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Note to Students

This *Student Solutions Manual and Study Guide* for the fourth edition of *Discrete Mathematics with Applications* contains complete solutions to every third exercise in the text that is not fully answered in Appendix B. It also contains additional explanation, and review material.

The specific topics developed in the book provide a foundation for virtually every other mathematics or computer science subject you might study in the future. Perhaps more important, however, is the book's recurring focus on the logical principles used in mathematical reasoning and the techniques of mathematical proof.

Why should it matter for you to learn these things? The main reason is that these principles and techniques are the foundation for all kinds of careful analyses, whether of mathematical statements, computer programs, legal documents, or other technical writing. A person who understands and knows how to develop basic mathematical proofs has learned to think in a highly disciplined way, is able to deduce correct consequences from a few basic principles, can build a logically connected chain of statements and appreciates the need for giving a valid reason for each statement in the chain, is able to move flexibly between abstract symbols and concrete objects, and can deal with multiple levels of abstraction. Mastery of these skills opens a host of interesting and rewarding possibilities in a person's life.

In studying the subject matter of this book, you are embarking on an exciting and challenging adventure. I wish you much success!

Acknowledgements

I am enormously indebted to the work of Tom Jenkyns, whose eagle eye, mathematical knowledge, and understanding of language made an invaluable contribution to this volume. I am also most grateful to my husband, Helmut Epp and my daughter, Caroline Epp, who constructed all the diagrams and, especially, to my husband, who provided much support and wise counsel over many years.

Susanna S. Epp

Chapter 1: Speaking Mathematically

The aim of this chapter is to provide some of the basic terminology that is used throughout the book. Section 1.1 introduces special terms that are used to describe aspects of mathematical thinking and contains exercises to help you start getting used to expressing mathematical statements both formally and informally. To be successful in mathematics, it is important to be able comfortably to translate from formal to informal and from informal to formal modes of expression. Sections 1.2 and 1.3 introduce the basic notions of sets, relations, and functions.

Section 1.1

6. *a.* s is negative *b.* negative; the cube root of s is negative (*Or*: $\sqrt[3]{s}$ is negative)
   c. is negative; $\sqrt[3]{s}$ is negative (*Or*: the cube root of s is negative)

9. *a.* have at most two real solutions *b.* has at most two real solutions *c.* has at most two real solutions *d.* is a quadratic equation; has at most two real solutions *e.* E has at most two real solutions

Section 1.2

6. T_2 and T_{-3} each have two elements, and T_0 and T_1 each have one element.
   Justification: $T_2 = \{2, 2^2\} = \{2, 4\}$, $T_{-3} = \{-3, (-3)^2\} = \{-3, 9\}$,
   $T_1 = \{1, 1^2\} = \{1, 1\} = \{1\}$, and $T_0 = \{0, 0^2\} = \{0, 0\} = \{0\}$.

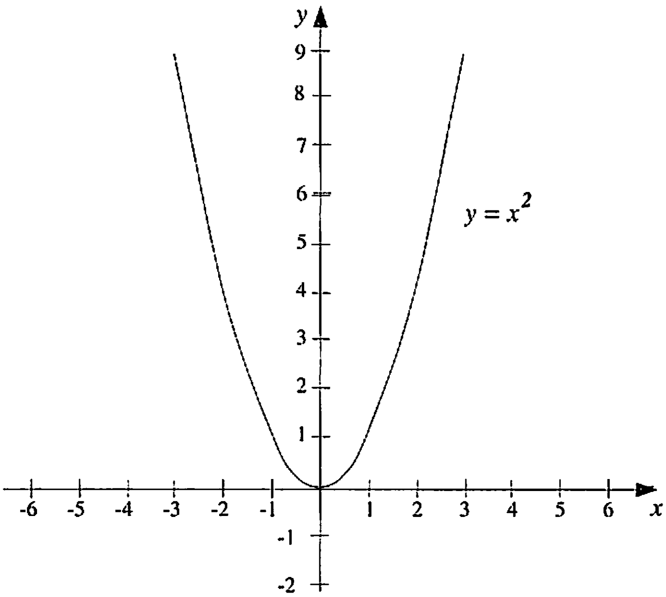
9. *c.* No: The only elements in $\{1, 2\}$ are 1 and 2, and $\{2\}$ is not equal to either of these.
   d. Yes: $\{3\}$ is one of the elements listed in $\{1, \{2\}, \{3\}\}$.
   e. Yes: $\{1\}$ is the set whose only element is 1.
   g. Yes: The only element in $\{1\}$ is 1, and 1 is an element in $\{1, 2\}$.
   h. No: The only elements in $\{\{1\}, 2\}$ are $\{1\}$ and 2, and 1 is not equal to either of these.
   j. Yes: The only element in $\{1\}$ is 1, which is is an element in $\{1\}$. So every element in $\{1\}$ is in $\{1\}$.

12. All four sets have nine elements.
    a. $S \times T = \{(2,1), (2,3), (2,5), (4,1), (4,3), (4,5), (6,1), (6,3), (6,5)\}$
    b. $T \times S = \{(1,2), (3,2), (5,2), (1,4), (3,4), (5,4), (1,6), (3,6), (5,6)\}$
    c. $S \times S = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$
    d. $T \times T = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)\}$

Section 1.3

6. *a.* $(2, 4) \in R$ because $4 = 2^2$.
   $(4, 2) \notin R$ because $2 \neq 4^2$.
   $(-3, 9) \in R$ because $9 = (-3)^2$.
   $(9, -3) \notin R$ because $-3 \neq 9^2$.

b.



12. T is not a function because, for example, both $(0, 1)$ and $(0, -1)$ are in T but $1 \neq -1$. Many other examples could be given showing that T does not satisfy property (2) of the definition of function.

15. *c.* This diagram does not determine a function because 4 is related to both 1 and 2, which violates property (2) of the definition of function.

    d. This diagram defines a function; both properties (1) and (2) are satisfied.

    e. This diagram does not determine a function because 2 is in the domain but it is not related to any element in the co-domain.

18. $h(-\frac{12}{5}) = h(\frac{0}{1}) = h(\frac{9}{17}) = 2$

Review Guide: Chapter 1

Variables and Mathematical Statements

- What are the two main ways variables are used? *(p. 1)*
- What is a universal statement? Give one example. *(p. 2)*
- What is a conditional statement? Give one example. *(p. 2)*
- What is an existential statement Give one example. *(p. 2)*
- Give an example of a universal conditional statement. *(p. 3)*
- Give an example of a universal existential statement. *(p. 3)*
- Give an example of an existential universal statement. *(p. 4)*

Sets

- What does the notation $x \in S$ mean? *(p. 7)*
- What does the notation $x \notin S$ mean? *(p. 7)*
- How is the set-roster notation used to define a set? *(p. 7)*
- What is the axiom of extension? *(p. 7)*
- What do the symbols **R**, **Z**, and **Q** stand for? *(p. 8)*
- What is the set builder notation? *(p. 8)*
- If S is a set and $P(x)$ is a property that elements may or may not satisfy, how should the following be read out loud: $\{x \in S \mid P(x)\}$? *(p. 8)*

Subsets

- If A and B are sets, what does it mean for A to be a subset of B? What is the notation that indicates that A is a subset of B? *(p. 9)*
- What does the notation $A \not\subseteq B$ mean? *(p. 9)*
- What does it mean for one set to be a proper subset of another? *(p. 9)*
- How are the symbols $\subseteq$ and $\in$ different from each other? *(p. 10)*

Cartesian Products

- What does it mean for an ordered pair (a, b) to equal an ordered pair (c, d)? *(p. 11)*
- Given sets A and B, what is the Cartesian product of A and B.? What is the notation for the Cartesian product of A and B? *(p. 11)*
- What is the Cartesian plane? *(p. 12)*

Relations

- What is a relation from a set A to a set B? *(p. 14)*
- If R is a relation from A to B, what is the domain of R? *(p. 14)*
- If R is a relation from A to B, what is the co-domain of R? *(p. 14)*
- If R is a relation from A to B, what does the notation $x\,R\,y$ mean? *(p. 14)*
- How should the following notation be read: $x \not R\, y$? *(p. 14)*
- How is the arrow diagram for a relation drawn? *(p. 16)*

Functions

- What is a function F from a set A to a set B? *(p. 17)*

- What are less formal/more formal ways to state the two properties a function F must satisfy? *(p. 17)*
- Given a function F from a set A to a set B and an element x in A, what is $F(x)$? *(p. 17)*
- What is the squaring function from **R** to **R**? *(p. 20)*
- What is the successor function from **Z** to **Z**? *(p. 20)*
- Give an example of a constant function. *(p. 20)*
- What is the difference between the notations f and $f(x)$? *(pp. 17, 20)*
- If f and g are functions from A to B, what does it mean for f to equal g? *(p. 20)*

Chapter 2: The Logic of Compound Statements

The ability to reason using the principles of logic is essential for solving problems in abstract mathematics and computer science and for understanding the reasoning used in mathematical proof and disproof. In this chapter the various rules used in logical reasoning are developed both symbolically and in the context of their somewhat limited but very important use in everyday language.

Exercise sets for Sections 2.1–2.3 and 3.1–3.4 contain sentences for you to negate, write the contrapositive for, and so forth. These are designed to help you learn to incorporate the rules of logic into your general reasoning processes. Chapters 2 and 3 also present the rudiments of symbolic logic as a foundation for a variety of upper-division courses. Symbolic logic is used in, among others, the study of digital logic circuits, relational databases, artificial intelligence, and program verification.

Section 2.1

9. $(n \vee k) \wedge \sim (n \wedge k)$

15. When you are filling out a truth table, a convenient way to remember the definitions of $\sim$ (not), $\wedge$ (and), and $\vee$ (or) is to think of them in words as follows.

   (1) A *not* statement has opposite truth value from that of the statement. So to fill out a column of a truth table for the negation of a statement, you look at the column representing the truth values for the statement. In each row where the truth value for the statement is T, you place an F in the corresponding row in the column representing the truth value of the negation. In each row where the truth value for the statement is F, you place a T in the corresponding row in the column representing the truth value of the negation.

   (2) The only time an *and* statement is true is when both components are true. Thus to fill out a column of a truth table where the word *and* links two component statements, just look at the columns with the truth values for the component statements. In any row where both columns have a T, you put a T in the same row in the column for the *and* statement. In every other row of this column, you put a F.

   (3) The only time an *or* statement is false is when both components are false. So to fill out a column of a truth table where the word *or* links two component statements, you look at the columns with the truth values for the component statements. In any row where both columns have a F, you put a F in the same row in the column for the *or* statement. In every other row of this column, you put a T.

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

   The truth table shows that $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ always have the same truth values. Therefore they are logically equivalent. This proves the distributive law for $\wedge$ over $\vee$.

24.

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \vee (p \wedge r)$	$(p \vee q) \wedge r$	
T	T	T	T	T	T	T	
T	T	F	T	F	T	F	←
T	F	T	T	T	T	T	←
T	F	F	T	F	T	F	
F	T	T	T	F	T	T	
F	T	F	T	F	T	F	←
F	F	T	F	F	F	F	
F	F	F	F	F	F	F	

different truth values

The truth table shows that $(p \vee q) \vee (p \wedge r)$ and $(p \vee q) \wedge r$ have different truth values in rows 2, 3, and 6. Hence they are not logically equivalent.

30. The dollar is not at an all-time high or the stock market is not at a record low.

33. $-10 \geq x$ or $x \geq 2$

39. The statement's logical form is $(p \wedge q) \vee ((r \wedge s) \wedge t)$, so its negation has the form

$$\begin{aligned} \sim ((p \wedge q) \vee ((r \wedge s) \wedge t)) &\equiv \; \sim (p \wedge q) \wedge \sim ((r \wedge s) \wedge t)) \\ &\equiv (\sim p \vee \sim q) \wedge (\sim (r \wedge s) \vee \sim t)) \\ &\equiv (\sim p \vee \sim q) \wedge ((\sim r \vee \sim s) \vee \sim t)). \end{aligned}$$

Thus a negation is (*num_orders* $\geq$ 50 or *num_instock* $\leq$ 300) and ((50 $>$ *num_orders* or *num_orders* $\geq$ 75) or *num_instock* $\leq$ 500).

42.

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge q$	$q \wedge r$	$((\sim p \wedge q) \wedge (q \wedge r))$	$((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	F	F	F	F
T	F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F	F
F	T	T	T	F	T	T	T	F
F	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F
F	F	F	T	T	F	F	F	F

all $F's$

Since all the truth values of $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$ are F, $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$ is a contradiction.

45. Let b be "Bob is a double math and computer science major," m be "Ann is a math major," and a be "Ann is a double math and computer science major." Then the two statements can be symbolized as follows: a. $(b \wedge m) \wedge \sim a$ and b. $\sim (b \wedge a) \wedge (m \wedge b)$. *Note*: The entries in the truth table assume that a person who is a double math and computer science major is also a math major and a computer science major.

b	m	a	$\sim a$	$b \wedge m$	$m \wedge b$	$b \wedge a$	$\sim (b \wedge a)$	$(b \wedge m) \wedge \sim a$	$\sim (b \wedge a) \wedge (m \wedge b)$
T	T	T	F	T	T	T	F	F	F
T	T	F	T	F	T	F	T	T	T
T	F	T	F	T	F	T	F	F	F
T	F	F	T	F	F	F	T	F	F
F	T	T	F	F	F	F	T	F	F
F	T	F	T	F	F	F	T	F	F
F	F	T	F	F	F	F	T	F	F
F	F	F	T	F	F	F	T	F	F

same truth values

The truth table shows that $(b \wedge m) \wedge \sim a$ and $\sim (b \wedge a) \wedge (m \wedge b)$ always have the same truth values. Hence they are logically equivalent.

51. *Solution 1*: $p \wedge (\sim q \vee p) \equiv p \wedge (p \vee \sim q)$ commutative law for $\vee$
$\equiv p$ absorption law

Solution 2: $p \wedge (\sim q \vee p) \equiv (p \wedge \sim q) \vee (p \wedge p)$ distributive law
$\equiv (p \wedge \sim q) \vee p$ identity law for $\wedge$
$\equiv p$ by exercise 50.

54. $(p \wedge (\sim (\sim p \vee q))) \vee (p \wedge q) \equiv (p \wedge (\sim (\sim p) \wedge \sim q)) \vee (p \wedge q)$ De Morgan's law
$\equiv (p \wedge (p \wedge \sim q)) \vee (p \wedge q)$ double negative law
$\equiv ((p \wedge p) \wedge \sim q)) \vee (p \wedge q)$ associative law for $\wedge$
$\equiv (p \wedge \sim q)) \vee (p \wedge q)$ idempotent law for $\wedge$
$\equiv p \wedge (\sim q \vee q)$ distributive law
$\equiv p \wedge (q \vee \sim q)$ commutative law for $\vee$
$\equiv p \wedge \mathbf{t}$ negation law for $\vee$
$\equiv p$ identity law for $\wedge$

Section 2.2

6.

p	q	$\sim p$	$\sim p \wedge q$	$p \vee q$	$(p \vee q) \vee (\sim p \wedge q)$	$(p \vee q) \vee (\sim p \wedge q) \to q$
T	T	F	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	F	T	F	F	F	T

15.

p	q	r	$q \to r$	$p \to q$	$p \to (q \to r)$	$(p \to q) \to r$	
T	T	T	T	T	T	T	
T	T	F	F	T	F	F	
T	F	T	T	F	T	T	
T	F	F	T	F	T	T	
F	T	T	T	T	T	T	
F	T	F	F	T	T	F	←
F	F	T	T	T	T	F	←
F	F	F	T	T	T	F	←

different truth values

The truth table shows that $p \to (q \to r)$ and $(p \to q) \to r$ do not always have the same truth values. (They differ for the combinations of truth values for p, q, and r shown in rows 6, 7, and 8.) Therefore they are not logically equivalent.

18. *Part 1*: Let p represent "It walks like a duck," q represent "It talks like a duck," and r represent "It is a duck." The statement "If it walks like a duck and it talks like a duck, then it is a duck" has the form $p \wedge q \to r$. And the statement "Either it does not walk like a duck or it does not talk like a duck or it is a duck" has the form $\sim p \vee \sim q \vee r$.

p	q	r	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$p \wedge q \to r$	$(\sim p \vee \sim q) \vee r$
T	T	T	F	F	T	F	T	T
T	T	F	F	F	T	F	F	F
T	F	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	F	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	F	T	T	T

same truth values

The truth table shows that $p \wedge q \to r$ and $(\sim p \vee \sim q) \vee r$ always have the same truth values. Thus the following statements are logically equivalent: "If it walks like a duck and it talks like a duck, then it is a duck" and "Either it does not walk like a duck or it does not talk like a duck or it is a duck."

Part 2: The statement "If it does not walk like a duck and it does not talk like a duck then it is not a duck" has the form $\sim p \wedge \sim q \to \sim r$.

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \wedge q$	$\sim p \wedge \sim q$	$p \wedge q \to r$	$(\sim p \wedge \sim q) \to \sim r$	
T	T	T	F	F	F	T	F	T	T	
T	T	F	F	F	T	T	F	F	T	←
T	F	T	F	T	F	F	F	T	T	
T	F	F	F	T	T	F	F	T	T	
F	T	T	T	F	F	F	F	T	T	
F	T	F	T	F	T	F	F	T	T	
F	F	T	T	T	F	F	T	T	F	←
F	F	F	T	T	T	F	T	T	T	

different truth values

The truth table shows that $p \wedge q \to r$ and $(\sim p \wedge \sim q) \to \sim r$ do not always have the same truth values. (They differ for the combinations of truth values of p, q, and r shown in rows 2 and 7.) Thus they are not logically equivalent, and so the statement "If it walks like a duck and it talks like a duck, then it is a duck" is not logically equivalent to the statement "If it does not walk like a duck and it does not talk like a duck then it is not a duck." In addition, because of the logical equivalence shown in Part 1, we can also conclude that the following two statements are not logically equivalent: "Either it does not walk like a duck or it does not talk like a duck or it is a duck" and "If it does not walk like a duck and it does not talk like a duck then it is not a duck."

21. By the truth table for $\to$, $p \to q$ is false if, and only if, p is true and q is false. Under these circumstances, (b) $p \vee q$ is true and (c) $q \to p$ is also true.

27.

p	q	$\sim p$	$\sim q$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

same truth values

The truth table shows that $q \to p$ *and* $\sim p \to \sim q$ always have the same truth values, so they are logically equivalent. Thus the converse and inverse of a conditional statement are logically equivalent to each other.

30. The corresponding tautology is $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \wedge (q \vee r) \leftrightarrow$ $(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	T	F	T	T	T	T
T	F	F	F	F	F	F	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	F	F	F	T
F	F	T	T	F	F	F	F	T
F	F	F	F	F	F	F	F	T

all T's

The truth table shows that $p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$ is always true. Hence it is a tautology.

33. If this integer is even, then it equals twice some integer, and if this integer equals twice some integer, then it is even.

36. The Personnel Director did not lie. By using the phrase "only if," the Personnel Director set forth conditions that were necessary but not sufficient for being hired: if you did not satisfy those conditions then you would not be hired. The Personnel Director's statement said nothing about what would happen if you did satisfy those conditions.

39. *b.* If a security code is not entered, then the door will not open.

45. If this computer program produces error messages during translation, then it is not correct.

If this computer program is correct, then it does not produce error messages during translation.

48. *a.*

$p \vee \sim q \to r \vee q$	$\equiv$	$\sim (p \vee \sim q) \vee (r \vee q)$	*[an acceptable answer]*
	$\equiv$	$(\sim p \wedge \sim (\sim q)) \vee (r \vee q)$	by De Morgan's law *[another acceptable answer]*
	$\equiv$	$(\sim p \wedge q) \vee (r \vee q)$	by the double negative law *[another acceptable answer]*

b.

$p \vee \sim q \to r \vee q$	$\equiv$	$(\sim p \wedge q) \vee (r \vee q)$	by part (a)
	$\equiv$	$\sim (\sim (\sim p \wedge q) \wedge \sim (r \vee q))$	by De Morgan's law
	$\equiv$	$\sim (\sim (\sim p \wedge q) \wedge (\sim r \wedge \sim q))$	by De Morgan's law

The steps in the answer to part (b) would also be acceptable answers for part (a).

51. Yes. As in exercises 47-50, the following logical equivalences can be used to rewrite any statement form in a logically equivalent way using only $\sim$ and $\wedge$:

$$p \to q \equiv \sim p \vee q \qquad p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$$
$$p \vee q \equiv \sim (\sim p \wedge \sim q) \qquad \sim (\sim p) \equiv p$$

The logical equivalence $p \wedge q \equiv \sim(\sim p \vee \sim q)$ can then be used to rewrite any statement form in a logically equivalent way using only $\sim$ and $\vee$.

Section 2.3

9.

p	q	r	$\sim q$	$\sim r$	$p \wedge q$	$p \wedge q \to \sim r$ (premise)	$p \vee \sim q$ (premise)	$\sim q \to p$ (premise)	$\sim r$ (conclusion)
T	T	T	F	F	T	F	T	T	
T	T	F	F	T	T	T	T	T	T ← critical row
T	F	T	T	F	F	T	T	T	F ← critical row
T	F	F	T	T	F	T	T	T	T ← critical row
F	T	T	F	F	F	T	F	T	
F	T	F	F	T	F	T	F	T	
F	F	T	T	F	F	T	T	F	
F	F	F	T	T	F	T	T	F	

Rows 2, 3, and 4 of the truth table are the critical rows in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

12. *b.*

p	q	$p \to q$ (premise)	$\sim p$ (premise)	$\sim q$ (conclusion)
T	T	T	F	
T	F	F	F	
F	T	T	T	F ← critical row
F	F	T	T	T ← critical row

Rows 3, and 4 of the truth table represent the situations in which all the premises are true, but row 3 shows that it is possible for an argument of this form to have true premises and a false conclusion. Hence the argument form is invalid.

