Exercises :

2.3.0 : True or False and prove or counter example.

a. If x_n is strictly decreasing and $0 \leq x_n < \frac{1}{2}$, then $x_n \to 0$ as $n \to \infty$ False

counter example $x_n = \frac{1}{4} + \frac{1}{n+4}$ is strictly decreasing and $|x_n| \leq \frac{1}{4} + \frac{1}{5} < \frac{1}{2}$

But $x_n \to \frac{1}{4}$ as $n \to \infty$.

b. If $x_n = \frac{(n-1)\cos(n^2+n+1)}{2n-1}$ then x_n has a convergence subsequance.

By blzano : proof x_n is bdd sequence then x_n is convergent subsequence.

since $\frac{n-1}{2n-1} \to \frac{1}{2}$ as $n \to \infty$ this factor is bdd. because it converge.

since $|\cos(n^2+n+1)| \leq 1$ it follows that $\{x_n\}$ is bounded.

So By Bolzano weierstrass Theorem x_n has convergent subsequence.

c. If x_n is a strictly increasing sequence and $|x_n| < 1 + \frac{1}{n}$ for $n = 1, 2, 3, \ldots$

then $x_n \to 1$ as $n \to \infty$. False.

$x_n = \frac{1}{2} - \frac{1}{n}$ is strictly increasing and $|x_n| \leq \frac{1}{2} < 1 + \frac{1}{n}$ But $x_n \to \frac{1}{2}$ as $n \to \infty$

d. If x_n has a convergent subsequence, then x_n is bounded. False

$x_n = (1+(-1)^n)n$ satisfies $x_n = 0$ for n odd and $x_n = 2n$ for n even

Thus, $x_{2k+1} \to 0$ as $k \to \infty$ But x_n is Not bounded.

← اثبت انها increasing و converge to 0 .

2.3.1 : suppose that $x_0 \in (-1,0)$ and $x_n = \sqrt{x_{n-1}+1} - 1$ for $n \in \mathbb{N}$, prove that $x_n \uparrow 0$ as $n \to \infty$. what happens when $x_0 \in [-1,0]$?

suppose that $-1 < x_{n-1} < 0$ for some $n \geq 0$

$$0 < x_{n-1} + 1 < 1 \rightarrow$$

طالما يكون الشيء بين صفر وواحد
فالجذر للشيء بكون أكبر
من الشيء نفسه

$$0 < x_{n-1} + 1 < \sqrt{x_{n-1}+1}$$

أي $0 < \frac{1}{4} < 1$

$$\sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\frac{1}{2} > \frac{1}{4}$$

and it follows that $x_{n-1} < \sqrt{x_{n-1}+1} - 1 = x_n$

$x_{n-1} < x_n$ i.e $\{x_n\}$ is increasing ✓

Moreover $\sqrt{x_{n-1}+1} - 1 \leq 1 - 1 = 0$ (bdd above by 0).

so x_n is increasing and bdd above by 0 (By MCT).

$$x_n \to L \text{ as } n \to \infty$$

Now, Find L :

$$L = \lim_{n \to \infty} \sqrt{x_{n-1}+1} - 1$$

2.3.2: suppose that $0 \le x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1-x_n}$ for $n \in \mathbb{N}$. prove that $x_n \downarrow 0$ as $n \to \infty$ and $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$ as $n \to \infty$.

case 1: If $x_1 = 0$ then $x_n = 0$ for all n

so its converges to 0.

case 2: If $0 < x_1 < 1$ then.

suppose $0 < x_n < 1$

$0 < x_{n+1} < x_n$ so its decreasing.

2.3.3 : suppose that $x_0 \geq 2$ and $x_n = 2 + \sqrt{x_{n-1} - 2}$ for $n \in \mathbb{N}$. use the Monotone convergence Theorem to prove that either $x_n \to 2$ or $x_n \to 3$ as $n \to \infty$.

case 1 : $x_0 = 2$ Then $x_n = 2$ for all n so $\lim = 2$.

case 2 : $2 < x_0 < 3$

suppose that $\underset{-2}{2} < \underset{-2}{x_{n-1}} < \underset{-2}{3}$

$$0 < x_{n-1} - 2 < 1$$

$$\sqrt{x_{n-1} - 2} \geq x_{n-1} - 2$$

i.e $x_n = 2 + \sqrt{x_{n-1} - 2} \geq x_{n-1}$.

