$\boxed{1.3}$ Matrix Arithmetic

* For matrix and vectors, we define:

⇒ arithmetic operations: addition, subtraction, multiplication.

⇒ two more operations: scalar multiplication, transposition.

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* **Matrix Notation**: we denote to matrices by $A, B, C, \ldots$

- If A is $m \times n$ matrix, then $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

  → a_{ij} is the entry of A in i^{th} row and j^{th} column.

  → we refer to this entry as the (i,j) entry of A.

- Sometimes we shorten this to $A = (a_{ij})$
- Similarly, we refere to the matrix $B = (b_{ij})$, $C=(c_{ij})$ ...

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* **Vectors**: are Matrices with only one row or one column.

- They represent solutions of linear systems.
- The solution can be written as row vector or column vector.

  Exp The solution of the linear system $\begin{aligned} x_1 + x_2 &= 5 \\ x_1 - x_2 &= 3 \end{aligned}$

  can be written as row vector $(4,1)$ or column vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

- Usually, we represent the solution of $m \times n$ linear system using column vector $n \times 1$. The set of all $n \times 1$ vectors of real numbers is called Euclidean n-space denoted by $\mathbb{R}^n$.

- we denote to the column vector by a bold face lowercase letter as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$
- we denote to the row vector by a bold face lowercase letter with a horizontal aray above: $\vec{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$

  $\vec{y} = (y_1, y_2, \cdots, y_n)$

- Given $m \times n$ matrix A.
  - → the j^{th} column vector of A is denoted by a_j
  - → the i^{th} row vector of A is denoted by $\vec{a}_i$

That is $a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ and $\vec{a}_j = (a_{j1}, a_{j2}, \dots, a_{jn})$

→ We can write the matrix A in terms of its <u>colum vectors</u> as $A = (a_1, a_2, \dots, a_n)$

→ We can write the matrix A in terms of its <u>row vectors</u> as $A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$

- Similarly, if B is $n \times r$ matrix, then

$$B = (b_1, b_2, \dots, b_r) = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_n \end{bmatrix}$$

<u>Exp</u> If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 6 \end{bmatrix}$, then $a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $a_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

and $\vec{a}_1 = (1, -2, 3)$, $\vec{a}_2 = (2, 0, 6)$

- Note also $A = (a_1, a_2, a_3) = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix}$

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* <u>Equality</u>: For two matrices to be equal, they must have the same dimensions "order" and their corresponding entries must agree.

<u>Def</u>: Two $m \times n$ matrices A and B are <u>equal</u> if $a_{ij} = b_{ij}$ for each i and j.

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* Scalar Multiplication:

Def: If A is $m \times n$ matrix and α is a scalar, then αA is the $m \times n$ matrix whose (i,j) entry is αa_{ij}.

Exp If $A = \begin{bmatrix} 2 & 6 & -2 \\ 4 & 8 & 10 \end{bmatrix}$, then $\frac{1}{2}A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 5 \end{bmatrix}$

and $3A = \begin{bmatrix} 6 & 18 & -6 \\ 12 & 24 & 30 \end{bmatrix}$

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* Matrix Addition

Def If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrix, then the sum $A + B$ is the $m \times n$ matrix whose (i,j) entry is $a_{ij} + b_{ij}$ for each ordered pair (i,j).

Exp If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 5 & 7 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 3 & -3 \\ 5 & 9 & 4 \end{bmatrix}$$

$$A - B = A + (-B) = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -3 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ -2 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -10 \end{bmatrix}$$

➡ We refer to the zero matrix by O.

If $O_{2\times 3}$, then $O + A = A + O = A$

➡ The additive inverse of the matrix A is $-A$

Since $A + (-A) = (-A) + A = O$

$$A + (-A) = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -3 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 1 \\ -3 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

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* Matrix Multiplication and Linear Systems

* Consider the $m \times n$ linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

→ we can write this system using Matrix Multiplication of the form: $\boxed{A\mathbf{x} = \mathbf{b}}$, where

$A = (a_{ij})$ is Known, $\mathbf{x}$ is unknow vector $\in \mathbb{R}^n$:

$$
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}
$$

→ The product Ax is $m \times 1$ vector in $\mathbb{R}^m$ given by

$$
Ax = \begin{bmatrix} \vec{a}_1 x \\ \vec{a}_2 x \\ \vdots \\ a_m x \end{bmatrix}_{m \times 1}
$$

where the i^{th} entry of Ax is

$$\vec{a}_i x = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

$$
= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}
$$

→ We can write Ax as sum of column vectors:

$$
Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}
$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Exp Write the following system of equation as a matrix of equation of the form $Ax=b$.

(1)
$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 2 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & -2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}}_{b}$$

[2] $3x_1 + 2x_2 + 5x_3 = 4$

$$\begin{bmatrix} 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4$$

Note that in exp [1] $\Rightarrow$
$$Ax = \begin{bmatrix} 2x_1 + x_2 + x_3 \\ x_1 - x_2 + 2x_3 \\ 3x_1 - 2x_2 - x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Def * If $a_1, a_2, \dots, a_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \dots, c_n$ are scalars, then the sum $c_1 a_1 + c_2 a_2 + \dots + c_n a_n$ is a linear combination of the vectors $a_1, a_2, \dots, a_n$.

* If A is $m\times n$ matrix and $x \in \mathbb{R}^n$, then
$$Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

Exp In the exp [1] above, if we choose $x_1 = x_2 = x_3 = 1$, then

$$(1)\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

So the vector $\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ is a linear combination of the three columns. It follows that the linear system is consistent and $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to this system.

Th The linear system $Ax=b$ is consistent iff b can be written as a linear combination of the column vectors of A.

Exp The linear system $\begin{matrix} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 1 \end{matrix}$ is inconsistent

- since the vector $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can not be written as a linear combination of the column vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

- $$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \text{must be the double of } 1$$

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* Matrix Multiplication :

$\Rightarrow$ To multiply the matrix A by the matrix B, the number of columns of A must $=$ the number of rows of B

Def If $A = (a_{ij})$ is $m \times n$ matrix and $B = (b_{ij})$ is $n \times r$ matrix, then

$AB = C = (c_{ij})$ is $m \times r$ matrix whose entries are

$$c_{ij} = \vec{a}_i \, b_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Exp If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 4 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}_{2\times 3}$ then

1)

$$AB = \begin{bmatrix} 1 & -1 \\ \boxed{2 \quad 3} \end{bmatrix} \begin{bmatrix} 4 & \boxed{\begin{matrix} -1 \\ -2 \end{matrix}} & 1 \\ 0 & & 3 \end{bmatrix} = \begin{bmatrix} 4+0 & -1+2 & 1-3 \\ 8+0 & \boxed{-2-6} & 2+9 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -2 \\ 8 & \boxed{-8} & 11 \end{bmatrix}$$

$\boxed{2}$ $BA = \underset{2\times 3}{\begin{bmatrix} 4 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}}$

Hence $AB \neq BA$ "Multiplication of Matrices is not commutative."

Exp if $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ then

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$ but

$BA = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

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* The Transpose of Matrix

Def. The transpose of $m\times n$ matrix A is $n\times m$ matrix B defined by $b_{ji} = a_{ij}$ for $j = 1, \dots, n$ and $i = 1, \dots, m$.

- The transpose of A is denoted by A^T.

Exp * If $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -1 & 5 \end{bmatrix}$

* If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$.

Def The $n\times n$ matrix A is symmetric if $A^T = A$.

Exp The following matrices are symmetric:

$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 3 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 4 \\ -1 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$