15.

p	q	q (premise)	$p \vee q$ (conclusion)
T	T	T	T ← critical row
T	F	F	
F	T	T	T ← critical row
F	F	F	

The truth table shows that in the two situations (represented by rows 1 and 3) in which the premise is true, the conclusion is also true. Therefore, the the second version of generalization is valid.

21.

p	q	r	$p \vee q$ (premise)	$p \to r$ (premise)	$q \to r$ (premise)	r (conclusion)
T	T	T	T	T	T	T ← critical row
T	T	F	T	F	F	
T	F	T	T	T	T	T ← critical row
T	F	F	T	F	T	
F	T	T	T	T	T	T ← critical row
F	T	F	T	T	F	
F	F	T	F	T	T	
F	F	F	F	T	T	

The truth table shows that in the three situations (represented by rows 1, 3, 5) in which all three premises are true, the conclusion is also true. Therefore, proof by division into cases is valid.

30. form: $p \to q$ invalid, converse error
q
$\therefore\ p$

33. A valid argument with a false conclusion must have at least one false premise. In the following example, the second premise is false. (The first premise is true because its hypothesis is false.)

If the square of every real number is positive, then no real number is negative.

The square of every real number is positive.

Therefore, no real number is negative.

42. (1) $q \to r$ premise b
$\sim r$ premise d
$\therefore\ \sim q$ by modus tollens

(2) $p \vee q$ premise a
$\sim q$ by (1)
$\therefore\ p$ by elimination

(3) $\sim q \to u \wedge s$ premise e
$\sim q$ by (1)
$\therefore\ u \wedge s$ by modus ponens

(4) $u \wedge s$ by (3)
$\therefore\ s$ by specialization

(5) p by (2)
s by (4)
$\therefore\ p \wedge s$ by conjunction

(6) $p \wedge s \to t$ premise c
$p \wedge s$ by (5)
$\therefore\ t$ by modus ponens

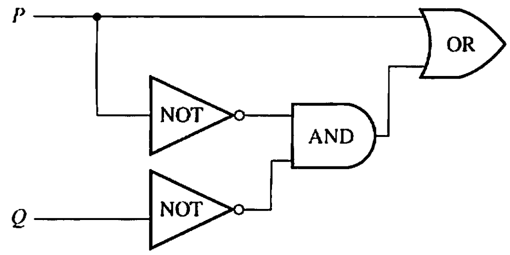
Section 2.4

6. The input/output table is as follows:

Input		Output
P	Q	R
1	1	0
1	0	1
0	1	0
0	0	0

12. $(P \vee Q) \vee \sim (Q \wedge R)$

15.



27. The Boolean expression for circuit (a) is $\sim P \wedge (\sim (\sim P \wedge Q))$ and for circuit (b) it is $\sim (P \vee Q)$. We must show that if these expressions are regarded as statement forms, then they are logically equivalent. But

$$\begin{array}{llll} \sim P \wedge (\sim (\sim P \wedge Q)) & \equiv & \sim P \wedge (\sim (\sim P) \vee \sim Q) & \text{by De Morgan's law} \\ & \equiv & \sim P \wedge (P \vee \sim Q) & \text{by the double negative law} \\ & \equiv & (\sim P \wedge P) \vee (\sim P \wedge \sim Q) & \text{by the distributive law} \\ & \equiv & \mathbf{c} \vee (\sim P \wedge \sim Q) & \text{by the negation law for } \wedge \\ & \equiv & \sim P \wedge \sim Q & \text{by the identity law for } \vee \\ & \equiv & \sim (P \vee Q) & \text{by De Morgan's law.} \end{array}$$

33. *a.*

$$\begin{array}{llll} (P \mid Q) \mid (P \mid Q) & \equiv & \sim [(P \mid Q) \wedge (P \mid Q)] & \text{by definition of } \mid \\ & \equiv & \sim (P \mid Q) & \text{by the idempotent law for } \wedge \\ & \equiv & \sim [\sim (P \wedge Q)] & \text{by definition of } \mid \\ & \equiv & P \wedge Q & \text{by the double negative law.} \end{array}$$

b.

$$\begin{array}{lll} P \wedge (\sim Q \vee R) & \equiv & (P|(\sim Q \vee R))|(P|(\sim Q \vee R)) \\ & & \qquad\qquad\qquad \text{by part (a)} \\ & \equiv & (P|[(\sim Q|\sim Q)|(R|R)])|(P|[(\sim Q|\sim Q)|(R|R)]) \\ & & \qquad\qquad\qquad \text{by Example 2.4.7(b)} \\ & \equiv & (P|[((Q|Q)|(Q|Q))|(R|R)])|(P|[((Q|Q)|(Q|Q))|(R|R)]) \\ & & \qquad\qquad\qquad \text{by Example 2.4.7(a)} \end{array}$$

Section 2.5

3. $287 = 256 + 16 + 8 + 4 + 2 + 1 = 100011111_2$

6. $1424 = 1024 + 256 + 128 + 16 = 10110010000_2$

9. $110110_2 = 32 + 16 + 4 + 2 = 54_{10}$

18.
$$\begin{array}{rr} & 11010_2 \\ - & 1101_2 \\ \hline & 1101_2 \end{array}$$

21. *b.* $S = 0, T = 1$ *c.* $S = 0, T = 0$

24. $67_{10} = (64 + 2 + 1)_{10} = 01000011_2 \longrightarrow 10111100 \longrightarrow 10111101.$

So the two's complement is 10111101.

30. $10111010 \longrightarrow -(01000101 + 1)_2 \longrightarrow -01000110_2 = -(64 + 4 + 2)_{10} = -70_{10}$

36. $123_{10} = (64+32+16+8+2+1)_{10} = 01111011_2$

$-94_{10} = -(64+16+8+4+2)_{10} = -01011110_2 \longrightarrow (10100001+1)_2 \longrightarrow 10100010$

So the 8-bit representations of 123 and −94 are 01111011 and 10100010. Adding the 8-bit representations gives

$$\begin{array}{rr} & 01111011 \\ + & 10100010 \\ \hline & 100011101 \end{array}$$

Truncating the 1 in the 2^8th position gives 00011101. Since the leading bit of this number is a 0, the answer is positive. Converting back to decimal form gives

$$00011101 \longrightarrow 11101_2 = (16+8+4+1)_{10} = 29_{10}.$$

So the answer is 29.

39. $E0D_{16} = 14 \cdot 16^2 + 0 + 13 = 3597_{10}$

42. $B53DF8_{16} = 1011\,0101\,0011\,1101\,1111\,1000_2$

45. $1011\,0111\,1100\,0101_2 = B7C5_{16}$

Review Guide: Chapter 2

Compound Statements

- What is a statement? *(p. 24)*
- If p and q are statements, how do you symbolize "p but q" and "neither p nor q"? *(p. 25)*
- What does the notation $a \leq x < b$ mean? *(p. 26)*
- What is the conjunction of statements p and q? *(p. 27)*
- What is the disjunction of statements p and q? *(p. 28)*
- What are the truth table definitions for $\sim p$, $p \wedge q$, $p \vee q$, $p \to q$, and $p \leftrightarrow q$? *(pp. 26-28,39,45)*
- How do you construct a truth table for a general compound statement form? *(p. 29)*
- What is exclusive or? *(p. 29)*
- What is a tautology, and what is a contradiction? *(p. 34)*
- What is a conditional statement? *(p. 40)*
- Given a conditional statement, what is its hypothesis (antecedent)? conclusion (consequent)? *(p. 39-40)*
- What is a biconditional statement? *(p. 45)*
- What is the order of operations for the logical operators? *(p. 46)*

Logical Equivalence

- What does it mean for two statement forms to be logically equivalent? *(p. 30)*
- How do you test to see whether two statement forms are logically equivalent? *(p. 30)*
- How do you annotate a truth table to explain how it shows that two statement forms are or are not logically equivalent? *(p. 30-31)*
- What is the double negative property? *(p. 31)*
- What are De Morgan's laws? *(p. 32)*
- How is Theorem 2.1.1 used to show that two statement forms are logically equivalent? *(p. 36)*
- What are negations for the following forms of statements? *(pp. 32,42)*
  - $p \wedge q$
  - $p \vee q$
  - $p \to q$ (if p then q)

Converse, Inverse, Contrapositive

- What is the contrapositive of a statement of the form "If p then q"? *(p. 43)*
- What are the converse and inverse of a statement of the form "If p then q"? *(p. 44)*
- Can you express converses, inverses, and contrapositives of conditional statements in ordinary English? *(pp. 43-44)*
- If a conditional statement is true, can its converse also be true? *(p. 44)*
- Given a conditional statement and its contrapositive, converse, and inverse, which of these are logically equivalent and which are not? *(p. 44)*

Necessary and Sufficient Conditions, Only If

- What does it mean to say that something is true only if something else is true? *(p. 45)*
- How are statements about only-if statements translated into if-then form.? *(p. 45)*
- What does it mean to say that something is a necessary condition for something else? *(p. 46)*
- What does it mean to say that something is a sufficient condition for something else? *(p. 46)*

- How are statements about necessary and sufficient conditions translated into if-then form.?*(p. 47)*

Validity and Invalidity

- How do you identify the logical form of an argument? *(p. 24)*
- What does it mean for a form of argument to be valid? *(p. 51)*
- How do you test to see whether a given form of argument is valid? *(p. 52)*
- How do you annotate a truth table to explain how it shows that an argument is or is not valid? *(pp. 52, 59, A9-A10)*
- What does it mean for an argument to be sound? *(p. 59)*
- What are modus ponens and modus tollens? *(pp. 52-53)*
- Can you give examples for and prove the validity of the following forms of argument? *(pp. 54-56)*
  - p, $\therefore p \vee q$ and q, $\therefore p \vee q$
  - $p \wedge q$, $\therefore p$ and $p \wedge q$, $\therefore q$
  - $p \vee q$, $\sim q$, $\therefore p$ and $p \vee q$, $\sim p$, $\therefore q$
  - $p \rightarrow q$, $q \rightarrow r$, $\therefore p \rightarrow r$
  - $p \vee q$, $p \rightarrow r$, $q \rightarrow r$, $\therefore r$
- What are converse error and inverse error? *(pp. 57-58)*
- Can a valid argument have a false conclusion? *(p. 58)*
- Can an invalid argument have a true conclusion? *(p. 59)*
- Which of modus ponens, modus tollens, converse error, and inverse error are valid and which are invalid? *(pp. 53,58)*
- What is the contradiction rule? *(p. 59)*
- How do you use valid forms of argument to solve puzzles such as those of Raymond Smullyan about knights and knaves? *(p. 60)*

Digital Logic Circuits and Boolean Expressions

- Given a digital logic circuit, how do you
  - find the output for a given set of input signals *(p. 68)*
  - construct an input/output table *(pp. 68-69)*
  - find the corresponding Boolean expression? *(pp. 69-70)*
- What is a recognizer? *(p. 70)*
- Given a Boolean expression, how do you draw the corresponding digital logic circuit? *(pp. 70-71)*
- Given an input/output table, how do you draw the corresponding digital logic circuit? *(p. 72)*
- What is disjunctive normal form? *(p. 72)*
- What does it mean for two circuits to be equivalent? *(p. 74)*

- What are NAND and NOR gates? *(p. 74)*
- What are Sheffer strokes and Peirce arrows? *(p. 74)*

Binary and Hexadecimal Notation

- How do you transform positive integers from decimal to binary notation and the reverse? *(pp. 79-80)*
- How do you add and subtract integers using binary notation? *(p. 81)*
- What is a half-adder? *(p. 82)*
- What is a full-adder? *(p. 83)*
- What is the 8-bit two's complement of an integer in binary notation? *(p. 84)*
- How do you find the 8-bit two's complement of a positive integer a that is at most 255? *(p. 85)*
- How do you find the decimal representation of the integer with a given 8-bit two's complement? *(p. 86)*
- How are negative integers represented using two's complements? *(p. 87)*
- How is computer addition with negative integers performed? *(pp. 87-90)*
- How do you transform positive integers from hexadecimal to decimal notation? *(p. 92)*
- How do you transform positive integers from binary to hexadecimal notation and the reverse? *(p. 93)*
- What is octal notation? *(p. 95)*

Chapter 3: The Logic of Quantified Statements

Ability to use the logic of quantified statements correctly is necessary for doing mathematics because mathematics is, in a very broad sense, about quantity. The main purpose of this chapter is to familiarize you with the language of universal and existential statements. The various facts about quantified statements developed in this chapter are used extensively in Chapter 4 and are referred to throughout the rest of the book. Experience with the formalism of quantification is especially useful for students planning to study LISP or Prolog, program verification, or relational databases.

One thing to keep in mind is the tolerance for potential ambiguity in ordinary language, which is typically resolved through context or inflection. For instance, as the "Caution" on page 111 of the text indicates, the sentence "All mathematicians do not wear glasses" is one way to phrase a negation to "All mathematicians wear glasses." (To see this, say it out loud, stressing the word "not.") Some grammarians ask us to avoid such phrasing because of its potentially ambiguity, but the usage is widespread even in formal writing in high-level publications ("All juvenile offenders are not alike," Anthony Lewis, The New York Times, 19 May 1997, Op-Ed page), or in literary works ("All that glisters is not gold," William Shakespeare, The Merchant of Venice, Act 2, Scene 7, 1596-1597).

Even rather complex sentences can be negated in this way. For instance, when asked to write a negation for "The sum of any two irrational numbers is irrational," many people instinctively write "The sum of any two irrational numbers is not irrational." This is an acceptable informal negation (again, say it out loud, stressing the word "not"), but it can lead to genuine mistakes in formal situations. To avoid such mistakes, tell yourself that simply inserting the word "not" is very rarely a good way to express the negation of a mathematical statement.

Section 3.1

6. *a.* When $m = 25$ and $n = 10$,the statement "m is a factor of n^2" is true because $n^2 = 100$ and $100 = 4 \cdot 25$. But the statement "m is a factor of n" is false because 10 is not a product of 25 times any integer. Thus the hypothesis is true and the conclusion is false, so the statement as a whole is false.

   b. $R(m,n)$ is also false when $m = 8$ and $n = 4$ because 8 is a factor of $4^2 = 16$, but 8 is not a factor of 4.

   c. When $m = 5$ and $n = 10$, both statements "m is a factor of n^2" and "m is a factor of n" are true because $n = 10 = 5 \cdot 2 = m \cdot 2$ and $n^2 = 100 = 5 \cdot 20 = m \cdot 20$. Thus both the hypothesis and conclusion of $R(m,n)$ are true, and so the statement as a whole is true.

   d. Here are examples of two kinds of correct answers:

   (1) Let $m = 2$ and $n = 6$. Then both statements "m is a factor of n^2" and "m is a factor of n" are true because $n = 6 = 2 \cdot 3 = m \cdot 3$ and $n^2 = 36 = 2 \cdot 18 = m \cdot 18$. Thus both the hypothesis and conclusion of $R(m,n)$ are true, and so the statement as a whole is true.

   (2) Let $m = 6$ and $n = 2$. Then both statements "m is a factor of n^2" and "m is a factor of n" are false because $n = 2 \neq 6 \cdot k$, for any integer k, and $n^2 = 4 \neq 6 \cdot j$, for any integer j. Thus both the hypothesis and conclusion of $R(m,n)$ are false, and so the statement as a whole is true.

12. Counterexample: Let $x = 1$ and $y = 1$, and note that

$$\sqrt{x+y} = \sqrt{1+1} = \sqrt{2}$$

whereas

$$\sqrt{x} + \sqrt{y} = \sqrt{1} + \sqrt{1} = 1 + 1 = 2,$$

and

$$2 \neq \sqrt{2}.$$

(This is one counterexample among many. Any real numbers x and y with $xy \neq 0$ will produce a counterexample.)

15. *a. Some acceptable answers:* All rectangles are quadrilaterals. If a figure is a rectangle then that figure is a quadrilateral. Every rectangle is a quadrilateral. All figures that are rectangles are quadrilaterals. Any figure that is a rectangle is a quadrilateral.

    b. Some acceptable answers: There is a set with sixteen subsets. Some set has sixteen subsets. Some sets have sixteen subsets. There is at least one set that has sixteen subsets.

18. *c.* $\forall s$, if $C(s)$ then $\sim E(s)$.

    d. $\exists x$ such that $C(s) \wedge M(s)$.

21. *b.* The base angles of T are equal, for any isosceles triangle T.

    d. f is not differentiable, for some continuous function f.

24. *b.* $\exists$ a question x such that x is easy.

    $\exists x$ such that x is a question and x is easy.

27. *c.* This statement translates as "There is a square that is above d." This is false because the only objects above d are a (a triangle) and b (a circle).

    d. This statement translates as "There is a triangle that has f above it," or, "f is above some triangle." This is true because g is a triangle and f is above g.

30. *a.* This statement translates as "There is a prime number that is not odd." This is true. The number 2 is prime and it is not odd.

    c. This statement translates as "There is a number that is both an odd number and a perfect square." This is true. For example, the number 9 is odd and it is also a perfect square (because $9 = 3^2$).

33. *c.* This statement translates as "For all real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$," which is true.

    d. This statement translates as "For all real numbers a, b, c, and d, if $a < b$ and $c < d$ then $ac < bd$," which is false.

    Counterexample: Let $a = -2$, $b = 1$, $c = -3$, and $d = 0$.

    Then $a < b$ because $-2 < 1$ and $c < d$ because $-3 < 0$, but $ac \not< bd$ because $ac = (-2)(-3) = 6$ and $bd = 1 \cdot 0 = 0$ and $6 \not< 0$.

Section 3.2

3. *b.* $\exists$ a computer C such that C does not have a CPU.

    d. $\forall$ bands b, b has won fewer than 10 Grammy awards.

6. *b. Formal negation*: $\exists$ a road r on the map such that r connects towns P and Q.

    Some acceptable informal negations: There is a road on the map that connects towns P and Q. Some road on the map connects towns P and Q. Towns P and Q are connected by a road on the map.

12. The proposed negation is not correct. *Correct negation:* There are an irrational number x and a rational number y such that xy is rational. Or: There are an irrational number and a rational number whose product is rational.

15. *b*. True *d*. True

e. False: $x = 36$ is a counterexample because the ones digit of x is 6 and the tens digit is neither 1 nor 2.

21. $\exists$ an integer n such that n is divisible by 6 and n is not divisible by 2 and n is not divisible by 3.

24. *b*. If an integer greater than 5 ends in 1, 3, 7, or 9, then the integer is prime.

If an integer greater than 5 is prime, then the integer ends in 1, 3, 7, or 9.

27. *Converse*: $\forall$ integers d, if $d = 3$ then $6/d$ is an integer.

Inverse: $\forall$ integers d, if $6/d$ is not an integer, then $d \neq 3$.

Contrapositive: $\forall$ integers d, if $d \neq 3$ then $6/d$ is not an integer.

Theconverse and inverse of the statement are both true, but both the statement and its contrapositive are false. For example, when $d = 2$, then $d \neq 3$ but $6/d = 3$ is an integer.

33. *Converse*: If a function is continuous, then it is differentiable.

Inverse: If a function is not differentiable, then it is not continuous.

Contrapositive: If a function is not continuous, then it is not differentiable.

The statement and its contrapositive are true, but both the converse and inverse are false. For example, take the function f defined by $f(x) = |x|$ for all real numbers x. This function is continuous for all real numbers, but it is not differentiable at $x = 0$.

36. *b*. *One possible answer*: Let $P(x)$ be "$x^2 \neq 2$." The statements "$\forall x \in \mathbf{Z}, x^2 \neq 2$" and "$\forall x \in \mathbf{Q}, x^2 \neq 2$" are true, but the statement "$\forall x \in \mathbf{R}, x^2 \neq 2$" is false.

42. If a person does not pass a comprehensive exam, then that person cannot obtain a master's degree. *Or*: If a person obtains a master's degree then that person passed a comprehensive exam.

Section 3.3

3. *c*. Let $y = \frac{4}{3}$. Then $xy = (\frac{3}{4})(\frac{4}{3}) = 1$.

6. True.

Given $x =$	Choose $y =$	Is y a circle above x, with a different color from x?
e	a, b, or c	yes ✓
g	a or c	yes ✓
h	a or c	yes ✓
j	b	yes ✓

9. *b*. True. *Solution 1*: Let $x = 0$. Then for any real number r, $x + r = r + x = r$ because 0 is an identity for addition of real numbers. Thus, because every element in E is a real number, $\forall y \in E, x + y = y$.

Solution 2: Let $x = 0$. Then $x + y = y$ is true for each individual element y of E:

Choose $x = 0$	Given $y =$	Is $x + y = y$?
	-2	yes: $0 + (-2) = -2$ ✓
	-1	yes: $0 + (-1) = -1$ ✓
	0	yes: $0 + 0 = 0$ ✓
	1	yes: $0 + 1 = 1$ ✓
	2	yes: $0 + 2 = 2$ ✓

12. *c. first version of negation*: $\exists\ x$ in D such that $\sim (\exists\ y$ in E such that $xy \geq y)$.

final version of negation: $\exists\ x$ in D such that $\forall\ y$ in E, $xy \not\geq y$. (*Or*: $\exists\ x$ in D such that $\forall\ y$ in E, $xy < y$.)

