7.8 Relative Rates of Growth.

Def: Rates of Growth as $x \to \infty$.

let $f(x)$ and $g(x)$ be positive for x sufficiently large

1) f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$

2) f and g grows at the same rate as $x \to \infty$ if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

where L is finite and positive

$\boxed{2}$ Which of the following functions grow faster than e^x as $x \to \infty$? which grow at the same rate as e^x? which grow slower?

a) $10x^4 + 30x + 1$

$$\lim_{x\to\infty} \frac{10x^4 + 30x + 1}{e^x} = \lim_{x\to\infty} \frac{40x^3 + 30}{e^x} = \lim_{x\to\infty} \frac{120x^2}{e^x}$$

$$= \lim_{x\to\infty} \frac{240x}{e^x} = \lim_{x\to\infty} \frac{240}{e^x} = 0$$

$10x^4 + 30x + 1$ grow slower than e^x as $x \to \infty$

b) $x \ln x - x$

$$\lim_{x\to\infty} \frac{x\ln x - x}{e^x} = \lim_{x\to\infty} \frac{x(\ln x - 1)}{e^x} \; \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{x \cdot \frac{1}{x} + (\ln x - 1)\cdot 1}{e^x}$$

$$= \lim_{x\to\infty} \frac{1 + \ln x - \cancel{1}}{e^x} = \lim_{x\to\infty} \frac{\ln x}{e^x} \; \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} =$$

$$= \lim_{x\to\infty} \frac{1}{x e^x} = 0$$ $\to$ $x\ln x - x$ grow slower than e^x

c) $\sqrt{1+x^4}$

$$\lim_{x\to\infty} \frac{\sqrt{1+x^4}}{e^x} = \lim_{x\to\infty} \sqrt{\frac{1+x^4}{e^{2x}}} = \sqrt{\lim_{x\to\infty} \frac{1+x^4}{e^x}} \overset{\frac{\infty}{\infty}}{=} \sqrt{\lim_{x\to\infty} \frac{4x^3}{e^x}} \quad \frac{\infty}{\infty}$$

$$= \sqrt{\lim_{x\to\infty} \frac{12x}{e^x}} \overset{\frac{\infty}{\infty}}{=} \sqrt{\lim_{x\to\infty} \frac{12}{e^x}} = \sqrt{0} = 0$$

$\sqrt{1+x^4}$ grow slower than e^x

d) $\left(\frac{5}{2}\right)^x$

$$\lim_{x\to\infty} \frac{(5/2)^x}{e^x} = \lim_{x\to\infty} \left(\frac{\frac{5}{2}}{e}\right)^x = \lim_{x\to\infty} \left(\overset{\text{كسر}}{\frac{5}{2e}}\right)^x = 0$$

$\left(\frac{5}{2}\right)^x$ grows slower than e^x

e) e^{-x} : $\lim_{x\to\infty} \frac{e^{-x}}{e^x} = \lim_{x\to\infty} \left(\frac{e^{-1}}{e}\right)^x = \lim_{x\to\infty} \left(\frac{\frac{1}{e}}{e}\right)^x$

$$= \lim_{x\to\infty} \left(\frac{1}{e^2}\right)^x = 0 \qquad \text{since } \frac{1}{e^2} < 1$$

e^{-x} grows slower than e^x

f) xe^x: $\lim_{x\to\infty} \frac{x\cancel{e^x}}{\cancel{e^x}} = \lim_{x\to\infty} x = \infty$

xe^x grows faster than e^x

g) $e^{\cos x}$

$\lim_{x\to\infty} \frac{e^{\cos x}}{e^x} =$

$-1 \leq \cos x \leq 1 \rightarrow e^{-1} \leq e^{\cos x} \leq e^{1} \rightarrow \frac{e^{-1}}{e^x} \leq \frac{e^{\cos x}}{e^x} \leq \frac{e}{e^x}$

$\lim_{x\to\infty} \frac{e^{-1}}{e^x} = \lim_{x\to\infty} \frac{1}{e\,e^x} = \lim_{x\to\infty} \frac{1}{e^{x+1}} = 0$

$\lim_{x\to\infty} \frac{e}{e^x} = e \lim_{x\to\infty} \frac{1}{e^x} = e(0) = 0$

so by sandwich theorem $\lim_{x\to\infty} \frac{e^{\cos x}}{e^x} = 0$

$\therefore e^{\cos x}$ grows slower than e^x

h) e^{x-1} : $\lim_{x\to\infty} \frac{e^{x-1}}{e^x} = \lim_{x\to\infty} \frac{\cancel{e^x}\cdot e^{-1}}{\cancel{e^x}} = \lim_{x\to\infty} e^{-1} = e^{-1}$

$\therefore e^{x-1}$ grows at the same rate as e^x.

