Exercises:

Q2: The m.g.f of a Random variable X is $\left(\frac{2}{3}+\frac{1}{3}e^{t}\right)^{9}$. show that
$$\Pr(\mu-2\sigma < X < \mu+2\sigma) = \sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{9-x}.$$

The m.g.f its m.g.f of binomial $(pe^{t}+1-p)^{n}$, $t\in\mathbb{R}$.

so $n=9$, $p=\frac{1}{3}$.

$\Rightarrow X\sim B(9,\frac{1}{3})$

$\rightarrow \mu = np = \frac{9}{3} = 3$

$\rightarrow \sigma^2 = np(1-p) = 3\left(\frac{2}{3}\right) = 2$

$\rightarrow \sigma = \sqrt{2}$

$\rightarrow \Pr(3-2\sqrt{2} < X < 3+2\sqrt{2}) = \Pr(0.17 < X < 5.8)$ تقريب ل integer

$= \Pr(1 \leq X \leq 5)$

$= \sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{9-x}$

Q3: if X is $b(n,p)$, show that

$E\left(\frac{X}{n}\right) = p$ and $E\left(\left(\frac{X}{n}-p\right)^2\right) = \frac{p(1-p)}{n}$

since it binomial $\rightarrow E(X) = np$

$\rightarrow E\left(\frac{X}{n}\right) = \frac{E(X)}{n} = \frac{np}{n} = p$.

$\rightarrow$

$\rightarrow E\left[\left(\frac{X}{n}-p\right)^2\right] = \frac{p(1-p)}{n}$

$\rightarrow var(X) = np(1-p)$, we have to compute $E\left(\frac{X}{n}-p\right)^2$

$\rightarrow E\left(\left(\frac{X}{n}-p\right)^2\right) = Var\left(\frac{X}{n}\right) = \frac{var X}{n^2} = \frac{np(1-p)}{n^2}$

$= \frac{p(1-p)}{n}$.

Q4: let the independent Random variables X_1, X_2, X_3 have the same p.d.f

$$f(x) = \begin{cases} 3x^2 , & 0<x<1 \\ 0 , & \text{elsewhere} \end{cases}$$

Find the probability that exactly two of these three variables exceed $\frac{1}{2}$.

Q7: let the indep. r.v X_1 and X_2 have binomial distribution with parameter $n_1 = 3$, $P_1 = \frac{2}{3}$ and $n_2 = 4$, $P_2 = \frac{1}{2}$. compute $Pr(X_1 = X_2)$.

$Pr(X_1 = X_2) = \sum_{k=0}^{3} P(X_1 = X_2 = k) = \sum_{k=0}^{3} P(X_1 = k)\, P(X_2 = k)$ since it indep.

$$= \sum_{0}^{3} \binom{3}{k}\left(\frac{2}{3}\right)^{k}\left(\frac{1}{3}\right)^{3-k} \cdot \binom{4}{k}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{4-k}$$

$$= \sum_{0}^{3} \binom{3}{k}(2)^{k}\left(\frac{1}{3}\right)^{3} \cdot \binom{4}{k}\left(\frac{1}{2}\right)^{4}$$

$$= \left(\frac{1}{3}\right)^{3}\left(\frac{1}{2}\right)^{4} \sum_{0}^{3} \binom{3}{k}(2)^{k}\binom{4}{k}.$$

$$= \frac{1}{27} \cdot \frac{1}{16}\left[\binom{3}{0}\binom{4}{0}(2)^{0} + \binom{3}{1}\binom{4}{1}(2)^{1} + \binom{3}{2}\binom{4}{2}(2)^{2} + \binom{3}{3}\binom{4}{3}(2)^{3}\right]$$

$$= \frac{1}{432}\left[1 + 24 + 72 + 32\right]$$

$$= \frac{129}{432}.$$

Note:

$$\rightarrow \left(\frac{2}{3}\right)^{k}\left(\frac{1}{3}\right)^{3-k} = 2^{k}\cancel{\left(\frac{1}{3}\right)^{k}}\left(\frac{1}{3}\right)^{3}\cancel{\left(\frac{1}{3}\right)^{-k}} = 2^{k}\left(\frac{1}{3}\right)^{3}$$

$$\rightarrow \left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{4-k} = \cancel{\left(\frac{1}{2}\right)^{k}}\left(\frac{1}{2}\right)^{4}\cancel{\left(\frac{1}{2}\right)^{-k}} = \left(\frac{1}{2}\right)^{4}.$$

Q6: let X be $b(\underset{n}{2}, p)$ and let Y be $b(\underset{n}{4}, p)$. if $Pr(X \geq 1) = \frac{5}{9}$

Find $Pr(Y \geq 1)$.

$\rightarrow Pr(X \geq 1) = 1 - Pr(X < 1) = 1 - Pr(X = 0)$.

$\rightarrow Pr(X=0) = \binom{2}{0} (p)^0 (1-p)^{2-0} = (1)(1)(1-p)^2$

$\rightarrow \frac{5}{9} = 1 - Pr(X=0)$

$\frac{5}{9} = 1 - (1-p)^2 \rightarrow \sqrt{(1-p)^2} = \sqrt{\frac{4}{9}} \rightarrow 1-p = \frac{2}{3}$

$\rightarrow \boxed{p = \frac{1}{3}}$

$X \sim (2, \frac{1}{3})$.

$\rightarrow Pr(Y \geq 1) = 1 - Pr(Y<1) = 1 - Pr(Y=0)$.

$\rightarrow Pr(Y<0) = \binom{4}{0} (p)^0 (1-p)^{4-0} = (1-p)^4$.

$\rightarrow Pr(Y \geq 1) = 1 - (1-p)^4$

$= 1 - \underbrace{\left(1 - \frac{1}{3}\right)}_{\frac{2}{3}}^4$

$= 1 - \frac{16}{81}$

$= \frac{65}{81}$