The statement is true. For each number x in D, you can find a y in D so that $xy \geq y$. Here is a table showing one way to do this: how all possible choices for x could be matched with a y so that $xy \geq y$.:

Given $x =$	you could take $y =$	Is $xy \geq y$?
-2	-2	$(-2)\cdot(-2) = 4 \geq -2$ ✓
-1	0	$(-1)\cdot 0 = 0 \geq -1$ ✓
0	1	$0 \cdot 1 = 0 \geq 0$ ✓
1	1	$1 \cdot 1 = 1 \geq 1$ ✓
2	2	$2 \cdot 2 = 4 \geq 2$ ✓

d. first version of negation: $\forall\ x$ in D, $\sim (\forall\ y$ in $E, x \leq y)$.

final version of negation: $\forall x$ in D, $\exists\ y$ in E such that $x \not\leq y$. (*Or*: $\forall x$ in D, $\exists\ y$ in E such that $x > y$.)

The statement is true. It says that there is a number in D that is less than or equal to every number in D. In fact, -2 is in D and -2 is less than or equal to every number in D (-2, -1, 0, 1, and 2).

21. *c*. Statement (1) is true because $x^2 - 2xy + y^2 = (x-y)^2$. Thus given any real number x, take $y = x$, then $x - y = 0$, and so $x^2 - 2xy + y^2 = 0$.

Statement (2) is false. Given any real number x, choose a real number y with $y \neq x$. Then $x^2 - 2xy + y^2 = (x-y)^2 \neq 0$.

d. Statement (1) is true because no matter what real number x might be chosen, y can be taken to be 1 so that $(x-5)(y-1) = (x-5)\cdot 0 = 0$.

Statement (2) is also true. Take $x = 5$. Then for all real numbers y, $(x-5)(y-1) = 0(y-1) = 0$.

e. Statements (1) and (2) are both false because all real numbers have nonnegative squares and the sum of any two nonnegative real numbers is nonnegative. Hence for all real numbers x and y, $x^2 + y^2 \neq -1$.

24. *b*.
$$\begin{aligned} \sim (\exists x \in D\ (\exists y \in E\ (P(x,y)))) &= \forall x \in D\ (\sim (\exists y \in E\ (P(x,y)))) \\ &= \forall x \in D\ (\forall y \in E\ (\sim P(x,y))) \end{aligned}$$

30. *a*. $\forall x \in \mathbf{R}$, $\exists y \in \mathbf{R}^-$ such that $x > y$.

b. The original statement says that there is a real number that is greater than every negative real number. This is true. For instance, 0 is greater than every negative real number.

The statement with interchanged quantifiers says that no matter what real number might be given, it is possible to find a negative real number that is smaller. This is also true. If the number x that is given is positive, y could be taken to be -1. Then $x > y$. On the other hand, if the number x that is given is 0 or negative, y could be taken to be $x - 1$. In this case also, $x > y$.

36. *a*. $\exists$ a person x such that $\forall$ people y, x trusts y.

b. *Negation*: $\forall$ people x, $\exists$ a person y such that x does not trust y.

Or: Nobody trusts everybody.

45. $\exists! x \in D$ such that $P(x) \equiv \exists x \in D$ such that $(P(x) \wedge (\forall y \in D$, if $P(y)$ then $y = x))$

Or: There exists a unique x in D such that $P(x)$.

Or: There is one and only one x in D such that $P(x)$.

54. *a.* The statement is false. It says that there are a circle and a triangle that have the same color, which is false because all the triangles are blue, and no circles are blue.

    b. $\exists x(\text{Circle}(x) \wedge (\exists y\ (\text{Triangle}(y) \wedge \text{SameColor}(x, y))))$

    c. $\forall x({\sim}\text{Circle}(x) \vee {\sim}\ (\exists y\ (\text{Triangle}(y) \wedge \text{SameColor}(x, y))))$

    $\equiv \forall x({\sim}\text{Circle}(x) \vee (\forall y\ ({\sim}\text{Triangle}(y) \vee {\sim}\ \text{SameColor}(x, y))))$

57. These statements do not necessarily have the same truth values. For example, let $D = \mathbf{Z}$, the set of all integers, let $P(x)$ be "x is even," and let $Q(x)$ be "x is odd." Then the statement "$\forall x \in D, (P(x) \vee Q(x))$" can be written "$\forall$ integers x, x is even or x is odd," which is true. On the other hand, "$(\forall x \in D, P(x)) \vee (\forall x \in D,\ Q(x))$" can be written "All integers are even or all integers are odd," which is false.

60. *a.* No *b.* No *c.* $X = g$

Section 3.4

6. This computer program is not correct.

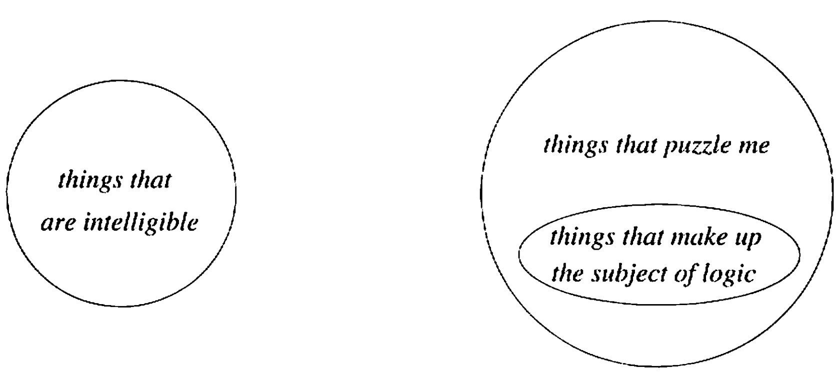
12. invalid, inverse error

15. invalid, converse error

18. valid, universal modus tollens

24. Valid. The only drawing representing the truth of the premises also represents the truth of the conclusion.

27. Valid. The only drawing representing the truth of the premises also represents the truth of the conclusion.



30. 3. If an object is black, then it is a square.

 2. (*contrapositive form*) If an object is a square, then it is above all the gray objects.

 4. If an object is above all the gray objects, then it is above all the triangles.

 1. If an object is above all the triangles, then it is above all the blue objects.

 $\therefore$ If an object is black, then it is above all the blue objects.

36. The universal form of elimination (part a) says that the following form of argument is valid:

$$\begin{array}{lll} & \forall x \text{ in } D, P(x) \vee Q(x). & \leftarrow \textit{major premise} \\ & \sim Q(c) \text{ for a particular } c \text{ in } D. & \leftarrow \textit{minor premise} \\ \therefore & P(c) & \end{array}$$

Proof of Validity:

Suppose the major and minor premises of the above argument form are both true.

[We must show that the conclusion $P(c)$ is also true.]

By definition of truth value for a universal statement, $\forall x$ in D, $P(x) \vee Q(x)$ is true if, and only if, the statement "$P(x) \vee Q(x)$" is true for each individual element of D.

So, by universal instantiation, it is true for the particular element c.

Hence "$P(c) \vee Q(c)$" is true.

And since the minor premise says that $\sim Q(c)$, it follows by the elimination rule that $P(c)$ is true.

[This is what was to be shown.]

Review Guide: Chapter 3

Quantified Statements

- What is a predicate? *(p. 97)*
- What is the truth set of a predicate? *(p. 97)*
- What is a universal statement, and what is required for such a statement to be true? *(p. 98)*
- What is required for a universal statement to be false? *(p. 98)*
- What is the method of exhaustion? *(p. 99)*
- What is an existential statement, and what is required for such a statement to be true? *(p. 99)*
- What is required for a existential statement to be false? *(p. 99)*
- What are some ways to translate quantified statements from formal to informal language? *(p. 100)*
- What are some ways to translate quantified statements from informal to formal language? *(p. 101)*
- What is a universal conditional statement? *(p. 101)*
- What are equivalent ways to write a universal conditional statements? *(pp. 101-103)*
- What are equivalent ways to write existential statements? *(p. 103)*
- What is a trailing quantifier? *(p. 101)*
- What does it mean for a statement to be quantified implicitly? *(p. 103)*
- What do the notations $\Rightarrow$ and $\Leftrightarrow$ mean? *(p. 104)*
- What is the relation among $\forall$, $\exists$, $\wedge$, and $\vee$? *(p. 112)*
- What does it mean for a universal statement to be vacuously true? *(p. 112)*
- What is the rule for interpreting a statement that contains both a universal and an existential quantifier? *(pp. 118-119)*
- How are statements expressed in the computer programming language Prolog? *(pp. 127-128)*

Negations: What are negations for the following forms of statements?

- $\forall x, Q(x)$ *(p. 109)*
- $\exists x$ such that $Q(x)$ *(p. 109)*
- $\forall x$, if $P(x)$ then $Q(x)$ *(p. 111)*
- $\forall x, \exists y$ such that $P(x, y)$ *(p. 123)*
- $\exists x$ such that $\forall y, P(x, y)$ *(p. 123)*

Variants of Conditional Statements

- What are the converse, inverse, and contrapositive of a statement of the form "$\forall x$, if $P(x)$ then $Q(x)$"? *(p. 113)*
- How are quantified statements involving necessary and sufficient conditions and the phrase only-if translated into if-then form? *(pp. 114-115)*

Validity and Invalidity

- What is universal instantiation? *(p. 132)*
- What are the universal versions of modus ponens, modus tollens, converse error, and inverse error, and which of these forms of argument are valid and which are invalid? *(pp. 133-135, 138-139)*
- How is universal modus ponens used in a proof? *(p. 134)*
- How can diagrams be used to test the validity of an argument with quantified statements? *(pp. 136-139)*

Chapter 4: Elementary Number Theory and Methods of Proof

One aim of this chapter is to introduce you to methods for evaluating whether a given mathematical statement is true or false. Throughout the chapter the emphasis is on learning to prove and disprove statements of the form "$\forall x$ in D, if $P(x)$ then $Q(x)$." To prove such a statement directly, you suppose you have a particular but arbitrarily chosen element x in D for which $P(x)$ is true and you show that $Q(x)$ must also be true. To disprove such a statement, you show that there is an element x in D (a counterexample) for which $P(x)$ is true and $Q(x)$ is false. To prove such a statement by contradiction, you show that no counterexample exists, that is, you suppose that there is an x in D for which $P(x)$ is true and $Q(x)$ is false and you show that this supposition leads to a contradiction. Direct proof, disproof by counterexample, and proof by contradiction can, therefore, all be viewed as three aspects of one whole. You arrive at one or the other by a thoughtful examination of the given statement, knowing what it means for a statement of that form to be true or false.

Another aim of the chapter is to help you obtain fundamental knowledge about numbers that is needed in mathematics and computer science. Note that the exercise sets contain problems of varying difficulty. Do not be discouraged if some of them are difficult for you.

Proofs given as solutions should be regarded as samples. Your instructor will probably discuss with you the particular range of proof styles that will be considered acceptable in your course.

Section 4.1

3. *a*. Yes, because

$$4rs = 2\cdot(2rs)$$

and $2rs$ is an integer since r and s are integers and products of integers are integers.

b. Yes, because

$$6r + 4s^2 + 3 = 2(3r + 2s^2 + 1) + 1$$

and $3r + 2s^2 + 1$ is an integer since r and s are integers and products and sums of integers are integers.

c. Yes, because

$$r^2 + 2rs + s^2 = (r + s)^2$$

and $r + s$ is an integer that is greater than or equal to 2 since both r and s are positive integers and thus each is greater than or equal to 1.

6. For example, let $a = 1$ and $b = 0$. Then

$$\sqrt{a+b} = \sqrt{1} = 1$$

and also

$$\sqrt{a} + \sqrt{b} = \sqrt{1} + \sqrt{0} = 1.$$

Hence for these values of a and b,

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}.$$

In fact, if a is any nonzero integer and $b = 0$, then

$$\sqrt{a+b} = \sqrt{a+0} = \sqrt{a} = \sqrt{a} + 0 = \sqrt{a} + \sqrt{0} = \sqrt{a} + \sqrt{b}.$$

12. Counterexample: Let $n = 5$. Then

$$\frac{n-1}{2} = \frac{5-1}{2} = \frac{4}{2} = 2,$$

which is not odd.

15. According to the order of operations for real numbers, $-a^n = -(a^n)$. The following table shows that the property is true for some values of a and n and false for other values.

a	n	$-a^n$	$(-a)^n$	Does $-a^n = (-a)^n$?
0	2	$-0^2 = -0 = 0$	$(-0)^2 = 0^2 = 0$	Yes
3	2	$-3^2 = -(3^2) = -9$	$(-3)^2 = (-3)(-3) = 9$	No
−2	3	$-(-2)^3 = -((-2)^3) = -(-8) = 8$	$(-(-2))^3 = 2^3 = 8$	Yes
−3	2	$-(-3)^2 = -((-3)^2) = -9$	$(-(-3))^3 = 3^2 = 9$	No

18. $1^2 - 1 + 11 = 11$, which is prime. $2^2 - 2 + 11 = 13$, which is prime.
$3^2 - 3 + 11 = 17$, which is prime. $4^2 - 4 + 11 = 23$, which is prime.
$5^2 - 5 + 11 = 31$, which is prime. $6^2 - 6 + 11 = 41$, which is prime.
$7^2 - 7 + 11 = 53$, which is prime. $8^2 - 8 + 11 = 67$, which is prime.
$9^2 - 9 + 11 = 83$, which is prime. $10^2 - 10 + 11 = 101$, which is prime.

21. If a real number is greater than 1, then its square it greater than itself.

Start of Proof: Suppose x is any *[particular but arbitrarily chosen]* real number such that $x > 1$.

Conclusion to be shown: $x^2 > x$.

27. Proof 1:

Suppose m and n are any *[particular but arbitrarily chosen]* odd integers. *[We must show that $m + n$ is even.]*

By definition of odd, there exist integers r and s such that $m = 2r + 1$ and $n = 2s + 1$. Then

$$\begin{aligned} m + n &= (2r+1) + (2s+1) && \text{by substitution} \\ &= 2r + 2s + 2 \\ &= 2(r + s + 1) && \text{by algebra.} \end{aligned}$$

Let $u = r + s + 1$.

Then u is an integer because r, s, and 1 are integers and a sum of integers is an integer.

Hence $m + n = 2u$, where u is an integer, and so by definition of even, $m + n$ is even *[as was to be shown]*.

Proof 2:

Suppose m and n are any *[particular but arbitrarily chosen]* odd integers. *[We must show that $m + n$ is even.]*

By definition of odd, there exist integers r and s such that $m = 2r + 1$ and $n = 2s + 1$. Then

$$\begin{aligned} m + n &= (2r+1) + (2s+1) && \text{by substitution} \\ &= 2r + 2s + 2 \\ &= 2(r + s + 1) && \text{by algebra.} \end{aligned}$$

But $r + s + 1$ is an integer because r, s, and 1 are integers and a sum of integers is an integer.

Hence $m + n$ equals twice an integer, and so by definition of even, $m + n$ is even *[as was to be shown]*.

30. Proof:

Suppose m is any *[particular but arbitrarily chosen]* even integer. *[We must show that $3m + 5$ is odd.]*

By definition of even, $m = 2r$ for some integer r. Then

$$\begin{aligned} 3m+5 &= 3(2r)+5 && \text{by substitution} \\ &= 6r+4+1 \\ &= 2(3r+2)+1 && \text{by algebra.} \end{aligned}$$

Let $t = 3r + 2$.

Then t is an integer because products and sums of integers are integers, and $3m + 5 = 2t + 1$.

Hence $3m + 5$ is odd by definition of odd *[as was to be shown]*.

36. To prove the given statement is false, we prove that its negation is true. The negation of the statement is "For all integers n, $6n^2 + 27$ is not prime."

Proof:

Suppose n is any integer. *[We must show that $6n^2 + 27$ is not prime.]*

Note that $6n^2 + 27$ is positive because $n^2 \geq 0$ for all integers n and products and sums of positive real numbers are positive. Then

$$6n^2 + 27 = 3(2n^2 + 9),$$

and both 3 and $2n^2 + 9$ are positive integers each greater than 1 and less than $6n^2 + 27$.

So $6n^2 + 27$ is not prime.

42. The mistake in the "proof" is that the same symbol, k, is used to represent two different quantities. By setting both m and n equal to $2k$, the "proof" specifies that $m = n$, and, therefore, it only deduces the conclusion in case $m = n$. If $m \neq n$, the conclusion is often false. For instance, $6 + 4 = 10$ but $10 \neq 4k$ for any integer k.

48. The statement is true.

Proof:

Let m and n be any even integers.

By definition of even, $m = 2r$ and $n = 2s$ for some integers r and s.

By substitution,

$$m - n = 2r - 2s = 2(r - s).$$

Since $r - s$ is an integer (being a difference of integers), then

$m - n$ equals twice some integer, and so

$m - n$ is even by definition of even.

51. The statement is false.

Counterexample: Let $n = 2$. Then n is prime but

$$(-1)^n = (-1)^2 = 1 \neq -1.$$

57. The statement is false.

Counterexample: Let $m = n = 3$. Then $mn = 3 \cdot 3 = 9$, which is a perfect square, but neither m nor n is a perfect square.

60. The statement is false.

Counterexample: When $a = 1$ and $b = 1$,

$$\sqrt{a+b} = \sqrt{1+1} = \sqrt{2} \qquad \text{and} \qquad \sqrt{a} + \sqrt{b} = \sqrt{1} + \sqrt{1} = 2.$$

But

$$\sqrt{2} \neq 2, \qquad \text{and so} \qquad \sqrt{a+b} \neq \sqrt{a} + \sqrt{b}.$$

63. Counterexample: Let $n = 5$. Then

$$2^{2^n} + 1 = 2^{32} + 1 = 4,294,967,297 = (641) \cdot (6700417),$$

and so $2^{2^n} + 1$ is not prime.

Section 4.2

18. The statement is true.

Proof: Suppose r and s are any two distinct rational numbers. *[We must show that* $\frac{r+s}{2}$ *is rational.]*

By definition of rational, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$.

By substitution and the laws of algebra,

$$\frac{r+s}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{\frac{ad+bc}{bd}}{2} = \frac{ad+bc}{2bd}.$$

Now $ad + bc$ and $2bd$ are integers because a, b, c, and d are integers and products and sums of integers are integers. And $2bd \neq 0$ by the zero product property.

Hence $\frac{r+s}{2}$ is a quotient of integers with a nonzero denominator, and so $\frac{r+s}{2}$ is rational *[as was to be shown]*.

30. Let the quadratic equation be

$$x^2 + bx + c = 0$$

where b and c are rational numbers. Suppose one solution, r, is rational. Call the other solution s. Then

$$x^2 + bx + c = (x - r)(x - s) = x^2 - (r + s)x + rs.$$

By equating the coefficients of x,

$$b = -(r + s).$$

Solving for s yields

$$s = -r - b = -(r + b).$$

Because s is the negative of a sum of two rational numbers, s also is rational (by Theorem 4.2.2 and exercise 13).

33. *a.* Note that $(x - r)(x - s) = x^2 - (r + s)x + rs$.

If both r and s are odd integers, then $r + s$ is even and rs is odd (by properties 2 and 3).

If both r and s are even integers, then both $r + s$ and rs are even (by property 1).

If one of r and s is even and the other is odd, then $r + s$ is odd and rs is even (by properties 4 and 5).

b. It follows from part (a) that $x^2 - 1253x + 255$ cannot be written as a product of the form $(x - r)(x - s)$ because if it could be then $r + s$ would equal 1253 and rs would equal 255, both of which are odd integers. But for none of the possible cases (both r and s odd, both r and s even, and one of r and s odd and the other even) are both $r + s$ and rs odd integers.

Note: In Section 4.4, we establish formally that any integer is either even or odd. The type of reasoning used in this solution is called argument by contradiction. It is introduced formally in Section 4.6.

36. This incorrect proof just shows the theorem to be true in the one case where one of the rational numbers is 1/4 and the other is 1/2. It is an example of the mistake of arguing from examples. A correct proof must show the theorem is true for *any* two rational numbers.

39. This incorrect proof assumes what is to be proved. The second sentence asserts that a certain conclusion follows if $r + s$ is rational, and the rest of the proof uses that conclusion to deduce that $r + s$ is rational.

Section 4.3

3. Yes: $0 = 0 \cdot 5$.

9. Yes: $2a \cdot 34b = 4(17ab)$ and $17ab$ is an integer because a and b are integers and products of integers are integers.

21. The statement is true.

    Proof:

    Let m and n be any two even integers. *[We must show that mn is a multiple of 4.]*

    By definition of even, $m = 2r$ and $n = 2s$ for some integers r and s. Then

$$\begin{aligned} mn &= (2r)(2s) && \text{by substitution} \\ &= 4(rs) && \text{by algebra.} \end{aligned}$$

    Since rs is an integer (being a product of integers), mn is a multiple of 4 (by definition of divisibility).

27. The statement is false.

    Counterexample:

    Let $a = 2$, $b = 3$, and $c = 1$.

    Then $a \mid (b + c)$ because $b + c = 4$ and $2 \mid 4$ but $a \nmid b$ because $2 \nmid 3$ and $a \nmid c$ because $2 \nmid 1$.

30. The statement is false.

    Counterexample:

    Let $a = 4$ and $n = 6$.

    Then $a \mid n^2$ and $a \leq n$ because $4 \mid 36$ and $4 \leq 6$, but $a \nmid n$ because $4 \nmid 6$.

33. No. The values of nickels, dimes, and quarters are all multiples of 5. It follows from exercise 15 that a sum of numbers each of which is divisible by 5 is also divisible by 5. So since $4.72 is not a multiple of 5, $4.72 cannot be obtained using only nickels, dimes, and quarters.

    Note: The form of reasoning used in this answer is called argument by contradiction. It is discussed formally in Section 4.6.

36. *b*. Let $N = 12,858,306,120,312$.

    The sum of the digits of N is 42, which is divisible by 3 but not by 9. Therefore, N is divisible by 3 but not by 9.

    The right-most digit of N is neither 5 nor 0, and so N is not divisible by 5.

    The two right-most digits of N are 12, which is divisible by 4. Therefore, N is divisible by 4.

    c. Let $N = 517,924,440,926,512$.

