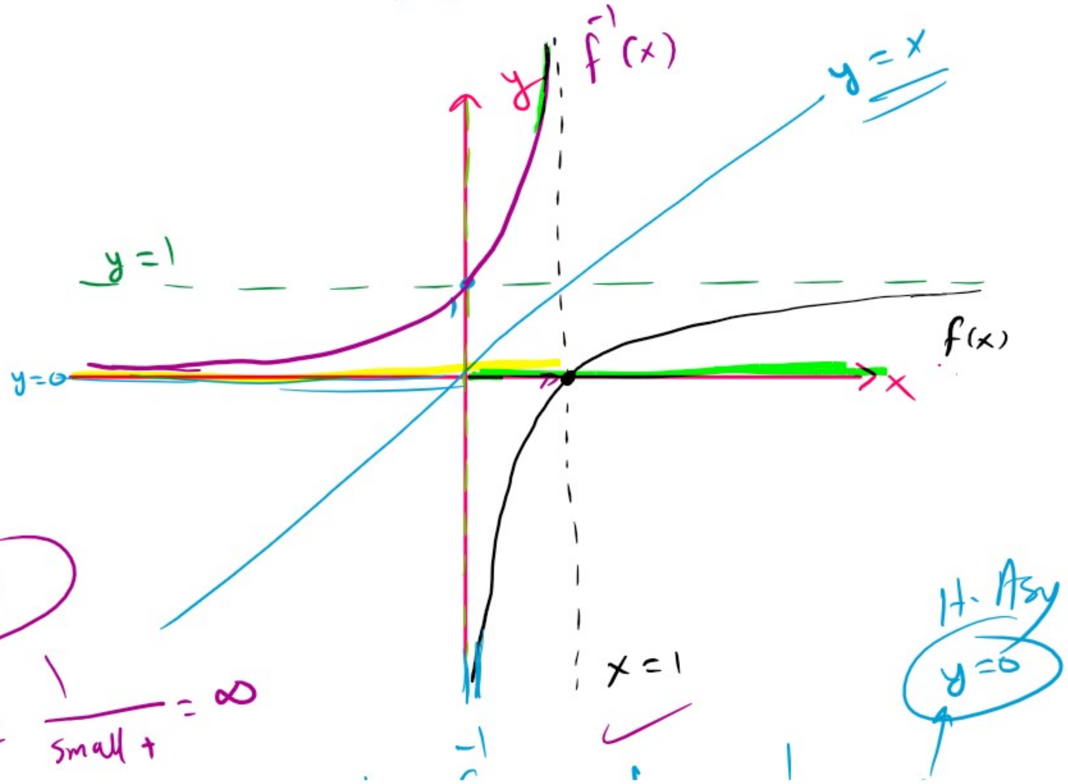
$$y = \frac{x-1}{x}$$

Keypoint $(1,0)$

$$f^{-1}(x) = \frac{1}{1-x}$$

$\boxed{x = 1 \quad \text{V. Asy}}$

$$\lim \frac{1}{\;} = \frac{1}{\text{small} +} = \infty$$



$\lim_{x\to 1^-} \frac{1}{1-x} = \frac{1}{\text{small} +} = \infty$

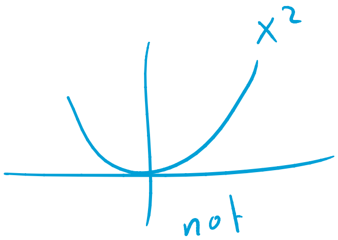
$\lim_{x\to\infty} f^{-1}(x) = \lim_{x\to\infty} \frac{1}{1-x} = 0$

$\boxed{f(x) = x^2 \quad \text{on} \quad x \geq 0}$

$\boxed{f(x) = x^2 \quad \text{on} \quad x \geq 2}$

(1)
(2) $D(f) = R(f^{-1})$
(3) $D(f^{-1}) = R(f)$
(4)

Only 1-1 functions have inverse

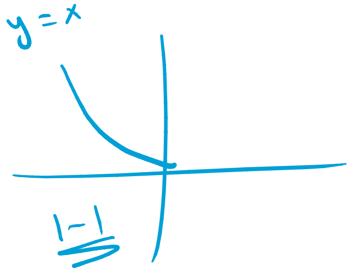



no inverse

$\Rightarrow$

$x \leq 0$

$y = x^2$




it has inverse

$D(f) = R(f^{-1})$

$D(f^{-1}) = R(f)$