6 Which of the following functions grow faster than $\ln x$ as $x \to \infty$? which grow at the same rate as $\ln x$? which grow slower?

a) $\log_2 x^2$:

$$\lim_{x\to\infty} \frac{\log_2 x^2}{\ln x} = \lim_{x\to\infty} \frac{2\log_2 x}{\ln x} = 2\lim_{x\to\infty} \frac{\frac{\ln x}{\ln 2}}{\ln x}$$

$$= 2 \lim_{x\to\infty} \frac{\ln x}{\ln 2} \cdot \frac{1}{\ln x} = 2 \lim_{x\to\infty} \frac{1}{\ln 2}$$

$$= 2\left(\frac{1}{\ln 2}\right)$$ constant $\Longrightarrow$ same rate

b) $\log_{10} 10x$:

$$\lim_{x\to\infty} \frac{\log_{10} 10x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\ln 10x}{\ln 10}}{\ln x}$$

$$= \lim_{x\to\infty} \frac{\ln 10x}{\ln 10 \cdot \ln x} = \frac{1}{\ln 10} \lim_{x\to\infty} \frac{\ln 10x}{\ln x} \quad \frac{\infty}{\infty}$$

$$= \frac{1}{\ln 10} \cdot \lim_{x\to\infty} \frac{\frac{10}{10x}}{\frac{1}{x}} = \frac{1}{\ln 10} \cdot 1 = \frac{1}{\ln 10}$$

constant $\Longrightarrow$ same rate

c) $\frac{1}{\sqrt{x}}$:

$$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\ln x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\cdot \ln x} = \frac{1}{\infty\cdot\infty} = 0$$

$\frac{1}{\sqrt{x}}$ grows slower than $\ln x$

d) $\frac{1}{x^2}$:

$$\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\ln x} = \lim_{x\to\infty} \frac{1}{x^2 \ln x} = 0$$

$\frac{1}{x^2}$ grows slower than $\ln x$

e) $x - 2\ln x$:

$$\lim_{x\to\infty} \frac{x - 2\ln x}{\ln x} = \lim_{x\to\infty}\left(\frac{x}{\ln x} - \frac{2\ln x}{\ln x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x}{\ln x} - 2\right) \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{x}} - \lim_{x\to\infty} 2$$

$$= \left(\lim_{x\to\infty} x\right) - 2 = \infty - 2 = \infty$$

$x - 2\ln x$ grows faster than $\ln x$

f) e^{-x} :

$$\lim_{x\to\infty} \frac{e^{-x}}{\ln x} = \lim_{x\to\infty} \frac{1}{e^{x}\ln x} = 0$$

e^{-x} grows slower than $\ln x$.

g) $\ln(\ln x)$:

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}\cdot\frac{1}{\ln x}}{\frac{1}{x}} = 0$$

or let $u = \ln x$, as $x \to \infty$, $u \longrightarrow \infty$

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{u\to\infty} \frac{\ln u}{u} \overset{\frac{\infty}{\infty}}{=} \lim_{u\to\infty} \frac{\frac{1}{u}}{1} = 0$$

$\ln(\ln x)$ grows slower than $\ln x$.

h) $\ln(2x+5)$.

$$\lim_{x\to\infty} \frac{\ln(2x+5)}{\ln x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{2}{2x+5}}{\frac{1}{x}}$$

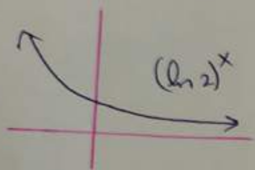
$$= \lim_{x\to\infty} \frac{2x}{2x+5} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2}{2+0} = \frac{2}{2} = 1$$

$\ln(2x+5)$ grows at the same rate as $\ln x$

8 Order the following functions from slowest growing to fastest growing as $x \to \infty$

$$2^x \ , \ (\ln 2)^x \ , \ x^2 \ , \ e^x$$

$(\ln 2)^x = (0.69)^x \Rightarrow$ decay function since $a<1$



$(\ln 2)^x \ , \ x^2$:-

$\lim_{x\to\infty} \frac{(\ln 2)^x}{x^2} = \frac{0}{\infty} = 0$ so $(\ln 2)^x$ grows slower than x^2

$2^x \ , \ e^x$:-

e^x grows faster than 2^x since $e>2$ (growth functions) also $\lim_{x\to\infty} \frac{e^x}{2^x} = \lim_{x\to\infty} \left(\frac{e}{2}\right)^x = \infty$

$2^x \ , \ x^2$:-

$\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{2^x \ln 2}{2x} = \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} = \infty$

2^x grows faster than x^2

$$\boxed{(\ln 2)^x \ , \ x^2 , \ 2^x \ , e^x}$$