    The sum of the digits of N is 61, which is not divisible by 3 (and hence not by 9 either). Therefore, N is not divisible by either 3 or 9.

The right-most digit of N is neither 5 nor 0, and so N is not divisible by 5.

The two right-most digits of N are 12, which is divisible by 4. Therefore, N is divisible by 4.

d. Let $N = 14,328,083,360,232$.

The sum of the digits of N is 45, which is divisible by 9 and hence also by 3. Therefore, N is divisible by 9 and by 3.

The right-most digit of N is neither 5 nor 0, and so N is not divisible by 5.

The two right-most digits of N are 32, which is divisible by 4. Therefore, N is divisible by 4.

39. *a.* $a^3 = p_1^{3e_1} \cdot p_2^{3e_2} \cdots p_k^{3e_k}$

b. To solve the problem, find the smallest powers of the factors 2, 3, 7, and 11 by which to multiply the given number so that the exponent of each factor becomes a multiple of 3. The value of k that works is $k = 2^2 \cdot 3 \cdot 7^2 \cdot 11$. Then

$$2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k = 2^6 \cdot 3^6 \cdot 7^3 \cdot 11^3 = (2^2 \cdot 3^2 \cdot 7 \cdot 11)^3 = 2772^3.$$

42. *b.*

$$\begin{aligned} 20! &= 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 2^2 \cdot 5 \cdot 19 \cdot 2 \cdot 3^2 \cdot 17 \cdot 2^4 \cdot 3 \cdot 5 \cdot 2 \cdot 7 \cdot 13 \cdot 2^2 \cdot 3 \cdot 11 \cdot 2 \cdot 5 \cdot 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \\ &= 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \end{aligned}$$

c. Squaring the result of part (b) gives

$$\begin{aligned} (20!)^2 &= (2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19)^2 \\ &= 2^{36} \cdot 3^{16} \cdot 5^8 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \end{aligned}$$

When $(20!)^2$ is written in ordinary decimal form, there are as many zeros at the end of it as there are factors of the form $2 \cdot 5$ $(= 10)$ in its prime factorization.

Thus, since the prime factorization of $(20!)^2$ contains eight 5's and more than eight 2's, $(20!)^2$ contains eight factors of 10 and hence eight zeros.

45. Proof:

Suppose n is a nonnegative integer whose decimal representation ends in 5.

Then $n = 10m + 5$ for some integer m.

By factoring out a 5, $n = 10m + 5 = 5(2m + 1)$, and $2m + 1$ is an integer since m is an integer.

Hence n is divisible by 5 by definition of divisibility.

48. Proof:

Suppose n is any nonnegative integer for which the sum of the digits of n is divisible by 3.

By definition of decimal representation, n can be written in the form

$$n = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_2 10^2 + d_1 10 + d_0$$

where k is a nonnegative integer and all the d_i are integers from 0 to 9 inclusive. Then

$$\begin{aligned} n &= d_k(\underbrace{99\ldots9}_{k\ 9\text{'s}} + 1) + d_{k-1}(\underbrace{99\ldots9}_{(k-1)\ 9\text{'s}} + 1) + \cdots + d_2(99+1) + d_1(9+1) + d_0 \\ &= d_k \cdot \underbrace{99\ldots9}_{k\ 9\text{'s}} + d_{k-1} \cdot \underbrace{99\ldots9}_{(k-1)\ 9\text{'s}} + \cdots + d_2 \cdot 99 + d_1 \cdot 9 + (d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0) \\ &= 9(d_k \cdot \underbrace{11\ldots1}_{k\ 1\text{'s}} + d_{k-1} \cdot \underbrace{11\ldots1}_{(k-1)\ 1\text{'s}} + \cdots + d_2 \cdot 11 + d_1) + (d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0) \\ &= 3[3(d_k \cdot \underbrace{11\ldots1}_{k\ 1\text{'s}} + d_{k-1} \cdot \underbrace{11\ldots1}_{(k-1)\ 1\text{'s}} + \cdots + d_2 \cdot 11 + d_1)] + (d_k + d_{k-1} + \cdots + d_2 + d_1 + d_0) \\ &= (\text{an integer divisible by 3}) + (\text{the sum of the digits of } n). \end{aligned}$$

Since the sum of the digits of n is divisible by 3, n can be written as a sum of two integers each of which is divisible by 3.

It follows from exercise 15 that n is divisible by 3.

Section 4.4

6. $q = -4$, $r = 5$

9. *a.* 5 *b.* 3

12. Let the days of the week be numbered from 0 (Sunday) through 6 (Saturday) and let $DayT$ and $DayN$ be variables representing the day of the week today and the day of the week N days from today. By the quotient-remainder theorem, there exist unique integers q and r such that

$$DayT + N = 7q + r \text{ and } 0 \le r < 7.$$

Now $DayT + N$ counts the number of days to the day N days from today starting last Sunday (where "last Sunday" is interpreted to mean today if today is a Sunday). Thus $DayN$ is the day of the week that is $DayT + N$ days from last Sunday. Because each week has seven days,

$DayN$ is the same as the day of the week $DayT + N - 7q$ days from last Sunday.

But

$$DayT + N - 7q = r \quad \text{and} \quad 0 \le r < 7.$$

Therefore,

$$DayN = r = (DayT + N) \ mod \ 7.$$

15. There are 13 leap year days between January 1, 2000 and January 1, 2050 (once every four years in 2000, 2004, 2008, 2012, . . . , 2048). So 13 of the years have 366 days and the remaining 38 years have 365 days. This gives a total of

$$13 \cdot 366 + 37 \cdot 365 = 18,263$$

days between the two dates. Using the formula $DayN = (DayT + N) \ mod \ 7$, and letting $DayT = 6$ (Saturday) and $N = 18,263$ gives

$$DayN = (6 + 18263) mod \ 7 = 18269 mod \ 7 = 6,$$

which is also a Saturday.

21. Given that b is an integer and $b \ mod \ 12 = 5$, it follows that $8b \ mod \ 12 = 4$.

<u>Proof</u>: When b is divided by 12, the remainder is 5. Thus there exists an integer m so that $b = 12m + 5$. Multiplying this equation by 8 gives

$$\begin{aligned} 8b &= 8(12m+5) && \text{by substitution} \\ &= 96m + 40 \\ &= 96m + 36 + 4 \\ &= 12(8m+3) + 4 && \text{by algebra.} \end{aligned}$$

Since $8m + 3$ is an integer and since $0 \le 4 < 12$, the uniqueness part of the quotient-remainder theorem guarantees that the remainder obtained when $8b$ is divided by 12 is 4.

24. <u>Proof</u>:

Suppose m and n are any *[particular but arbitrarily chosen]* integers such that $m \ mod \ 5 = 2$ and $n \ mod \ 5 = 1$.

Then the remainder obtained when m is divided by 5 is 2 and the remainder obtained when n is divided by 5 is 1, and so $m = 5q + 2$ and $n = 5r + 1$ for some integers q and r.

$$\begin{aligned} mn &= (5q+2)(5r+1) && \text{by substitution} \\ &= 25qr + 5q + 10r + 2 \\ &= 5(5qr + q + 2r) + 2 && \text{by algebra.} \end{aligned}$$

Because products and sums of integers are integers, $5qr + q + 2r$ is an integer, and hence $mn = 5\cdot$ (an integer) $+ 2$.

Thus, since $0 \le 2 < 5$, the remainder obtained when mn is divided by 5 is 2, and so $mn \bmod 5 = 2$.

30. *a.* <u>Proof</u>: Suppose n and $n + 1$ are any two consecutive integers. By the quotient-remainder theorem with $d = 3$, we know that $n = 3q$, or $n = 3q + 1$, or $n = 3q + 2$ for some integer q.

Case 1 (n = 3q for some integer q): In this case,

$$\begin{aligned} n(n+1) &= 3q(3q+1) && \text{by substitution} \\ &= 3[q(3q+1)] && \text{by algebra.} \end{aligned}$$

Let $k = q(3q + 1)$. Then k is an integer because sums and products of integers are integers. Hence $n(n + 1) = 3k$ for some integer k.

Case 2 (n = 3q + 1 for some integer q): In this case,

$$\begin{aligned} n(n+1) &= (3q+1)(3q+2) && \text{by substitution} \\ &= 9q^2 + 9q + 2 \\ &= 3(3q^2 + 3q) + 2 && \text{by algebra.} \end{aligned}$$

Let $k = 3q^2 + 3q$. Then k is an integer because sums and products of integers are integers. Hence $n(n + 1) = 3k + 2$ for some integer k.

Case 3 (n = 3q for some integer q): In this case,

$$\begin{aligned} n(n+1) &= (3q+2)(3q+3) && \text{by substitution} \\ &= 3[(3q+2)(q+1)] \\ &= 3[(3q+2)(q+1)] && \text{by algebra.} \end{aligned}$$

Let $k = (3q + 2)(q + 1)$. Then k is an integer because sums and products of integers are integers. Hence $n(n + 1) = 3k$ for some integer k.

Conclusion: In all three cases, the product of the two consecutive integers either equals $3k$ or it equals $3k + 2$ for some integer k *[as was to be shown]*.

b. Given any integers m and n, $mn \bmod 3 = 0$ or $mn \bmod 3 = 2$.

(Or, equivalently, Given any integer n, $mn \bmod 3 \neq 1$.)

33. Given any integers a, b, and c if $a - b$ is odd and $b - c$ is even, then $a - c$ is odd.

<u>Proof</u>: Suppose a, b, and c are any integers such that $a - b$ is odd and $b - c$ is even. Then $(a - b) + (b - c)$ is a sum of an odd integer and an even integer and hence is odd (by property 5 in Example 4.2.3). But $(a - b) + (b - c) = a - c$, and thus $a - c$ is odd.

36. <u>Proof</u>:

Suppose n is any integer. *[We must show that* $8 \mid n(n + 1)(n + 2)(n + 3)$.*]*

By the quotient-remainder theorem with $d = 4$, there is an integer k such that

$$n = 4k \quad \text{or} \quad n = 4k+1 \quad \text{or} \quad n = 4k+2 \quad \text{or} \quad n = 4k+3.$$

Case 1 (n = 4k for some integer k): In this case,

$$\begin{aligned} n(n+1)(n+2)(n+3) &= 4k(4k+1)(4k+2)(4k+3) && \text{by substitution} \\ &= 8[k(4k+1)(2k+1)(4k+3)] && \text{by algebra,} \end{aligned}$$

and this is divisible by 8 (because k is an integer and sums and products of integers are integers).

Case 2 (n = 4k + 1 for some integer k): In this case,

$$\begin{aligned} n(n+1)(n+2)(n+3) &= (4k+1)(4k+2)(4k+3)(4k+4) && \text{by substitution} \\ &= 8[(4k+1)(2k+1)(4k+3)(k+1)] && \text{by algebra,} \end{aligned}$$

and this is divisible by 8 (because k is an integer and sums and products of integers are integers).

Case 3 (n = 4k + 2 for some integer k): In this case,

$$\begin{aligned} n(n+1)(n+2)(n+3) &= (4k+2)(4k+3)(4k+4)(4k+5) && \text{by substitution} \\ &= 8[(2k+1)(4k+3)(k+1)(4k+5)] && \text{by algebra,} \end{aligned}$$

and this is divisible by 8 (because k is an integer and sums and products of integers are integers).

Case 4 (n = 4k + 3 for some integer k): In this case,

$$\begin{aligned} n(n+1)(n+2)(n+3) &= (4k+3)(4k+4)(4k+5)(4k+6) && \text{by substitution} \\ &= 8[(4k+3)(k+1)(4k+5)(2k+3)] && \text{by algebra,} \end{aligned}$$

and this is divisible by 8 (because k is an integer and sums and products of integers are integers).

Conclusion: In all four possible cases, $8 \mid n(n+1)(n+2)(n+3)$ *[as was to be shown]*.

Note: One can make use of exercise 17 to produce a proof that only requires two cases: n is even and n is odd. Then, since both $n(n+1)$ and $(n+2)(n+3)$ are products of consecutive integers, by the result of exercise 17, both products are even and hence contain a factor of 2. Multiplying those two factors shows that there is a factor of 4 in $n(n+1)(n+2)(n+3)$.

39. Proof: Consider any four consecutive integers. Call the smallest n. Then the sum of the four integers is

$$n + (n+1) + (n+2) + (n+3) = 4n + 6 = 4(n+1) + 2.$$

Let $k = n + 1$. Then k is an integer because it is a sum of integers. Hence n can be written in the required form.

42. Proof: Let p be any prime number except 2 or 3. By the quotient-remainder theorem, there is an integer k so that p can be written as

$$p = 6k \quad \text{or} \quad p = 6k+1 \quad \text{or} \quad p = 6k+2 \quad \text{or} \quad p = 6k+3 \quad \text{or} \quad p = 6k+4 \quad \text{or} \quad p = 6k+5.$$

Since p is prime and $p \neq 2$, p is not divisible by 2. Consequently, $p \neq 6k$, $p \neq 6k+2$, and $p \neq 6k+4$ for any integer k *[because all of these numbers are divisible by 2]*.

Furthermore, since p is prime and $p \neq 3$, p is not divisible by 3. Thus $p \neq 6k+3$ *[because this number is divisible by 3]*.

Therefore, $p = 6k+1$ or $p = 6k+5$ for some integer k.

45. Proof: Let c be any positive real number and let r be any real number. Suppose that $-c \leq r \leq c$.(*) By the trichotomy law (see Appendix A, T17), either $r \geq 0$ or $r < 0$.

Case 1 ($r \geq 0$): In this case $|r| = r$, and so by substitution into (*), $-c \leq |r| \leq c$. In particular, $|r| \leq c$.

Case 2 ($r < 0$): In this case $|r| = -r$, and so $r = -|r|$. Hence by substitution into (*), $-c \leq -|r| \leq c$. In particular, $-c \leq -|r|$. Multiplying both sides by -1 gives $c \geq |r|$, or, equivalently, $|r| \leq c$.

Therefore, regardless of whether $r \geq 0$ or $r < 0$, $|r| \leq c$ *[as was to be shown]*.

48. *Solution 1*: We are given that M is a matrix with m rows and n columns, stored in row major form at locations $N + k$, where $0 \leq k < mn$. Given a value for k, we want to find indices r and s so that the entry for M in row r and column s, a_{rs}, is stored in location $N + k$. By the quotient-remainder theorem, $k = nQ + R$, where Q and R are integers and $0 \leq R < n$. The first Q rows of M (each of length n) are stored in the first nQ locations: $N + 0, N + 1, \ldots, N + nQ - 1$ with a_{Qn} stored in the last of these. Consider the next row. When $r = Q + 1$,

$$\begin{array}{ll} & a_{r1} \text{ will be in location } N + nQ \\ & a_{r2} \text{ will be in location } N + nQ + 1 \\ & a_{r3} \text{ will be in location } N + nQ + 2 \\ & \vdots \\ & a_{rs} \text{ will be in location } N + nQ + (s-1) \\ & \vdots \\ \text{and} & a_{rn} \text{ will be in location } N + nQ + (n-1) = N + n(Q+1) - 1. \end{array}$$

Thus location $N + k$ contains a_{rs} where $r = Q + 1$ and $R = s - 1$. But $Q = k \; div \; n$ and $R = k \; mod \; n$, and hence $r = (k \; div \; n) + 1$ and $s = (k \; mod \; n) + 1$.

Solution 2: *[After the floor notation has been introduced, the following solution can be considered as an alternative.]* To find a formula for r, note that for $1 \leq a \leq m$,

$$\text{when} \quad (a-1)n \leq k < an, \quad \text{then} \quad r = a.$$

Dividing through by n gives that

$$\text{when} \quad (a-1) \leq \frac{k}{n} < a, \quad \text{then} \quad r = a$$

or, equivalently,

$$\text{when} \quad \left\lfloor \frac{k}{n} \right\rfloor = a - 1, \quad \text{then} \quad r = a.$$

But this implies that

$$\text{when} \quad a = \left\lfloor \frac{k}{n} \right\rfloor + 1 = r, \quad \text{then} \quad r = a.$$

and since

$$\left\lfloor \frac{k}{n} \right\rfloor = k \; div \; n, \quad \text{we have that} \quad r = (k \; div \; n) + 1.$$

To find a formula for s, note that

$$\text{when} \quad k = n \cdot (\text{an integer}) + b \quad \text{and} \quad 0 \leq b < n, \quad \text{then} \quad s = b + 1.$$

Thus by the quotient-remainder theorem, $s = (k \; mod \; n) + 1$.

51. Answer to the first question: not necessarily

Counterexample: Let $m = n = 3$, $d = 2$, $a = 1$, and $b = 1$. Then

$$m \bmod d = n \bmod d = 3 \bmod 2 = 1 = a = b.$$

But $a + b = 1 + 1 = 2$, whereas $(m + n) \bmod d = 6 \bmod 2 = 0$.

Answer to the second question: yes.

Proof: Suppose m, n, a, b, and d are integers and $m \bmod d = a$ and $n \bmod d = b$. By definition of *mod*, $m = dq_1 + a$ and $n = dq_2 + b$ for some integers q_1 and q_2. Then

$$\begin{aligned} m + n &= (dq_1 + a) + (dq_2 + b) && \text{by substitution} \\ &= d(q_1 + q_2) + (a + b) \;(*) && \text{by algebra.} \end{aligned}$$

Apply the quotient-remainder theorem to $a + b$ to obtain unique integers q_3 and r such that $a + b = dq_3 + r$ (**) and $0 \le r < d$. By definition of *mod*, $r = (a + b) \bmod d$. Then

$$\begin{aligned} m + n &= d(q_1 + q_2) + (a + b) && \text{by } (*) \\ &= d(q_1 + q_2) + (dq_3 + r) && \text{by } (**) \\ &= d(q_1 + q_2 + q_3) + r && \text{by algebra,} \end{aligned}$$

where $q_1 + q_2 + q_3$ and r are integers and $0 \le r < d$. Hence by definition of *mod*,

$$r = (m + n) \bmod d, \text{ and so } (m + n) \bmod d = (a + b) \bmod d.$$

Section 4.5

6. If k is an integer, then $\lceil k \rceil = k$ because $k - 1 < k \le k$ and $k - 1$ and k are integers.

9. If the remainder obtained when n is divided by 36 is positive, an additional box beyond those containing exactly 36 units will be needed to hold the extra units.

   So, since the ceiling notation rounds each number up to the nearest integer, the number of boxes required will be $\lceil \frac{n}{36} \rceil$.

   Also, because the ceiling of an integer is itself, if the number of units is a multiple of 36, the number of boxes required will be $\lceil \frac{n}{36} \rceil$ as well.

   Thus the ceiling notation is more appropriate for this problem because the answer is simply $\lceil \frac{n}{36} \rceil$ regardless of the value of n.

   If the floor notation is used, the answer is more complicated: if $\frac{n}{36}$ is not an integer, the answer is $\lfloor \frac{n}{36} \rfloor + 1$, but if n is an integer, it is $\lfloor \frac{n}{36} \rfloor$.

18. Counterexample: Let $x = y = 1.5$. Then

$$\lceil x + y \rceil = \lceil 1.5 + 1.5 \rceil = \lceil 3 \rceil = 3 \quad \text{whereas} \quad \lceil x \rceil + \lceil y \rceil = \lceil 1.5 \rceil + \lceil 1.5 \rceil = 2 + 2 = 4,$$

   and $3 \ne 4$.

21. Proof: Let n be any odd integer. *[We must show that* $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$*.]* By definition of odd, $n = 2k + 1$ for some integer k. The left-hand side of the equation to be proved is

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil &= \left\lceil \frac{2k+1}{2} \right\rceil && \text{by substitution} \\ &= \left\lceil k + \frac{1}{2} \right\rceil && \text{by algebra} \\ &= k + 1 && \text{by definition of ceiling because } k \text{ is an integer and } k < k + 1/2 \le k + 1 \end{aligned}$$

On the other hand, the right-hand side of the equation to be proved is

$$\begin{aligned} \frac{n+1}{2} &= \frac{(2k+1)+1}{2} && \text{by substitution} \\ &= \frac{2k+2}{2} \\ &= \frac{2(k+1)}{2} \\ &= k+1 && \text{by algebra.} \end{aligned}$$

Thus both the left- and right-hand sides of the equation to be proved equal $k+1$, and so both are equal to each other. In other words, $\lceil n/2 \rceil = (n+1)/2$ *[as was to be shown]*.

24. Proof:

Suppose m is any integer and x is any real number that is not an integer.

By definition of floor, $\lfloor x \rfloor = n$ where n is an integer and $n \le x < n+1$.

Since x is not an integer, $x \neq n$, and so

$$n < x < n+1.$$

Multiply all parts of this inequality by -1 to obtain

$$-n > -x > -n-1.$$

Then add m to all parts to obtain

$$m-n > m-x > m-n-1, \quad \text{or, equivalently,} \quad m-n-1 < m-x < m-n.$$

But $m-n-1$ and $m-n$ are both integers, and so by definition of floor, $\lfloor m-x \rfloor = m-n-1$. By substitution,

$$\lfloor x \rfloor + \lfloor m-x \rfloor = n + (m-n-1) = m-1$$

[as was to be shown].

27. Proof: Suppose x is any real number such that

$$x - \lfloor x \rfloor \ge 1/2.$$

Multiply both sides by 2 to obtain

$$2x - 2\lfloor x \rfloor \ge 1 \quad \text{or, equivalently,} \quad 2x \ge 2\lfloor x \rfloor + 1.$$

Now by definition of floor,

$$x < \lfloor x \rfloor + 1 \quad \text{and hence} \quad 2x < 2\lfloor x \rfloor + 2.$$

Put the two inequalities involving x together to obtain

$$2\lfloor x \rfloor + 1 \le 2x < 2\lfloor x \rfloor + 2.$$

By definition of floor, then, $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$.