$x_n \geq x_{n-1}$ so it is increasing.

$x_n = 2 + \sqrt{x_{n-1} - 2} \leq 2 + 1 = 3$ (bdd above by 3).

so By MCT $x_n \to L$ as $n \to \infty$

Now Find L:

$$L = \lim_{n \to \infty} 2 + \sqrt{x_{n-1} - 2} =$$

2.3.4: suppose that $x_0 \in \mathbb{R}$ and $x_n = \dfrac{1 + x_{n-1}}{2}$ for $n \in \mathbb{N}$ use the monotone convergence Theorem to prove that $x_n \to 1$ as $n \to \infty$.

Case 1 : $x_0 < 1$

suppose $x_{n-1} < 1$ Then

$$x_{n-1} = \frac{2 x_{n-1}}{2} < \frac{1 + x_{n-1}}{2} = x_n < \frac{2}{2} = 1$$

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i.e $x_{n-1} < x_n$ i.e $\{x_n\}$ is increasing and bdd abov

so $x_n \to x$ as $n \to \infty$.

Now Find x :

$$\lim_{n \to \infty} \frac{1 + x_{n-1}}{2}$$

Case 2 : $x_0 \geq 1$.

If $x_{n-1} \geq 1$ then $x_n = \dfrac{1 + x_{n-1}}{2} \leq \dfrac{2 x_{n-1}}{2} = x_{n-1}$

$$1 \leq \frac{2}{2} \leq x_n \leq x_{n-1}$$

i.e $x_n \leq x_{n-1}$ Thus, $\{x_n\}$ is decreasing and bdd below.

So $x_n \to x$ as $n \to \infty$.

Now Find x :

2.3.5: prove that $\lim_{n\to\infty} x^{\frac{1}{2n-1}} = \begin{cases} 1 & , x>0 \\ 0 & , x=0 \\ -1 & , x<0 \end{cases}$

case 1 : $x=0$

The result is obvious when $x=0$: $\lim_{n\to\infty} 0 = 0$ ☒

case 2 : If $x>0$:

$$\lim_{n\to\infty} x^{\frac{1}{2n-1}} = \lim_{m\to\infty} x^{\frac{1}{m}} = 1$$

case 3 : If $x<0$ then

since $2n-1$ is odd we have by the previous case that

$$x^{\frac{1}{2n-1}} = -(-x)^{\frac{1}{2n-1}} \to -1 \text{ as } n\to\infty .$$

2.3.6 :

a. suppose that $\{x_n\}$ is monotone increasing sequence in $\mathbb{R}$ (not necessarily bdd above) prove that there is an extended real number x s.t $x_n \to x$ as $n\to\infty$.

suppose that $\{x_n\}$ is increasing.

If $\{x_n\}$ is bounded above then there is an $x\in\mathbb{R}$ s.t $x_n \to x$ (By MCT).

If $\{x_n\}$ (otherwise) not bdd above given any $M>0$ there is an $N\in\mathbb{N}$ s.t $x_N > M$

Since $\{x_n\}$ is increasing, $n\geq N$ implies $x_n \geq x_N > M$

Hence $x_n \to \infty$ as $n\to\infty$.

b. State and prove an analogous result for decreasing sequences.

Suppose that $\{x_n\}$ is a monotone decreasing sequence in $\mathbb{R}$ (not necessarily bdd below) prove that there is an extended real number x s.t $x_n \to x$ as $n \to \infty$.

If $\{x_n\}$ is bdd below, then $\exists\, x \in \mathbb{R}$ s.t $x_n \to x$ as $n \to \infty$ (By MCT).

If $\{x_n\}$ is not bdd below, then $\exists\, M<0$, $\exists\, N \in \mathbb{N}$ s.t $x_N < M$

$\{x_n\}$ decreasing $\rightsquigarrow$ $x_n \leq x_N < M$, $\forall n \geq N$.

$\rightsquigarrow$ $x_n < M$, $\forall n \geq N$

So $x_n \to -\infty$ as $n \to \infty$ ☒