Section 4.6

6. *Negation of Statement*: There is a greatest negative real number.

Proof of statement (by contradiction):

Suppose not. That is, suppose there is a greatest negative real number. Call it a. *[We must show that this supposition leads logically to a contradiction.]*

By supposition,

$$a < 0 \quad \text{and} \quad a \geq x \quad \text{f or every negative real number } x.$$

Let $b = a/2$. Then b is a real number because b is a quotient of two real numbers (with a nonzero denominator). Also

$$\text{because} \quad 0 < \frac{1}{2} < 1 \quad \text{then} \quad 0 > \frac{a}{2} > a$$

by multiplying all parts of the inequality by a, which is negative. Thus

$$a < \frac{a}{2} < 0 \quad \text{and so, by substitution,} \quad a < b < 0.$$

Thus b is a negative real number that is greater than a. This contradicts the supposition that a is the greatest negative real number.

[Hence the supposition is false and the given statement is true.]

9. *b.* Proof by contradiction:

Suppose not. That is, suppose there is an irrational number and a rational number whose difference is rational. In other words, suppose there are real numbers x and y such that x is irrational, y is rational and $x - y$ is rational. *[We must show that this supposition leads logically to a contradiction.]*

By definition of rational,

$$y = \frac{a}{b} \quad \text{and} \quad x - y = \frac{c}{d} \quad \text{for some integers } a, b, c, \text{ and } d \text{ with } b \neq 0 \text{ and } d \neq 0.$$

Then, by substitution,

$$x - \frac{a}{b} = \frac{c}{d}.$$

Solve this equation for x to obtain

$$x = \frac{c}{d} + \frac{a}{b} = \frac{bc}{bd} + \frac{ad}{bd} = \frac{bc + ad}{bd}.$$

But both $bc + ad$ and bd are integers because products and sums of integers are integers, and $bd \neq 0$ by the zero product property.

Hence x is a ratio of integers with a nonzero denominator, and so x is rational by definition of rational.

This contradicts the supposition that x is irrational. *[Hence the supposition is false and the given statement is true.]*

Note: The fact that order matters in subtraction implies that the truth of the statement in exercise 8 does not automatically imply the truth of the statement in this exercise.

12. Proof by contradiction:

Suppose not. That is, suppose there are rational numbers a and b such that $b \neq 0$, r is an irrational number, and $a + br$ is rational. *[We must show that this supposition leads logically to a contradiction.]*

By definition of rational,

$$a = \frac{i}{j}, \quad b = \frac{k}{l} \quad \text{and} \quad a + br = \frac{m}{n}$$

where i, j, k, l, m, and n are integers and $j \neq 0$, $l \neq 0$, and $n \neq 0$. Since $b \neq 0$, we also have that $k \neq 0$. By substitution

$$a + br = \frac{i}{j} + \frac{k}{l} \cdot r = \frac{m}{n}, \quad \text{or, equivalently,} \quad \frac{k}{l} \cdot r = \frac{m}{n} - \frac{i}{j}$$

Solving for r gives

$$r = \frac{mj - in}{nj} \cdot \frac{l}{k} = \frac{mjl - inl}{njk}.$$

Now $mjl - inl$ and njk are both integers *[because products and differences of integers are integers]* and $njk \neq 0$ because $n \neq 0, j \neq 0$, and $k \neq 0$.

It follows, by definition of rational, that r is a rational number, which contradicts the supposition that r is irrational. *[Hence the supposition is false and the given statement is true.]*

15. Yes.

Proof by contradiction:

Suppose not. That is, suppose there exist integers a, b, and c such that a and b are both odd and $a^2 + b^2 = c^2$. *[We must show that this supposition leads logically to a contradiction.]*

By definition of odd, $a = 2k + 1$ and $b = 2m + 1$ for some integers k and m. Then,

$$\begin{aligned} c^2 &= a^2 + b^2 && \text{by supposition} \\ &= (2k+1)^2 + (2m+1)^2 && \text{by substitution} \\ &= 4k^2 + 4k + 1 + 4m^2 + 4m + 1 && \\ &= 4(k^2 + k + m^2 + m) + 2 && \text{by algebra.} \end{aligned}$$

Let $t = k^2 + k + m^2 + m$. Then t is an integer because products and sums of integers are integers, and so, by substitution and algebra,

$$c^2 = 4t + 2 = 2(2t + 1).$$

Thus c^2 is even by definition of even, and hence c is even by Proposition 4.6.4. It follows by definition of even again that $c = 2r$ for some integer r. Substituting $c = 2r$ into the equation $c^2 = 4t + 2$ gives

$$(2r)^2 = 4t + 2, \quad \text{and so} \quad 4r^2 = 4t + 2.$$

Dividing by 2 gives

$$2r^2 = 2t + 1.$$

But since r^2 is an integer, $2r^2$ is even, by definition of even, and hence $2t + 1$ is even. However, since t is an integer, $2t + 1$ is odd by definition of odd. So $2t + 1$ is both even and odd, which contradicts Theorem 4.6.2 . It follows that the supposition that there exist integers a, b, and c such that a and b are both odd and $a^2 + b^2 = c^2$ is false, and thus the given statement is true.

24. a. *Proof by contraposition*:

Suppose x is a nonzero real number and the reciprocal of x, namely $\frac{1}{x}$, is rational. *[We must show that x is rational.]*

Because $\frac{1}{x}$ is rational, there are integers a and b with $b \neq 0$ such that

$$\frac{1}{x} = \frac{a}{b} \ (*).$$

Now

$$\text{since } 1 \cdot (\frac{1}{x}) = 1, \quad \text{it follows that} \quad \frac{1}{x} \quad \text{cannot equal zero, and so} \quad a \neq 0.$$

Thus we may solve equation (*) for x to obtain

$$x = \frac{b}{a} \quad \text{where } b \text{ and } a \text{ are integers and } a \neq 0.$$

Hence, by definition of rational, $\frac{1}{x}$ is rational *[as was to be shown]*.

b. Proof by contradiction:

Suppose not. That is, suppose there exists a nonzero irrational number x such that the reciprocal of x, namely $\frac{1}{x}$, is rational. *[We must show that this supposition leads logically to a contradiction.]*

By definition of rational,

$$\frac{1}{x} = \frac{a}{b} \ (*).$$

where a and b are integers with $b \neq 0$. Now

$$\text{since } 1 \cdot (\frac{1}{x}) = 1, \quad \text{it follows that } \frac{1}{x} \text{ cannot equal zero, and so } a \neq 0.$$

Thus we may solve equation (*) for x to obtain

$$x = \frac{b}{a} \quad \text{where } b \text{ and } a \text{ are integers and } a \neq 0.$$

Hence, by definition of rational, $\frac{1}{x}$ is rational, which contradicts the supposition that x is irrational. *[Hence the supposition is false and the given statement is true.]*

27. *a.* Proof 1 by contraposition: *[This proof is based only on the definitions of even and odd integers and the parity property.]*

Suppose m and n are integers such that one of m and n is even and the other is odd. *[We must show that $m + n$ is not even.]*

Case 1 (m is even and n is odd): In this case there exists integers r and s such that $m = 2r$ and $n = 2s + 1$. Then

$$\begin{aligned} m + n &= 2r + (2s + 1) && \text{by substitution} \\ &= 2(r + s) + 1 && \text{by algebra.} \end{aligned}$$

Let $k = r + s$. Then k is an integer because it is a product of integers, and thus $m + n$ is odd by definition of odd.

Case 2 (m is odd and n is even): In this case, by Theorem 4.4.3, there is an integer m such that $n^2 = 8m + 1$. Then

$$\begin{aligned} m + n &= (2r + 1) + 2s && \text{by substitution} \\ &= 2(r + s) + 1 && \text{by algebra.} \end{aligned}$$

Let $k = r + s$. Then k is an integer because it is a product of integers, and thus $m + n$ is odd by definition of odd.

Conclusion: In both cases, $m + n$ is odd, and so by the parity property, $m + n$ is not even *[as was to be shown]*.

Proof 2 by contraposition: *[This proof uses a previously established property of even and odd integers.]*

Suppose m and n are integers such that one of m and n is even and the other is odd. *[We must show that $m+n$ is not even.]*

By property 5 of Example 4.2.3, the sum of any even integer and any odd integer is odd. Hence $m+n$ is odd, and so, by the parity property, $m+n$ is not even *[as was to be shown]*.

b. Proof 1 by contradiction: *[This proof is based only on the definitions of even and odd integers and the parity property.]*

Suppose not. That is, suppose there exist integers m and n such that $m+n$ is even and either m is even and n is odd or m is odd and n is even. *[We must show that this supposition leads logically to a contradiction.]*

Case 1 (m is even and n is odd): In this case there exists integers r and s such that $m = 2r$ and $n = 2s+1$. Then

$$\begin{aligned} m+n &= 2r+(2s+1) && \text{by substitution} \\ &= 2(r+s)+1 && \text{by algebra.} \end{aligned}$$

Let $k = r+s$. Then k is an integer because it is a product of integers, and thus $m+n$ is odd by definition of odd.

Case 2 (m is odd and n is even): In this case, by Theorem 4.4.3, there is an integer m such that $n^2 = 8m+1$. Then

$$\begin{aligned} m+n &= (2r+1)+2s && \text{by substitution} \\ &= 2(r+s)+1 && \text{by algebra.} \end{aligned}$$

Let $k = r+s$. Then k is an integer because it is a product of integers, and thus $m+n$ is odd by definition of odd.

Conclusion: In both cases, $m+n$ is odd, whereas, by supposition, $m+n$ is even. This result contradicts the parity property, which says that an integer cannot be both even and odd. Thus the supposition is false, and the given statement is true.

Proof 2 by contradiction: *[This proof uses a previously established property of even and odd integers.]*

Suppose not. That is, suppose there exist integers m and n such that $m+n$ is even and either m is even and n is odd or m is odd and n is even. *[We must show that this supposition leads logically to a contradiction.]*

By property 5 of Example 4.2.3, the sum of any even integer and any odd integer is odd. Thus both when m is even and n is odd and when m is odd and n is even, the sum $m+n$ is odd. But, by supposition, $m+n$ is even. This result contradicts the parity property, which says that an integer cannot be both even and odd. Thus the supposition is false, and the given statement is true.

33. After crossing out all multiples of 2, 3, 5, and 7 (the prime numbers less than $\sqrt{100}$), the remaining numbers are prime. They are circled in the following diagram.

(2)	(3)	~~4~~	(5)	~~6~~	(7)	~~8~~	~~9~~	~~10~~	(11)	~~12~~	(13)	~~14~~	~~15~~
~~16~~	(17)	~~18~~	(19)	~~20~~	~~21~~	~~22~~	(23)	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	(29)
~~30~~	(31)	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	(37)	~~38~~	~~39~~	~~40~~	(41)	~~42~~	(43)
~~44~~	~~45~~	~~46~~	(47)	~~48~~	~~49~~	~~50~~	~~51~~	~~52~~	(53)	~~54~~	~~55~~	~~56~~	~~57~~
~~58~~	(59)	~~60~~	(61)	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	(67)	~~68~~	~~69~~	~~70~~	(71)
~~72~~	(73)	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	(79)	~~80~~	~~81~~	~~82~~	(83)	~~84~~	~~85~~
~~86~~	~~87~~	~~88~~	(89)	~~90~~	~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	(97)	~~98~~	~~99~~

Section 4.7

6. False.

   Proof 1 (by using a previous result): $\sqrt{2}/6 = (1/6)\cdot\sqrt{2}$, which is a product of a nonzero rational number and an irrational number. By exercise 11 of Section 4.6, such a product is irrational.

   Proof 2 (by contradiction):

   Suppose not. That is, suppose $\sqrt{2}/6$ is rational. *[We must show that this supposition leads logically to a contradiction.]*

   By definition of rational, there exist integers a and b with

   $$\sqrt{2}/6 = a/b \quad \text{and} \quad b \neq 0.$$

   Solving for $\sqrt{2}$ gives $\sqrt{2} = 6a/b$. But $6a$ is an integer (because products of integers are integers) and b is a nonzero integer.

   Therefore, by definition of rational, $\sqrt{2}$ is rational. This contradicts Theorem 4.7.1 which states that $\sqrt{2}$ is irrational. Hence the supposition is false. In other words $\sqrt{2}/6$ is irrational.

12. Counterexample: $\sqrt{2}$ is irrational. Also $\sqrt{2}\cdot\sqrt{2} = 2$ and 2 is rational because $2 = 2/1$. Thus there exist irrational numbers whose product is rational.

15. *a.* Proof by contraposition:

   Let n be any integer such that n is not even. *[We must show that n^3 is not even.]*

   By Theorem 4.6.2 n is odd, and so n^3 is also odd (by property 3 of Example 4.2.3 applied twice).

   Thus (again by Theorem 4.6.2), n^3 is not even *[as was to be shown]*.

   b. Proof by contradiction:

   Suppose not. That is, suppose $\sqrt[3]{2}$ is rational. *[We must show that this supposition leads logically to a contradiction.]*

   By definition of rational,

   $$\sqrt[3]{2} = a/b \text{ for some integers } a \text{ and } b \text{ with } b \neq 0.$$

   By cancelling any common factors if necessary, we may assume that a and b have no common factors. Cubing both sides of equation $\sqrt[3]{2} = a/b$ gives

   $$2 = a^3/b^3, \quad \text{and so} \quad 2b^3 = a^3.$$

Thus a^3 is even. By part (a) of this question, a is even, and thus $a = 2k$ for some integer k. By substitution

$$a^3 = (2k)^3 = 8k^3 = 2b^3,$$

and so

$$b^3 = 4k^3 = 2(2k^3).$$

It follows that b^3 is even, and hence (also by part (a)) b is even.

Thus both a and b are even which contradicts the assumption that a and b have no common factor. Therefore, the supposition is false, and so $\sqrt[3]{2}$ is irrational.

18. Proof :

Suppose that a and d are integers with $d > 0$ and that q_1, q_2, r_1, and r_2 are integers such that

$$a = dq_1 + r_1 \quad \text{and} \quad a = dq_2 + r_2, \quad \text{where} \quad 0 \le r_1 < d \quad \text{and} \quad 0 \le r_2 < d.$$

[We must show that $r_1 = r_2$ and $q_1 = q_2$.]

Then $$dq_1 + r_1 = dq_2 + r_2,$$

and so $$r_2 - r_1 = dq_1 - dq_2 = d(q_1 - q_2).$$

This implies that $$d \mid (r_2 - r_1)$$

because $q_1 - q_2$ is an integer (since it is a difference of integers).

But both r_1 and r_2 lie between 0 and d, and thus the difference $r_2 - r_1$ lies between $-d$ and d. *[For, by properties T23 and T26 of Appendix A, because $0 \le r_1 < d$ and $0 \le r_2 < d$, then multiplying the first inequality by -1 gives $0 \ge -r_1 > -d$ or, equivalently, $-d < -r_1 \le 0$, and adding the inequalities $-d < -r_1 \le 0$ and $0 \le r_2 < d$ gives $-d < r_2 - r_1 < d$.]*

Since $r_2 - r_1$ is a multiple of d and yet lies between $-d$ and d, the only possibility is that $r_2 - r_1 = 0$, or, equivalently, that $r_1 = r_2$.

Substituting back into the original expressions for a and equating the two gives

$$dq_1 + r_1 = dq_2 + r_1 \qquad \text{because } r_1 = r_2.$$

Subtracting r_1 from both sides gives $dq_1 = dq_2$, and since $d \ne 0$, we have that $q_1 = q_2$.

24. Proof by contradiction :

Suppose not. That is, suppose that $\log_5(2)$ is rational. *[We will show that this supposition leads logically to a contradiction.]*

By definition of rational,

$$\log_5(2) = \frac{a}{b} \quad \text{for some integers} \quad a \text{ and } b \text{ with } b \ne 0.$$

Since logarithms are always positive, we may assume that a and b are both positive. By definition of logarithm,

$$5^{\frac{a}{b}} = 2, \quad \text{and so} \quad (5^{\frac{a}{b}})^b = 2^b \quad \text{or, equivalently,} \quad 5^a = 2^b$$

Let

$$N = 5^a = 2^b.$$

Since $b \ge 0$, $N > 2^0 = 1$. Thus we may consider the prime factorization of N.

Because $N = 5^a$, the prime factors of N are all 5. On the other hand, because $N = 2^b$, the prime factors of N are all 2.

This contradicts the unique factorization of integers theorem, which states that the prime factors of any integer greater than 1 are unique except for the order in which they are written.

Hence the supposition is false, and so $\log_5(2)$ is irrational.

27. *a*. All of the following are prime numbers: $N_1 = 2+1 = 3$, $N_2 = 2{\cdot}3+1 = 7$, $N_3 = 2{\cdot}3{\cdot}5+1 = 31$, $N_4 = 2{\cdot}3{\cdot}5{\cdot}7 + 1 = 211$, $N_5 = 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11 + 1 = 2311$. However,

$$N_6 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509.$$

Thus the smallest non-prime integer of the given form is 30,031.

b. Each of N_1, N_2, N_3, N_4, and N_5 is prime, and so each is its own smallest prime divisor. Thus $q_1 = N_1$,$q_2 = N_2$, $q_3 = N_3$, $q_4 = N_4$, and $q_5 = N_5$. However, N_6 is not prime and $N_6 = 30031 = 59{\cdot}509$. Since 59 and 509 are primes, the smallest prime divisor of N_6 is $q_6 = 59$.

30. Proof: Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers with $p_1 = 2$ and $n > 1$. *[We must show that $p_1p_2 \cdots p_n + 1 = 4k + 3$ for some integer k.]* Let

$$N = p_1p_2 \cdots p_n + 1.$$

By the quotient-remainder theorem, there is an integer k such that

$$N = 4k,\ 4k+1,\ 4k+2, \text{ or } 4k+3.$$

But N is odd (because $p_1 = 2$); hence

$$N = 4k+1 \quad \text{or} \quad 4k+3$$

Suppose $N = 4k + 1$. *[We will show that this supposition leads to a contradiction.]*

By substitution,

$$4k + 1 = p_1p_2 \cdots p_n + 1 \quad \text{and so} \quad 4k = p_1p_2 \cdots p_n.$$

Thus

$$4 \mid p_1p_2 \cdots p_n.$$

But $p_1 = 2$ and all of p_2, p_3, $\ldots$, p_n are odd (being prime numbers that are greater than 2). Consequently, there is only one factor of 2 in the prime factorization of $p_1p_2 \cdots p_n$, and so

$$4 \nmid p_1p_2 \cdots p_n,$$

which results in a contradiction. Therefore the supposition that $N = 4k + 1$ for some integer k is false, and so *[by elimination]* $N = 4k + 3$ for some integer k *[as was to be shown]*.

33. Existence Proof: When $n = 2$, then

$$n^2 + 2n - 3 = 2^2 + 2 \cdot 2 - 3 = 5,$$

which is prime. Thus there is a prime number of the form $n^2 + 2n - 3$, where n is a positive integer.

Uniqueness Proof (by contradiction): By the existence proof above, we know that when $n = 2$, then n^2+2n-3 is prime. Suppose there is another positive integer m, not equal to 2, such that $m^2 + 2m - 3$ is prime. *[We will show that this supposition leads logically to a contradiction.]* By factoring, we see that

$$m^2 + 2m - 3 = (m + 3)(m - 1).$$

Now $m \neq 1$ because otherwise $m^2+2m-3 = 0$, which is not prime. Also $m \neq 2$ by supposition. Thus $m > 2$. Consequently, $m + 3 > 5$ and $m - 1 > 1$, and so $m^2 + 2m - 3$ can be written as a product of two positive integers neither of which is 1 (namely $m + 3$ and $m - 1$). This contradicts the supposition that $m^2 + 2m - 3$ is prime. Hence the supposition is false: there is no integer m other than 2 such that $m^2 + 2m - 3$ is prime.

Uniqueness Proof (direct): Suppose m is any positive integer such that $m^2 + 2m - 3$ is prime. *[We will show that $m = 2$.]* By factoring,

$$m^2 + 2m - 3 = (m + 3)(m - 1).$$

Since $m^2 + 2m - 3$ is prime, either $m + 3 = 1$ or $m - 1 = 1$. Now $m + 3 \neq 1$ because m is positive and if $m + 3 = 1$ then $m = -2$. Thus $m - 1 = 1$, which implies that $m = 2$ *[as was to be shown.]*.

Section 4.8

3. *b.* $z = 6$

12. *Solution 1*: $\gcd(48, 54) = \gcd(6 \cdot 8, 6 \cdot 9) = 6$

    Solution 2: $\gcd(48, 54) = \gcd(2^4 \cdot 3, 2 \cdot 3^3) = 2 \cdot 3 = 6$

15.
```
       13
832 ) 10933
      10816
        117
```
So $10933 = 832 \cdot 13 + 117$, and hence $\gcd(10933, 832) = \gcd(832, 117)$

```
        7
117 ) 832
      819
       13
```
So $832 = 117 \cdot 7 + 13$, and hence $\gcd(832, 117) = \gcd(117, 13)$

```
        9
 13 ) 117
      117
        0
```
So $117 = 13 \cdot 9 + 0$, and hence $\gcd(117, 13) = \gcd(13, 0)$

But $\gcd(13, 0) = 13$. So $\gcd(10933, 832) = 13$.

18.

A	5859						
B	1232						
r	1232	931	301	28	21	7	0
a	5859	1232	931	301	28	21	7
b	1232	931	301	28	21	7	0
gcd							7

21. Proof:

Suppose a and b are any integers with $b \neq 0$, and suppose q and r are any integers such that

$$a = bq + r.$$

We must show that

$$\gcd(b, r) \leq \gcd(a, b).$$

***Step 1 (proof that any common divisor of b and r is also a common divisor of a and b)*:**

Let c be a common divisor of b and r. Then $c \mid b$ and $c \mid r$, and so by definition of divisibility, there are integers n and m so that

$$b = nc \quad \text{and} \quad r = mc$$

Substitute these values into the equation $a = bq + r$ to obtain

$$a = (nc)q + mc = c(nq + m).$$

But $nq + m$ is an integer, and so by definition of divisibility $c \mid a$. Because we already know that $c \mid b$, we can conclude that c is a common divisor of a and b.

Step 2 (proof that $\gcd(b, r) \leq \gcd(a, b)$***):***

By step 1, every common divisor of b and r is a common divisor of a and b. It follows that the greatest common divisor of b and r is a common divisor of a and b. But then $\gcd(b, r)$ (being one of the common divisors of a and b) is less than or equal to the greatest common divisor of a and b:

$$\gcd(b, r) \leq \gcd(a, b)$$

[as was to be shown].

24. *a.* Proof: Suppose a and b are integers and $a \geq b > 0$.

Part 1 (proof that every common divisor of a and b is a common divisor of b and a − b):

Suppose

$$d \mid a \quad \text{and} \quad d \mid b.$$

Then, by exercise 16 of Section 4.3,

$$d \mid (a - b).$$

Hence d is a common divisor of a and $a - b$.

Part 2 (proof that every common divisor of b and a − b is a common divisor of a and b):

Suppose

$$d \mid b \quad \textit{and} \quad d \mid (a - b).$$

Then, by exercise 15 of Section 4.3,

$$a \mid [b + (a - b)].$$

But $b + (a - b) = a$, and so

$$d \mid a.$$

Hence d is a common divisor of a and b.

Part 3 (end of proof):

Because every common divisor of a and b is a common divisor of b and $a - b$, the greatest common divisor of a and b is a common divisor of b and $a - b$ and so is less than or equal to the greatest common divisor of a and $a - b$. Thus

$$\gcd(a, b) \leq \gcd(b, a - b).$$

By similar reasoning,

$$\gcd(b, a - b) \leq \gcd(a, b) \quad \text{and, therefore,} \quad \gcd(a, b) = \gcd(b, a - b).$$

c.

A	768											
B	348											
a	768	420	72					12				0
b	348			276	204	132	60		48	36	24	12
gcd												12

27. Proof: Let a and b be any positive integers.

Part 1 (proof that if $\mathbf{lcm}(a, b) = b$ ***then*** $a \mid b$***)***: Suppose that

$$\text{lcm}(a, b) = b.$$

By definition of least common multiple,

$$a \mid \text{lcm}(a, b),$$

and so by substitution, $a \mid b$.

Part 2 (proof that if $a \mid b$ ***then*** $\mathbf{lcm}(a, b) = b$***)***: Suppose that

$$a \mid b.$$

Then since it is also the case that

$$b \mid b,$$

b is a common multiple of a and b. Moreover, because b divides any common multiple of both a and b,

$$\text{lcm}(a, b) = b.$$

Review Guide: Chapter 4

Definitions

- Why is the phrase "if, and only if" used in a definition? *(p. 147)*
- How are the following terms defined?
  - even integer *(p. 147)*
  - odd integer *(p. 147)*
  - prime number *(p. 148)*
  - composite number *(p. 148)*
  - rational number *(p. 163)*
  - divisibility of one integer by another *(p. 170)*
  - n *div* d and n *mod* d *(p. 181)*
  - the floor of a real number *(p. 191)*
  - the ceiling of a real number *(p. 191)*
  - greatest common divisor of two integers *(p. 220)*

Proving an Existential Statement/Disproving a Universal Statement

- How do you determine the truth of an existential statement? *(p. 148)*
- What does it mean to "disprove" a statement? *(p. 149)*
- What is disproof by counterexample? *(p. 149)*
- How do you establish the falsity of a universal statement? *(p. 149)*

Proving a Universal Statement/Disproving an Existential Statement

- If a universal statement is defined over a small, finite domain, how do you use the method of exhaustion to prove that it is true? *(p. 150)*
- What is the method of generalizing from the generic particular? *(p. 151)*
- If you use the method of direct proof to prove a statement of the form "$\forall x$, if $P(x)$ then $Q(x)$", what do you suppose and what do you have to show? *(p. 152)*
- What are the guidelines for writing proofs of universal statements? *(pp. 155-156)*
- What are some common mistakes people make when writing mathematical proofs? *(pp. 157-158)*
- How do you disprove an existential statement? *(p. 159)*
- What is the method of proof by division into cases? *(p. 184)*
- What is the triangle inequality? *(p. 188)*
- If you use the method of proof by contradiction to prove a statement, what do you suppose and what do you have to show? *(p. 198)*
- If you use the method of proof by contraposition to prove a statement of the form "$\forall x$, if $P(x)$ then $Q(x)$", what do you suppose and what do you have to show? *(p. 202)*
- Are you able to use the various methods of proof and disproof to establish the truth or falsity of statements about odd and even integers *(pp. 154,199)*, prime numbers *(pp. 159,210)*, rational and irrational numbers *(pp. 165,166,201,208,209)*, divisibility of integers *(pp. 171,173-175,184,186,202,203)*, absolute value *(pp. 187-188)*, and the floor and ceiling of a real number *(pp. 194-196)*?

Some Important Theorems and Algorithms

- What is the transitivity of divisibility theorem? *(p. 173)*
- What is the theorem about divisibility by a prime number? *(p. 174)*

- What is the unique factorization of integers theorem? (This theorem is also called the fundamental theorem of arithmetic.) *(p. 176)*
- What is the quotient-remainder theorem? Can you apply it to specific situations? *(p. 180)*
- What is the theorem about the irrationality of the square root of 2? Can you prove this theorem? *(p. 208)*
- What is the theorem about the infinitude of the prime numbers? Can you prove this theorem? *(p. 210)*
- What is the division algorithm ? *(p. 219)*
- What is the Euclidean algorithm? *(pp. 220,224)*
- How do you use the Euclidean algorithm to compute the greatest common divisor of two positive integers? *(p. 223)*

Notation for Algorithms

- How is an assignment statement executed? *(p. 214)*
- How is an **if-then** statement executed? *(p. 215)*
- How is an **if-then-else** statement executed? *(p. 215)*
- How are the statements **do** and **end do** used in an algorithm? *(p. 215)*
- How is a **while** loop executed? *(p. 216)*
- How is a **for-next** loop executed? *(p. 217)*
- How do you construct a trace table for a segment of an algorithm?*(pp. 217,219)*

Chapter 5: Sequences, Mathematical Induction, and Recursion

The first section of this chapter introduces the notation for sequences, summations, products, and factorial. The section is intended to help you learn to recognize patterns so as to be able, for instance, to transform expanded versions of sums into summation notation, and to handle subscripts, particularly to change variables for summations and to distinguish index variables from variables that are constant with respect to a summation.

The second, third, and fourth sections of the chapter treat mathematical induction. The ordinary form is discussed in Sections 5.2 and 5.3 and the strong form in Section 5.4. Because of the importance of mathematical induction in discrete mathematics, a wide variety of examples is given to help you become comfortable with using the technique in many different situations. Section 5.5 then shows how to use a variation of mathematical induction to prove the correctness of an algorithm. Sections 5.6-5.8 deal with recursively defined sequences, both how to analyze a situation using recursive thinking to obtain a sequence that describes the situation and how to find an explicit formula for the sequence once it has been defined recursively. Section 5.9 applies recursive thinking to the question of defining a set, and it describes the technique of structural induction, which is the variation of mathematical induction that can be used to verify properties of a set that has been defined. Section 5.9 also introduces the concept of a recursively defined function.

The logic of ordinary mathematical induction can be described by relating it to the logic discussed in Chapters 2 and 3. The main point is that the inductive step establishes the truth of a sequence of if-then statements. Together with the basis step, this sequence gives rise to a chain of inferences that lead to the desired conclusion. More formally:

Suppose
1. $P(1)$ is true; and
2. for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true.

The truth of statement (2) implies, according to the law of universal instantiation, that no matter what particular integer $k \geq 1$ is substituted in place of k, the statement "If $P(k)$ then $P(k+1)$" is true. The following argument, therefore, has true premises, and so by modus ponens it has a true conclusion:

	If $P(1)$ then $P(2)$.	by 2 and universal instantiation
	$P(1)$	by 1
$\therefore$	$P(2)$	by modus ponens

Similar reasoning gives the following chain of arguments, each of which has a true conclusion by modus ponens:

If $P(2)$ then $P(3)$.
$P(2)$
$\therefore$ $P(3)$
If $P(3)$ then $P(4)$.
$P(3)$
$\therefore$ $P(4)$
If $P(4)$ then $P(5)$.
$P(4)$
$\therefore$ $P(5)$
And so forth.

Thus no matter how large a positive integer n is specified, the truth of $P(n)$ can be deduced as the final conclusion of a (possibly very long) chain of arguments continuing those shown above.

Section 5.1

6. $f_1 = \left\lfloor \frac{1}{4} \right\rfloor \cdot 4 = 0 \cdot 4 = 0,\quad f_2 = \left\lfloor \frac{2}{4} \right\rfloor \cdot 4 = 0 \cdot 4 = 0,\quad f_3 = \left\lfloor \frac{3}{4} \right\rfloor \cdot 4 = 0 \cdot 4 = 4,$
$f_4 = \left\lfloor \frac{4}{4} \right\rfloor \cdot 4 = 1 \cdot 4 = 4$

9.
$$\begin{array}{lclcl}
h_1 & = & 1 \cdot \lfloor \log_2 1 \rfloor & = & 1 \cdot 0 \\
h_2 & = & 2 \cdot \lfloor \log_2 2 \rfloor & = & 2 \cdot 1 \\
h_3 & = & 3 \cdot \lfloor \log_2 3 \rfloor & = & 3 \cdot 1 \\
h_4 & = & 4 \cdot \lfloor \log_2 4 \rfloor & = & 4 \cdot 2 \\
h_5 & = & 5 \cdot \lfloor \log_2 5 \rfloor & = & 5 \cdot 2 \\
h_6 & = & 6 \cdot \lfloor \log_2 6 \rfloor & = & 6 \cdot 2 \\
h_7 & = & 7 \cdot \lfloor \log_2 7 \rfloor & = & 7 \cdot 2 \\
h_8 & = & 8 \cdot \lfloor \log_2 8 \rfloor & = & 8 \cdot 3 \\
h_9 & = & 9 \cdot \lfloor \log_2 9 \rfloor & = & 9 \cdot 3 \\
h_{10} & = & 10 \cdot \lfloor \log_2 10 \rfloor & = & 10 \cdot 3 \\
h_{11} & = & 11 \cdot \lfloor \log_2 11 \rfloor & = & 11 \cdot 3 \\
h_{12} & = & 12 \cdot \lfloor \log_2 12 \rfloor & = & 12 \cdot 3 \\
h_{13} & = & 13 \cdot \lfloor \log_2 13 \rfloor & = & 13 \cdot 3 \\
h_{14} & = & 14 \cdot \lfloor \log_2 14 \rfloor & = & 14 \cdot 3 \\
h_{15} & = & 15 \cdot \lfloor \log_2 15 \rfloor & = & 15 \cdot 3
\end{array}$$

When n is an integral power of 2, h_n is n times the exponent of that power. For instance, $8 = 2^3$ and $h_8 = 8 \cdot 3$. If m and n are integers and $2^m \le n < 2^{m+1}$, then $h_n = n \cdot m$.

15. $a_n = (-1)^{n-1}\left(\frac{n-1}{n}\right)$ for all integers $n \ge 1$ (There are other correct answers for this exercise.)

18. e. $\prod_{k=2}^{2} a_k = a_2 = -2$

21. $\displaystyle\sum_{m=0}^{3} \frac{1}{2^m} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

24. $\displaystyle\sum_{j=0}^{0} (j+1) \cdot 2^j = (0+1) \cdot 2^0 = 1 \cdot 1 = 1$

30. $\displaystyle\sum_{j=1}^{n} j(j+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1)$ (There are other correct answers for this exercise.)

36. $\left(\frac{1 \cdot 2}{3 \cdot 4}\right) = \frac{1}{6}$

39. $\displaystyle\sum_{m=1}^{n+1} m(m+1) = \sum_{m=1}^{n} m(m+1) + (n+1)((n+1)+1)$

42. $\displaystyle\sum_{m=0}^{n} (m+1)2^m + (n+2)2^{n+1} = \sum_{m=0}^{n+1} (m+1)2^m$

Exercises 45 and 48 have more than one correct answer.

45. $\displaystyle\prod_{i=2}^{4} (i^2 - 1)$

48. $\prod_{j=1}^{4}(1-t^j)$

54. When $k=1$, $i=1+1=2$. When $k=n$, $i=n+1$. Since $i=k+1$, then $k=i-1$. So

$$\frac{k}{k^2+4}=\frac{(i-1)}{(i-1)^2+4}=\frac{i-1}{i^2-2i+1+4}=\frac{i-1}{i^2-2i+5}.$$

Therefore,

$$\prod_{k=1}^{n}\left(\frac{k}{k^2+4}\right)=\prod_{i=2}^{n+1}\left(\frac{i-1}{i^2-2i+5}\right).$$

57. When $i=1$, $j=1-1=0$. When $i=n-1$, $j=n-2$. Since $j=i-1$, then $i=j+1$. So

$$\frac{i}{(n-i)^2}=\frac{j+1}{(n-(j+1))^2}=\frac{j+1}{(n-j-1)^2}.$$

Therefore,

$$\sum_{i=1}^{n-1}\left(\frac{i}{(n-i)^2}\right)=\sum_{j=0}^{n-2}\left(\frac{j+1}{(n-j-1)^2}\right).$$

60. By Theorem 5.1.1,

$$\begin{aligned}2\cdot\sum_{k=1}^{n}(3k^2+4)+5\cdot\sum_{k=1}^{n}(2k^2-1) &= \sum_{k=1}^{n}2(3k^2+4)+\sum_{k=1}^{n}5(2k^2-1)\\ &= \sum_{k=1}^{n}(6k^2+8)+\sum_{k=1}^{n}(10k^2-5)\\ &= \sum_{k=1}^{n}(6k^2+8+10k^2-5)\\ &= \sum_{k=1}^{n}(16k^2+3).\end{aligned}$$

63. $\frac{6!}{8!}=\frac{6!}{8\cdot 7\cdot 6!}=\frac{1}{56}$

72. $\binom{7}{4}=\frac{7!}{4!(7-4)!}=\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(4\cdot 3\cdot 2\cdot 1)(3\cdot 2\cdot 1)}=\frac{7\cdot 6\cdot 5}{(3\cdot 2\cdot 1)}=35$

87. Let a nonnegative integer a be given. Divide a by 16 using the quotient-remainder theorem to obtain a quotient $q[0]$ and a remainder $r[0]$. If the quotient is nonzero, divide by 16 again to obtain a quotient $q[1]$ and a remainder $r[1]$. Continue this process until a quotient of 0 is obtained. The remainders calculated in this way are the hexadecimal digits of a:

$$a_{10}=(r[k]r[k-1]\ldots r[2]r[1]r[0])_{16}.$$

90.

	0	R. $8=8_{16}$
16	8	R. $15=F_{16}$
16	143	R. $13=D_{16}$
16	2301	

Hence $2301_{10}=8FD_{16}$.

Section 5.2

9. Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}. \qquad \leftarrow Pn)$$

Show that P(3) is true: $P(3)$ is true because the left-hand side is $4^3 = 64$ and the right-hand side is $\frac{4(4^3-16)}{3} = \frac{4(64-16)}{3} = \frac{4 \cdot 48}{3} = 64$ also.

Show that for all integers $k \geq 3$, if P(k) is true then P(k+1) is true: Let k be any integer with $k \geq 3$, and suppose that

$$4^3 + 4^4 + 4^5 + \cdots + 4^k = \frac{4(4^k - 16)}{3}. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}. \qquad \leftarrow P(k+1)$$

Now the left-hand side of $P(k+1)$ is

$$\begin{aligned}
4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} &= 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} \\
&\quad \text{by making the next-to-last term explicit} \\
&= \frac{4(4^k - 16)}{3} + 4^{k+1} \\
&\quad \text{by inductive hypothesis} \\
&= \frac{4^{k+1} - 64}{3} + \frac{3 \cdot 4^{k+1}}{3} \\
&\quad \text{by creating a common denominator} \\
&= \frac{4 \cdot 4^{k+1} - 64}{3} \\
&\quad \text{by adding the fractions} \\
&= \frac{4(4^{k+1} - 16)}{3} \\
&\quad \text{by factoring out the 4,}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown]*.

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for all integers $n \geq 3$.]

12. Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}. \qquad \leftarrow Pn)$$

Show that P(1) is true: $P(1)$ is true because the left-hand side equals $\frac{1}{1 \cdot 2} = \frac{1}{2}$ and the right-hand side equals $\frac{1}{1+1} = \frac{1}{2}$ also.

Show that for all integers $k \geq 1$, if P(k) is true then P(k+1) is true: Let k be any integer with $k \geq 1$ and suppose that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)((k+1)+1)}=\frac{k+1}{(k+1)+1},$$

or, equivalently,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}. \quad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}\\
&= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} && \text{by making the next-to-last term explicit}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} && \text{by inductive hypothesis}\\
&= \frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)(k+2)} && \text{by creating a common denominator}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)} && \text{by adding the fractions}\\
&= \frac{(k+1)^2}{(k+1)(k+2)} && \text{because } k^2+2k+1=(k+1)^2\\
&= \frac{k+1}{k+2} && \text{by canceling } (k+1) \text{ from numerator and denominator,}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown]*

15. Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n} i(i!)=(n+1)!-1. \qquad \leftarrow Pn)$$

Show that $P(1)$ is true: We must show that $\sum_{i=1}^{1} i(i!)=(1+1)!-1$. But the left-hand side of this equation is $\sum_{i=1}^{1} i(i!)=1\cdot(1!)=1$ and the right-hand side is $(1+1)!-1=2!-1=2-1=1$ also. So $P(1)$ is true.

Show that for all integers $k\geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k\geq 1$, and suppose that

$$\sum_{i=1}^{k} i(i!)=(k+1)!-1. \qquad \leftarrow \begin{array}{l}P(k)\\ \text{inductive hypothesis}\end{array}$$

We must show that

$$\sum_{i=1}^{k+1} i(i!)=((k+1)+1)!-1,$$

or, equivalently,

$$\sum_{i=1}^{k+1} i(i!)=(k+2)!-1. \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i!) &= \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!) && \text{by writing the } (k+1)\text{st term separately}\\
&= [(k+1)! - 1] + (k+1)((k+1)!) && \text{by inductive hypothesis}\\
&= ((k+1)!)(1+(k+1)) - 1 && \text{by combining the terms with the common factor } (k+1)!\\
&= (k+1)!(k+2) - 1 &&\\
&= (k+2)! - 1 && \text{by algebra,}
\end{aligned}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown]*.

18. Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$\sin x + \sin 3x + \cdots + \sin(2n-1)x = \frac{1-\cos 2nx}{2\sin x}. \qquad \leftarrow Pn)$$

Show that $P(1)$ is true: $P(1)$ is true because the left-hand side equals $\sin x$, and the right-hand side equals

$$\frac{1-\cos 2x}{2\sin x} = \frac{1-\cos^2 + \sin^2 x}{2\sin x} = \frac{2\sin^2 x}{2\sin x} = \sin x.$$

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$ and suppose that

$$\sin x + \sin 3x + \cdots + \sin(2k-1)x = \frac{1-\cos 2kx}{2\sin x}. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis}\end{array}$$

We must show that

$$\sin x + \sin 3x + \cdots + \sin(2(k+1)-1)x = \frac{1-\cos 2(k+1)x}{2\sin x},$$

or, equivalently,

$$\sin x + \sin 3x + \cdots + \sin(2k+1)x = \frac{1-\cos 2(k+1)x}{2\sin x}. \qquad \leftarrow P(k+1)$$

When the next-to-last term of the the left-hand side of $P(k+1)$ is made explicit, the left-hand side becomes

$$\begin{aligned}
&\sin x + \sin 3x + \cdots + \sin(2k-1)x + \sin(2k+1)x\\
&= \frac{1-\cos 2kx}{2\sin x} + \sin(2k+1)x && \text{by inductive hypothesis}\\
&= \frac{1-\cos 2kx}{2\sin x} + \frac{2\sin x\sin(2kx+x)}{2\sin x} && \text{by creating a common denominator}\\
&= \frac{1-\cos 2kx + 2\sin x\sin(2kx+x)}{2\sin x} && \text{by adding fractions}\\
&= \frac{1-\cos 2kx + 2\sin x[\sin(2kx)\cos x + \cos(2kx)\sin x]}{2\sin x} && \text{by the addition formula for sine}\\
&= \frac{1-\cos 2kx + 2\sin x\sin(2kx)\cos x + 2\sin^2 x\cos(2kx)}{2\sin x} && \text{by multiplying out}\\
&= \frac{1+\cos 2kx(2\sin^2 x - 1) + 2\sin x\cos x\sin(2kx)}{2\sin x} && \text{by combining like terms}\\
&= \frac{1+\cos 2kx(-\cos 2x) + \sin 2x\sin(2kx)}{2\sin x} && \text{by the formulas for } \cos 2x \text{ and } \sin 2x\\
&= \frac{1-(\cos 2kx\cos 2x - \sin 2x\sin(2kx))}{2\sin x} && \text{by factoring out } -1\\
&= \frac{1-\cos(2kx+2x)}{2\sin x} && \text{by the addition formula for cosine}\\
&= \frac{1-\cos(2(k+1)x)}{2\sin x} && \text{by factoring out } 2x,
\end{aligned}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown]*.

21. $5 + 10 + 15 + 20 + \cdots + 300 = 5(1 + 2 + 3 + \cdots + 60) = 5\left(\frac{60 \cdot 61}{2}\right) = 9150.$

27. *Solution 1:*

$$\begin{aligned} 5^3 + 5^4 + 5^5 + \cdots + 5^k &= 5^3(1 + 5 + 5^2 + \cdots + 5^{k-3}) \\ &= 5^3\left(\frac{5^{(k-3)+1} - 1}{5 - 1}\right) = \frac{5^3(5^{k-2} - 1)}{4} \end{aligned}$$

Solution 2:

$$\begin{aligned} 5^3 + 5^4 + 5^5 + \cdots + 5^k &= 1 + 5 + 5^2 + \cdots + 5^k - (1 + 5 + 5^2) \\ &= \frac{5^{k+1} - 1}{5 - 1} - 31 = \frac{5^{k+1} - 1}{4} - 31 \end{aligned}$$

Note that the expression obtained in solution 2 can be transformed into the one obtained in solution 1:

$$\frac{5^{k+1} - 1}{4} - 31 = \frac{5^{k+1} - 1}{4} - \frac{31 \cdot 4}{4} = \frac{5^{k+1} - 125}{4} = \frac{5^{k+1} - 5^3}{4} = \frac{5^3(5^{k-2} - 1)}{4}.$$

30. $(a + md) + (a + (m + 1)d) + (a + (m + 2)d) + \cdots + (a + (m + n)d)$

$$\begin{aligned} &= (a + md) + (a + md + d) + (a + md + 2d) + \cdots + (a + md + nd) \\ &= \underbrace{((a + md) + (a + md) + \cdots + (a + md))}_{n+1 \text{ terms}} + d(1 + 2 + 3 + \cdots + n) \\ &= (n + 1)(a + md) + d\left(\frac{n(n + 1)}{2}\right) \qquad \text{by Theorem 5.2.2} \\ &= (a + md + \frac{n}{2}d)(n + 1) \\ &= [a + (m + \frac{n}{2})d](n + 1) \end{aligned}$$

Any one of the last three equations or their algebraic equivalents could be considered a correct answer.

36. Proof: Suppose m and n are any positive integers such that m is odd. By definition of odd, $m = 2q + 1$ for some integer k, and so, by Theorems 5.1.1 and 5.2.2,

$$\sum_{k=0}^{m-1}(n + k) = \sum_{k=0}^{(2q+1)-1}(n + k) = \sum_{k=0}^{2q}(n + k) = \sum_{k=0}^{2q} n + \sum_{k=0}^{2q} k = (2q + 1)n + \sum_{k=1}^{2q} k$$

$$= (2q + 1)n + \frac{2q(2q + 1)}{2} = (2q + 1)n + q(2q + 1) = (2q + 1)(n + q) = m(n + q).$$

But $n + q$ is an integer because it is a sum of integers. Hence, by definition of divisibility, $\sum_{k=0}^{m-1}(n + k)$ is divisible by m.

Note: If m is even, the property is no longer true. For example, if $n = 1$ and $m = 2$, then $\sum_{k=0}^{m-1}(n + k) = \sum_{k=0}^{2-1}(1 + k) = 1 + 2 = 3$, and 3 is not divisible by 2.

Section 5.3

9. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$7^n - 1$ is divisible by 6.

We will prove that $P(n)$ is true for all integers $n \geq 0$.

Show that P(0) is true: $P(0)$ is true because because $7^0 - 1 = 1 - 1 = 0$ and 0 is divisible by 6 (since $0 = 0 \cdot 6$).

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose

$$7^k - 1 \text{ is divisible by } 6. \quad \leftarrow \text{ inductive hypothesis}$$

We must show that

$$7^{k+1} - 1 \text{ is divisible by } 6.$$

By definition of divisibility, the inductive hypothesis is equivalent to the statement

$$7^k - 1 = 6r$$

for some integer r. Then

$$\begin{aligned} 7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\ &= (6+1)7^k - 1 \\ &= 6 \cdot 7^k + (7^k - 1) && \text{by algebra} \\ &= 6 \cdot 7^k + 6r && \text{by inductive hypothesis} \\ &= 6(7^k + r) && \text{by algebra.} \end{aligned}$$

Now $7^k + r$ is an integer because products and sums of integers are integers.Thus, by definition of divisibility, $7^{k+1} - 1$ is divisible by 6 *[as was to be shown]*.

12. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$7^n - 2^n \text{ is divisible by } 5.$$

We will prove that $P(n)$ is true for all integers $n \geq 0$.

Show that P(0) is true: $P(0)$ is true because $7^0 - 2^0 = 0 - 0 = 0$ and 0 is divisible by 5 (since $0 = 5 \cdot 0$).

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose

$$7^k - 2^k \text{ is divisible by } 5. \quad \leftarrow \text{ inductive hypothesis}$$

We must show that

$$7^{k+1} - 2^{k+1} \text{ is divisible by } 5.$$

By definition of divisibility, the inductive hypothesis is equivalent to the statement $7^k - 2^k = 5r$ for some integer r. Then

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= (5+2) \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2(7^k - 2^k) && \text{by algebra} \\ &= 5 \cdot 7^k + 2 \cdot 5r && \text{by inductive hypothesis} \\ &= 5(7^k + 2r) && \text{by algebra.} \end{aligned}$$

Now $7^k + 2r$ is an integer because products and sums of integers are integers. Therefore, by definition of divisibility, $7^{k+1} - 2^{k+1}$ is divisible by 5 *[as was to be shown]*.

15. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$n(n^2 + 5) \text{ is divisible by } 6.$$

We will prove that $P(n)$ is true for all integers $n \geq 0$.

Show that $P(0)$ is true: $P(0)$ is true because $0(0^2+5)=0$ and 0 is divisible by 6.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose

$$k(k^2+5) \text{ is divisible by } 6. \qquad \leftarrow \text{ inductive hypothesis}$$

We must show that

$$(k+1)((k+1)^2+5) \text{ is divisible by } 6.$$

By definition of divisibility $k(k^2+5)=6r$ for some integer r. Then

$$\begin{aligned}(k+1)((k+1)^2+5) &= (k+1)(k^2+2k+1+5)\\ &= (k+1)(k^2+2k+6)\\ &= k^3+2k^2+6k+k^2+2k+6\\ &= k^3+3k^2+8k+6\\ &= (k^3+5k)+(3k^2+3k+6)\\ &= k(k^2+5)+(3k^2+3k+6) && \text{by algebra}\\ &= 6r+3(k^2+k)+6, && \text{by inductive hypothesis.}\end{aligned}$$

Now $k(k+1)$ is a product of two consecutive integers. By Theorem 4.4.2 one of these is even, and so *[by properties 1 and 4 of Example 4.2.3]* the product $k(k+1)$ is even. Hence $k(k+1)=2s$ for some integer s. Thus

$$6r+3(k^2+k)+6=6r+3(2s)+6=6(r+s+1).$$

By substitution, then,

$$(k+1)((k+1)^2+5)=6(r+s+1),$$

which is divisible by 6 because $r+s+1$ is an integer. Therefore, $(k+1)((k+1)^2+5)$ is divisible by 6 *[as was to be shown]*.

18. Proof (by mathematical induction): Let the property $P(n)$ be the inequality $5^n+9<6^n$.

We will prove that $P(n)$ is true for all integers $n \geq 2$.

Show that $P(2)$ is true: $P(2)$ is true because the left-hand side is $5^2+9=25+9=34$ and the right-hand side is $6^2=36$, and $34<36$.

Show that for all integers $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$5^k+9<6^k. \qquad \leftarrow \text{ inductive hypothesis}$$

We must show that

$$5^{k+1}+9<6^{k+1}.$$

Multiplying both sides of the inequality in the inductive hypothesis by 5 gives

$$5(5^k+9)<5\cdot 6^k. \ (*)$$

Note that

$$5^{k+1}+9<5^{k+1}+45=5(5^k+9) \quad \text{and} \quad 5\cdot 6^k<6^{k+1}. \ (**)$$

Thus, by by the transitive property of order, (*), and (**),

$$5^{k+1}+9<5(5^k+9) \quad \text{and} \quad 5(5^k+9)<5\cdot 6^k \quad \text{and} \quad 5\cdot 6^k<6^{k+1}.$$

So, by the transitive property of order,

$$5^{k+1}+9<6^{k+1}$$

[as was to be shown].

21. Proof (by mathematical induction): Let the property $P(n)$ be the inequality

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}.$$

We will prove that $P(n)$ is true for all integers $n \geq 2$.

Show that P(2) is true: To show that $P(2)$ is true we must show that

$$\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}.$$

But this inequality is true if, and only if,

$$2 < \sqrt{2} + 1$$

(by multiplying both sides by $\sqrt{2}$). And this is true if, and only if,

$$1 < \sqrt{2}$$

(by subtracting 1 on both sides). But $1 < \sqrt{2}$, and so $P(2)$ is true.

Show that for all integers $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}}. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k+1}}.$$

But for each integer $k \geq 2$,

$$\sqrt{k} < \sqrt{k+1} \; (*),$$

and multiplying both sides of (*) by $\sqrt{k}$ gives

$$k < \sqrt{k} \cdot \sqrt{k+1}.$$

Adding 1 to both sides gives

$$k + 1 < \sqrt{k} \cdot \sqrt{k+1} + 1,$$

and dividing both sides by $\sqrt{k+1}$ gives

$$\sqrt{k+1} < \sqrt{k} + \frac{1}{\sqrt{k+1}}.$$

By substitution from the inductive hypothesis, then,

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

[as was to be shown].

(*) *Note*: Strictly speaking, the reason for this claim is that $k < k+1$ and for all positive real numbers a and b, if $a < b$, then $\sqrt{a} < \sqrt{b}$.

27. Proof (by mathematical induction): According to the definition of $d_1, d_2, d_3, \ldots$, we have that $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$ for all integers $k \geq 2$. Let the property $P(n)$ be the equation

$$d_n = \frac{2}{n!}.$$

We will prove that $P(n)$ is true for all integers $n \geq 1$.

Show that $P(1)$ is true: To show that $P(1)$ is true we must show that $d_1 = \frac{2}{1!}$. But $\frac{2}{1!} = 2$ and $d_1 = 2$ (by definition of $d_1, d_2, d_3, \ldots$). So the property holds for $n = 1$.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$d_k = \frac{2}{k!}. \qquad \leftarrow \text{ inductive hypothesis}$$

We must show that

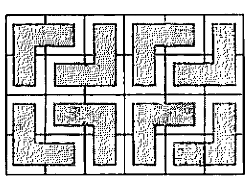
$$d_{k+1} = \frac{2}{(k+1)!}.$$

But the left-hand side of this equation is

$$\begin{aligned} d_{k+1} &= \frac{d_k}{k+1} && \text{by definition of } d_1, d_2, d_3, \ldots \\ &= \frac{\frac{2}{k!}}{k+1} && \text{by inductive hypothesis} \\ &= \frac{2}{(k+1)k!} \\ &= \frac{2}{(k+1)!} && \text{by the algebra of fractions,} \end{aligned}$$

which is the right-hand side of the equation. *[This is what was to be shown].*

33.



36. Proof by mathematical induction: Let the property $P(n)$ be the sentence

> In any round-robin tournament involving n teams, it is possible to label the teams $T_1, T_2, T_3, \ldots, T_n$ so that for all $i = 1, 2, 3, \ldots, n-1$, T_i beats T_{i+1}.

We will prove that $P(n)$ is true for all integers $n \geq 2$.

Show that $P(2)$ is true: Consider any round-robin tournament involving two teams. By definition of round-robin tournament, these teams play each other exactly once. Let T_1 be the winner and T_2 the loser of this game. Then T_1 beats T_2, and so the labeling is as required for $P(2)$ to be true.

Show that for all integers $k \geq 2$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 2$ and suppose that

In any round-robin tournament involving k teams, it is possible to label the teams $T_1, T_2, T_3, \ldots, T_k$ so that for all $i = 1, 2, 3, \ldots, k-1$, T_i beats T_{i+1}. $\leftarrow$ inductive hypothesis

We must show that

> In any round-robin tournament involving $k+1$ teams, it is possible to label the teams $T_1, T_2, T_3, \ldots, T_{k+1}$ so that for all $i = 1, 2, 3, \ldots, k$, T_i beats T_{i+1}.

Consider any round-robin tournament with $k+1$ teams. Pick one and call it T'. Temporarily remove T' and consider the remaining k teams. Since each of these teams plays each other team exactly once, the games played by these k teams form a round-robin tournament. It follows by inductive hypothesis that these k teams may be labeled $T_1, T_2, T_3, \ldots, T_k$ where T_i beats T_{i+1} for all $i = 1, 2, 3, \ldots, k-1$.

Case 1 (T' beats T_1): In this case, relabel each T_i to be T_{i+1}, and let $T_1 = T'$. Then T_1 beats the newly labeled T_2 (because T' beats the old T_1), and T_i beats T_{i+1} for all $i = 2, 3, \ldots, k$ (by inductive hypothesis).

Case 2 (T' loses to $T_1, , T_2, T_3, \ldots, T_m$ and beats T_{m+1} where $1 \le m \le k-1$): In this case, relabel teams $T_{m+1}, T_{m+2}, \ldots, T_k$ to be $T_{m+2}, T_{m+3}, \ldots, T_{k+1}$ and let $T_{m+1} = T'$. Then for each i with $1 \le i \le m-1$, T_i beats T_{i+1} (by inductive hypothesis), T_m beats T_{m+1} (because T_m beats T'), T_{m+1} beats T_{m+2} (because T' beats the old T_{m+1}), and for each i with $m+2 \le i \le k$, T_i beats T_{i+1} (by inductive hypothesis).

Case 3 (T' loses to T_i for all $i = 1, 2, \ldots, k$): In this case, let $T_{k+1} = T'$. Then for all $i = 1, 2, \ldots, k-1$, T_i beats T_{i+1}(by inductive hypothesis) and T_k beats T_{k+1} (because T_k beats T').

Thus in all three cases the teams may be relabeled in the way specified *[as was to be shown]*.

39, Proof (by mathematical induction): Let the property $P(n)$ be the sentence

> The interior angles of any n-sided convex polygon add up to $180(n-2)$ degrees.

We will prove that $P(n)$ is true for all integers $n \ge 3$.

Show that $P(3)$ is true: $P(3)$ is true because any convex 3-sided polygon is a triangle, the sum of the interior angles of any triangle is 180 degrees, and $180(3-2) = 180$. So the angles of any 3-sided convex polygon add up to $180(3-2)$ degrees.

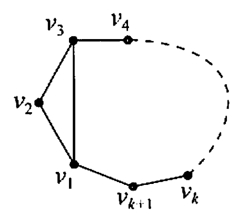
Show that for all integers $k \ge 3$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \ge 3$ and suppose that

> The interior angles of any k-sided convex polygon add up to $180(k-2)$ degrees. ← inductive hypothesis

We must show that

> The interior angles of any $(k+1)$-sided convex polygon add up to $180((k+1)-2) = 180(k-1)$ degrees.

Let p be any $(k+1)$-sided convex polygon. Label the vertices of p as $v_1, v_2, v_3, \ldots, v_k, v_{k+1}$, and draw a straight line from v_1 to v_3. Because the angles at v_1, v_2, and v_3 are all less than 180 degrees, this line lies entirely inside the polygon. Thus polygon p is split in two pieces: (1) the polygon p' obtained from p by using all of its vertices except v_2, and (2) triangle t with vertices v_1, v_2, v_3. This situation is illustrated in the diagram below.



Note that p' has k vertices, and it is convex because the sizes of the angles at v_1 and v_3 in p' are less than their sizes in p. Because p' is a convex polygon with k vertices, by inductive hypothesis, the sum of its interior angles is $180(k-2)$ degrees. Now polygon p is obtained by joining p' and t, and since the sum of the interior angles in t is 180 degrees,

$$
\begin{aligned}
\text{the sum of the interior angles in } p &= \text{the sum of the interior angles in } p' \\
&\quad + \text{the sum of the interior angles in } t \\
&= 180(k-2) \text{ degrees} + 180 \text{ degrees} \\
&= 180(k-2+1) \text{ degrees} \\
&= 180(k-1) \text{ degrees,}
\end{aligned}
$$

as was to be shown.

Section 5.4

3. Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

$$c_n \text{ is even.}$$

We will prove that $P(n)$ is true for all integers $n \geq 0$.

Show that $P(0)$, $P(1)$, and $P(2)$ are true: By definition of $c_0, c_1, c_2, \ldots$, we have that $c_0 = 2$, $c_1 = 2$, and $c_2 = 6$ and 2, 2, and 6 are all even. So $P(0)$, $P(1)$, and $P(2)$ are all true.

Show that if $k \geq 2$ and $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$c_i \text{ is even for all integers } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$c_{k+1} \text{ is even.}$$

But by definition of $c_0, c_1, c_2, \ldots$, $c_{k+1} = 3c_{k-2}$. Since $k \geq 2$, we have that $0 \leq k-2 \leq k$, and so, by inductive hypothesis, c_{k-2} is even. But the product of an even integer with any integer is even *[properties 1 and 4 of Example 4.2.3]*, and hence $3c_{k-2}$, which equals c_{k+1}, is also even *[as was to be shown]*.

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for all integers $n \geq 0$.]

6. Proof (by strong mathematical induction): Let the property $P(n)$ be the equation

$$f_n = 3 \cdot 2^n + 2 \cdot 5^n.$$

We will prove that $P(n)$ is true for all integers $n \geq 0$.

Show that $P(0)$ and $P(1)$ are true: By definition of $f_0, f_1, f_2, \ldots$, we have that $f_0 = 5$ and $f_1 = 16$. Since $3 \cdot 2^0 + 2 \cdot 5^0 = 3 + 2 = 5$ and $3 \cdot 2^1 + 2 \cdot 5^1 = 6 + 10 = 16$, $P(0)$ and $P(1)$ are both true.

Show that if $k \geq 1$ and $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose

$$f_i = 3 \cdot 2^i + 2 \cdot 5^i \text{ for all integers } i \text{ with } 0 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$f_{k+1} = 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}.$$

But

$$
\begin{array}{rcll}
f_{k+1} & = & 7f_k - 10f_{k-1} & \text{by definition of } f_0, f_1, f_2, \ldots \\
& = & 7(3\cdot 2^k + 2\cdot 5^k) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) & \text{by inductive hypothesis} \\
& = & 7(6\cdot 2^{k-1} + 10\cdot 5^{k-1}) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) & \text{because } 2^k = 2\cdot 2^{k-1} \text{ and } 5^k = 5\cdot 5^{k-1} \\
& = & (42\cdot 2^{k-1} + 70\cdot 5^{k-1}) - (30\cdot 2^{k-1} + 20\cdot 5^{k-1}) & \\
& = & (42-30)\cdot 2^{k-1} + (70-20)\cdot 5^{k-1} & \\
& = & 12\cdot 2^{k-1} + 50\cdot 5^{k-1} & \\
& = & 3\cdot 2^2\cdot 2^{k-1} + 2\cdot 5^2\cdot 5^{k-1} & \\
& = & 3\cdot 2^{k+1} + 2\cdot 5^{k+1} & \text{by algebra,}
\end{array}
$$

[as was to be shown].

[Since both the basis and the inductive steps have been proved, we conclude that $P(n)$ is true for all integers $n \geq 0$.]

9. Proof (by strong mathematical induction): Let the property $P(n)$ be the inequality

$$a_n \leq \left(\frac{7}{4}\right)^n.$$

We will prove that $P(n)$ is true for all integers $n \geq 1$.

Show that $P(1)$ and $P(2)$ are true: By definition of $a_1, a_2, a_3, \ldots$, we have that $a_1 = 1$ and $a_2 = 3$. But

$$\frac{7}{4} > 1 \quad \text{and} \quad \left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\frac{1}{16} > 3$$

So $a_1 \leq \frac{7}{4}$ and $a_2 \leq \left(\frac{7}{4}\right)^2$, and thus $P(1)$ and $P(2)$ are both true.

Show that if $k \geq 2$ and $P(i)$ is true for all integers i from 1 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose

$$a_i \leq \left(\frac{7}{4}\right)^i \quad \text{for all integers } i \text{ with } 0 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that $$a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}.$$

Since $k \geq 2$,

$$
\begin{array}{rrcll}
& a_{k+1} & = & a_k + a_{k-1} & \text{by definition of } a_1, a_2, a_3, \ldots \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} & \text{by inductive hypothesis} \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}+1\right) & \text{by factoring out } (\tfrac{7}{4})^{k-1} \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^{k-1}\left(\frac{11}{4}\right) & \text{by adding } \tfrac{7}{4} \text{ and } 1 \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^{k-1}\left(\frac{44}{16}\right) & \text{by multiplying numerator and denominator of } \tfrac{11}{4} \text{ by } 4 \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^{k-1}\left(\frac{49}{16}\right) & \text{because } \tfrac{44}{16} < \tfrac{49}{16} \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2 & \text{because } (\tfrac{49}{16}) = (\tfrac{7}{4})^2 \\
\Rightarrow & a_k + a_{k-1} & \leq & \left(\frac{7}{4}\right)^{k+1} & \text{by a law of exponents.}
\end{array}
$$

Thus $a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$ *[as was to be shown].*

12. *Note*: This problem can be solved with ordinary mathematical induction.

Proof (by mathematical induction): Let the property $P(n)$ be the sentence

> Given any sequence of n cans of gasoline, deposited around a circular track in such a way that the total amount of gasoline is enough for a car to make one complete circuit of the track, it is possible to find an inital location for the car so that it will be able to traverse the entire track by using the various amounts of gasoline in the cans that it encounters along the way.

We will prove that $P(n)$ is true for all integers $n \geq 1$.

Show that P(1) is true: When there is just one can, the car should be placed next to it. By hypothesis, the can contains enough gasoline to enable the car to make one complete circuit of the track. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

> For all integers i with $1 \leq i \leq k$, given any sequence of n cans of gasoline, deposited around a circular track in such a way that the total amount of gasoline is enough for a car to make one complete circuit of the track, it is possible to find an inital location for the car so that it will be able to traverse the entire track by using the various amounts of gasoline in the cans that it encounters along the way. ← inductive hypothesis

We must show that

> Given any sequence of $k+1$ cans of gasoline, deposited around a circular track in such a way that the total amount of gasoline is enough for a car to make one complete circuit of the track, it is possible to find an inital location for the car so that it will be able to traverse the entire track by using the various amounts of gasoline in the cans that it encounters along the way.

Now, because the total amount of gasoline in all $k+1$ cans is enough for a car to make one complete circuit of the track, there must be at least one can, call it C, that contains enough gasoline to enable the car to reach the next can, say D, in the direction of travel along the track. Imagine pouring all the gasoline from D into C. The result would be k cans deposited around the track in such a way that the total amount of gasoline would be enough for a car to make one complete circuit of the track. By inductive hypothesis, it is possible to find an initial location for the car so that it could traverse the entire track by using the various amounts of gasoline in the cans that it encounters along the way. Use that location as the starting point for the car. When the car reaches can C, the amount of gasoline in C is enough to enable it to reach can D, and once the car reaches D, the additional amount of gasoline in D enables it to complete the circuit. *[This is what was to be shown.]*

15. *Note*: This solution makes free use of the properties from Chapter 4 about sums and differences for two even and odd integers.

Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

> Any sum of n even integers is even.

We will prove that $P(n)$ is true for all integers $n \geq 2$.

Show that $P(2)$ is true: $P(2)$ is true because any sum of two even integers is even.

Show that if $k \geq 2$ and $P(i)$ is true for all integers i from 2 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose that

> For all integers i from 2 through k, any sum of i even integers is even. ← inductive hypothesis

We must show that

any sum of $k+1$ even integers is even.

Consider any sum S of $k+1$ even integers. Some addition is the final one that is used to obtain S. Thus there are integers A and B such that $S = A + B$, A is a sum of r even integers, and B is a sum of $(k+1) - r$ even integers, where $1 \le r \le k$ and $1 \le (k+1) - r \le k$.

Case 1 (both $2 \le r \le k$ and $2 \le k+1) - r \le k$): In this case, by inductive hypothesis, both A and B are even, and hence $S = A + B$ is even.

Case 2 ($r = 1$ or $(k+1) - r = 1$): In this case, since $A + B = B + A$, we may assume without loss of generality that $r = 1$. Then A is a single even integer, and, since $2 \le (k+1) - r \le k$, B is even by inductive hypothesis. Hence $S = A + B$ is even.

Conclusion: It follows from cases 1 and 2 that any sum of $k+1$ even integers is even *[as was to be shown]*.

18. Conjecture: For all integers $n \ge 0$, the units digit of 9^n is 1 if n is even and is 9 if n is odd.

Proof (by strong mathematical induction): Let the property $P(n)$ be the sentence

The units digit of 9^n is 1 if n is even and is 9 if n is odd.

We will prove that $P(n)$ is true for all integers $n \ge 0$.

Show that $P(0)$ and $P(1)$ are true: $P(0)$ is true because 0 is even and the units digit of $9^0 = 1$. $P(1)$ is true because 1 is odd and the units digit of $9^1 = 9$.

Show that if $k \ge 1$ and $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \ge 1$, and suppose that

The units digit of 9^n is 1 if n is even and is 9 if n is odd. ← inductive hypothesis

We must show that

The units digit of 9^{k+1} is 1 if $k+1$ is even and is 9 if $k+1$ is odd.

Case 1 ($k+1$ is even): In this case k is odd, and so, by inductive hypothesis, the units digit of 9^k is 9. This implies that there is an integer a so that $9^k = 10a + 9$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+9) && \text{by substitution} \\ &= 90a + 81 \\ &= 90a + 80 + 1 \\ &= 10(9a+8) + 1 && \text{by algebra.} \end{aligned}$$

Because $9a + 8$ is an integer, it follows that the units digit of 9^{k+1} is 1.

Case 2 ($k+1$ is odd): In this case k is even, and so, by inductive hypothesis, the units digit of 9^{k-1} is 1. This implies that there is an integer a so that $9^k = 10a + 1$, and hence

$$\begin{aligned} 9^{k+1} &= 9^1 \cdot 9^k && \text{by algebra (a law of exponents)} \\ &= 9(10a+1) && \text{by substitution} \\ &= 90a + 9 \\ &= 10(9a) + 9 && \text{by algebra.} \end{aligned}$$

Because $9a$ is an integer, it follows that the units digit of 9^{k+1} is 9.

Hence in both cases the units digit of 9^{k+1} is as specified *[as was to be shown]*.

21. Proof by contradiction: Suppose not. That is, suppose that there exists an integer that is greater than 1, that is not prime, and that is not a product of primes. *[We will show that this supposition leads to a contradiction.]*

Let S be the set of all integers that are greater than 1, are not prime, and are not a product of primes. That is,

$$S = \{n \in \mathbf{Z} \mid n > 1, n \text{ is not prime, and } n \text{ is not a product of primes }\}.$$

Then, by supposition, S has one or more elements. By the well-ordering principle for the integers, S has a least element; call it m. Then m is greater than 1, is not prime, and is not a product of primes.

Now because m is greater than 1 and is not prime, $m = rs$ for some integers r and s with $1 < r < m$ and $1 < s < m$. Also, because both r and s are less than m, which is the least element of S, neither r nor s is in S. Thus both r and s are either prime or products of primes.

But this implies that m is a product of primes because m is a product of r and s. Thus m is not in S. So m is in S and m is not in S, which is a contradiction. *[Hence the supposition is false, and so every integer greater than 1 is either prime or a product of primes.]*

27. Suppose $P(n)$ is a property that is defined for integers n and suppose the following statement can be proved using strong mathematical induction:

$$P(n) \text{ is true for all integers } n \geq a.$$

Then for some integer $b \geq a$ the following two statements are true:

1. $P(a)$, $P(a+1)$, $P(a+2)$, ..., $P(b)$ are all true.

2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true.

We will show that we can reach the conclusion that $P(n)$ is true for all integers $n \geq a$ using ordinary mathematical induction.

Proof by mathematical induction: Let $Q(n)$ be the property

$$P(j) \text{ is true for all integers } j \text{ from } a \text{ through } n.$$

Show that $Q(b)$ is true: For $n = b$, the property is "$P(j)$ is true for all integers j with $a \leq j \leq b$." But this is true by (1) above.

Show that for all integers $k \geq b$, if $Q(k)$ is true then $Q(k+1)$ is true: Let k be any integer with $k \geq b$, and suppose that $Q(k)$ is true. In other words, suppose that

$$P(j) \text{ is true for all integers } j \text{ from } a \text{ through } k. \quad \leftarrow \text{ inductive hypothesis}$$

We must show that $Q(k+1)$ is true. In other words, we must show that

$$P(j) \text{ is true for all integers } j \text{ from } a \text{ through } k+1.$$

Since, by inductive hypothesis, $P(j)$ is true for all integers j from a through k, it follows from (2) above that $P(k+1)$ is also true. Hence $P(j)$ is true for all integers j from a through $k+1$, *[as was to be shown]*.

It follows by the principle of ordinary mathematical induction that $P(j)$ is true for all integers j from a through n for all integers $n \geq b$. From this and from (1) above, we conclude that $P(n)$ is true for all integers $n \geq a$.

30. Theorem: Given any nonnegative integer n and any positive integer d, there exist integers q and r such that $n = dq + r$ and $0 \le r < d$.

Proof by (ordinary) mathematical induction: Let a nonnegative integer d be given, and let the property $P(n)$ be the sentence

$$\begin{array}{ll} \text{There exist integers } q \text{ and } r \text{ such that} & \leftarrow P(n) \\ n = dq + r \quad \text{and} \quad 0 \le r < d. & \end{array}$$

We will prove that $P(n)$ is true for all integers $n \ge 0$.

Show that P(0) is true: We must show that there exist integers q and r such that

$$0 = dq + r \quad \text{and} \quad 0 \le r < d.$$

Let $q = r = 0$. Then

$$0 = d \cdot 0 + 0 \quad \text{and} \quad 0 \le 0 < d.$$

Hence $P(0)$ is true.

Show that for all integers $k \ge 0$, if $P(k)$ is true, then $P(k+1)$ is true: Let k be any integer with $k \ge 0$, and suppose that

$$\begin{array}{ll} \text{There exist integers } q' \text{ and } r' \text{ such that} & \leftarrow P(k) \\ k = dq' + r' \quad \text{and} \quad 0 \le r' < d. & \quad\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\begin{array}{ll} \text{There exist integers } q \text{ and } r \text{ such that} & \leftarrow P(k+1) \\ k+1 = dq + r \quad \text{and} \quad 0 \le r < d. & \end{array}$$

Adding 1 to both sides of the equation in the inductive hypothesis gives

$$k + 1 = (dq' + r') + 1.$$

Note that since r' is an integer and $0 \le r' < d$, then either $r' < d - 1$ or $r' = d - 1$.

Case 1 ($r' < d - 1$): In this case

$$k + 1 = (dq' + r') + 1 = dq' + (r' + 1).$$

Let $q = q'$ and $r = r' + 1$. Then, by substitution,

$$k + 1 = dq + r.$$

Since

$$r' < d - 1 \quad \text{then} \quad r = r' + 1 < d,$$

and since

$$r = r' + 1 \quad \text{and} \quad r' \ge 0 \quad \text{then} \quad r \ge 0.$$

Hence

$$0 \le r < d.$$

Case 2 ($r' = d - 1$): In this case

$$\begin{aligned} k + 1 &= (dq' + r') + 1 \\ &= dq' + (r' + 1) \\ &= dq' + ((d - 1) + 1) \\ &= dq' + d = d(q' + 1). \end{aligned}$$

Let $q = q' + 1$ and $r = 0$. Then by substitution

$$k + 1 = dq + r,$$

and since

$$r = 0 \quad \text{and} \quad d > 0,$$

then

$$0 \leq r < d.$$

Thus in either case there exist integers q and r such that $k + 1 = dq + r$ and $0 \leq r < d$ *[as was to be shown]*.

Section 5.5

9. Proof

I. Basis Property: $I(0)$ is the statement

both a and A are even integers or both are odd integers and, in either case, $a \geq -1$.

According to the pre-condition this statement is true.

II. Inductive Property: Suppose k is any nonnegative integer such that $G \wedge I(k)$ is true before an iteration of the loop. Then when execution comes to the top of the loop, $a_{\text{old}} > 0$ and

both a_{old} and A are even integers or both are odd integers and, in either case, $a_{\text{old}} \geq -1$.

Execution of statement 1 sets a_{new} equal to $a_{\text{old}} - 2$. Hence a_{new} has the same parity as a_{old} which is the same as A. Also since $a_{\text{old}} > 0$, then

$$a_{\text{new}} = a_{\text{old}} - 2 > 0 - 2 = -2.$$

But since $a_{\text{new}} > -2$ and since a_{new} is an integer, $a_{\text{new}} \geq -1$. Hence after the loop iteration, $I(k+1)$ is true.

III. Eventual Falsity of Guard: The guard G is the condition $a > 0$. After each iteration of the loop,

$$a_{\text{new}} = a_{\text{old}} - 2 < a_{\text{old}},$$

and so successive iterations of the loop give a strictly decreasing sequence of integer values of a which eventually becomes less than or equal to zero, at which point G becomes false.

IV. Correctness of the Post-Condition: Suppose that N is the least number of iterations after which G is false and $I(N)$ is true. Then (since G is false) $a \leq 0$ and (since $I(N)$ is true) both a and A are even integers or both are odd integers, and $a \geq -1$. Putting the inequalities together gives

$$-1 \leq a \leq 0,$$

and so since a is an integer, $a = -1$ or $a = 0$. Since a and A have the same parity, then, $a = 0$ if A is even and $a = -1$ if A is odd. This is the post-condition.

12. *a*. Suppose the following condition is satisfied before entry to the loop: "there exist integers u, v, s, and t such that $a = uA + vB$ and $b = sA + tB$." Then

$$a_{\text{old}} = u_{\text{old}}A + v_{\text{old}}B \quad \text{and} \quad b_{\text{old}} = s_{\text{old}}A + t_{\text{old}}B,$$

for some integers u_{old}, v_{old}, s_{old}, and t_{old}. Observe that $b_{new} = r_{new} = a_{old} \mod b_{old}$. So by the quotient-remainder theorem, there exists a unique integer q_{new} with $a_{old} = b_{old} \cdot q_{new} + r_{new} = b_{old} \cdot q_{new} + b_{new}$. Solving for b_{new} gives

$$\begin{aligned} b_{new} &= a_{old} - b_{old} \cdot q_{new} \\ &= (u_{old}A + v_{old}B) - (s_{old}A + t_{old}B)q_{new} \\ &= (u_{old} - s_{old}q_{new})A + (v_{old} - t_{old}q_{new})B. \end{aligned}$$

Therefore, let

$$s_{new} = u_{old} - s_{old}q_{new} \quad \text{and} \quad t_{new} = v_{old} - t_{old}q_{new}.$$

Also since

$$a_{new} = b_{old} = s_{old}A + t_{old}B,$$

let

$$u_{new} = s_{old} \quad \text{and} \quad v_{new} = t_{old}.$$

Hence

$$a_{new} = u_{new} \cdot A + v_{new} \cdot B \quad \text{and} \quad b_{new} = s_{new} \cdot A + t_{new} \cdot B,$$

and so the condition is true after each iteration of the loop and hence after exit from the loop.

b. Initially $a = A$ and $b = B$. Let $u = 1$, $v = 0$, $s = 0$, and $t = 1$. Then before the first iteration of the loop,

$$a = uA + vB \quad \text{and} \quad b = sA + tB,$$

as was to be shown.

c. By part (b) there exist integers u, v, s, and t such that before the first iteration of the loop,

$$a = uA + vB \quad \text{and} \quad b = sA + tB.$$

So by part (a), after each subsequent iteration of the loop, there exist integers u, v, s, and t such that

$$a = uA + vB \quad \text{and} \quad b = sA + tB.$$

Now after the final iteration of the **while** loop in the Euclidean algorithm, the variable gcd is given the current value of a. (See page 224.) But by the correctness proof for the Euclidean algorithm, $gcd = \gcd(A, B)$. Hence there exist integers u and v such that

$$\gcd(A, B) = uA + vB.$$

d. The method discussed in part (a) gives the following formulas for u, v, s, and t:

$$u_{new} = s_{old}, \quad v_{new} = t_{old}, \quad s_{new} = u_{old} - s_{old}q_{new}, \quad \text{and} \quad t_{new} = v_{old} - t_{old}q_{new},$$

where in each iteration q_{new} is the quotient obtained by dividing a_{old} by b_{old}. The trace table below shows the values of a, b, r, q, gcd, and u, v, s, and t for the iterations of the **while** loop from the Euclidean algorithm. By part (b) the initial values of u, v, s, and t are $u = 1$, $v = 0$, $s = 0$, and $t = 1$.

r		18	12	6	0
q		2	8	1	2
a	330	156	18	12	6
b	156	18	12	6	0
gcd					6
u	1	0	1	-8	9
v	0	1	-2	17	-19
s	0	1	-8	9	-26
t	1	-2	17	-19	55

Since the final values of gcd, u, and v are 6, 9 and -19 and since $A = 330$ and $B = 156$, we have $\gcd(330, 156) = 6 = 330u + 156v = 330 \cdot 9 + 156 \cdot (-19)$, which is true.

Section 5.6

6. $t_0 = -1, \quad t_1 = 2, \quad t_2 = t_1 + 2 \cdot t_0 = 2 + 2 \cdot (-1) = 0, \quad t_3 = t_2 + 2 \cdot t_1 = 0 + 2 \cdot 2 = 4$

12. For all integers $n \geq 0$, $s_n = \frac{(-1)^n}{n!}$. Thus for any integer k with $k \geq 1$,

$$s_k = \frac{(-1)^k}{k!} \quad \text{and} \quad s_{k-1} = \frac{(-1)^{k-1}}{(k-1)!}.$$

It follows that for any integer k with $k \geq 1$,

$$\begin{aligned} \frac{-s_{k-1}}{k} &= \frac{-\frac{(-1)^{k-1}}{(k-1)!}}{k} && \text{by substitution} \\ &= \frac{-(-1)^{k-1}}{k(k-1)!} \\ &= \frac{(-1)^k}{k!} \\ &= s_k && \text{by algebra.} \end{aligned}$$

Thus $s_k = \dfrac{-s_{k-1}}{k}$.

15. **<u>Proof</u>** : Let n be an integer with $n \geq 1$. Then

$$\begin{aligned} \frac{1}{4n+2}\binom{2n+2}{n+1} &= \left(\frac{1}{2(2n+1)}\right)\left(\frac{(2n+2)!}{(n+1)!((2n+2)-(n+1))!}\right) \\ &= \left(\frac{1}{2(2n+1)}\right)\left(\frac{(2n+2)!}{(n+1)!(n+1)!}\right) \\ &= \left(\frac{1}{2(2n+1)}\right)\left(\frac{(2n+2)(2n+1)(2n)!}{(n+1)\cdot n! \cdot (n+1)\cdot n!}\right) \\ &= \frac{1}{2}\left(\frac{2(n+1)}{(n+1)\cdot(n+1)}\right)\left(\frac{(2n)!}{n!\cdot n!}\right) \\ &= \frac{1}{n+1}\binom{2n}{n} \\ &= C_n. \end{aligned}$$

Thus $C_n = \dfrac{1}{4n+2}\dbinom{2n+2}{n+1}$.

18. *a.* $b_1 = 1, \quad b_2 = 1 + 1 + 1 + 1 = 4, \quad b_3 = 4 + 4 + 1 + 4 = 13$

c. Note that it takes just as many moves to move a stack of disks from the middle pole to an outer pole as from an outer pole to the middle pole: the moves are the same except that their order and direction are reversed. For all integers $k \geq 2$,

$$\begin{aligned} b_k &= a_{k-1} && \text{(moves to transfer the top } k-1 \text{ disks from pole } A \text{ to pole } C) \\ &\quad +1 && \text{(move to transfer the bottom disk from pole } A \text{ to pole } B) \\ &\quad +b_{k-1} && \text{(moves to transfer the top } k-1 \text{ disks from pole } C \text{ to pole } B). \\ &= a_{k-1} + 1 + b_{k-1}. \end{aligned}$$

d. One way to transfer a tower of k disks from pole A to pole B is first to transfer the top $k-1$ disks from pole A to pole B *[this requires b_{k-1} moves]*, then transfer the top $k-1$ disks from pole B to pole C *[this also requires b_{k-1} moves]*, then transfer the bottom disk from pole A to pole B *[this requires one move]*, and finally transfer the top $k-1$ disks from pole C to pole B *[this again requires b_{k-1} moves]*. This sequence of steps need not necessarily, however, result in a minimum number of moves. Therefore, at this point, all we can say for sure is that for all integers $k \geq 2$,

$$b_k \leq b_{k-1} + b_{k-1} + 1 + b_{k-1} = 3b_{k-1} + 1.$$

e. Proof (by mathematical induction): Let the property $P(k)$ be the equation

$$b_k = 3b_{k-1} + 1.$$

Show that $P(2)$ is true: The property is true for $k = 2$ because for $k = 2$ the left-hand side is 4 (by part (a)) and the right-hand side is $3 \cdot 1 + 1 = 4$ also.

Show that for all integers $i \geq 2$, if $P(i)$ is true then $P(i+1)$ is true: Let i be any integer with $i \geq 2$, and suppose that

$$b_i = 3b_{i-1} + 1. \qquad \leftarrow \begin{array}{l} P(i) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$b_{i+1} = 3b_i + 1. \qquad \leftarrow P(i+1)$$

But the left-hand side of $P(i+1)$ is

$$\begin{aligned} b_{i+1} &= a_i + 1 + b_i && \text{by part (c)} \\ &= a_i + 1 + 3b_{i-1} + 1 && \text{by inductive hypothesis} \\ &= (3a_{i-1} + 2) + 1 + 3b_{i-1} + 1 && \text{by exercise 17 (c)} \\ &= 3a_{i-1} + 3 + 3b_{i-1} + 1 && \\ &= 3(a_{i-1} + 1 + b_{i-1}) + 1 && \text{by algebra} \\ &= 3b_i + 1 && \text{by part (c) of this exercise,} \end{aligned}$$

which is the right-hand side of $P(i+1)$ *[as was to be shown]*.

21. *a.* $t_1 = 2, \quad t_2 = 2 + 2 + 2 = 6$

c. For all integers $k \geq 2$,

$$\begin{aligned} t_k = \;& t_{k-1} && \text{(moves to transfer the top } 2k-2 \text{ disks from pole } A \text{ to pole } B) \\ & +2 && \text{(moves to transfer the bottom two disks from pole } A \text{ to pole } C) \\ & +t_{k-1} && \text{(moves to transfer the top } 2k-2 \text{ disks from pole } B \text{ to pole } C) \\ = \;& 2t_{k-1} + 2. && \end{aligned}$$

Note that transferring the stack of $2k$ disks from pole A to pole C requires at least two transfers of the top $2(k-1)$ disks: one to transfer them off the bottom two disks to free the bottom disks so that they can be moved to pole C and another to transfer the top $2(k-1)$ disks back on top of the bottom two disks. Thus at least $2t_{k-1}$ moves are needed to effect these two transfers. Two more moves are needed to transfer the bottom two disks from pole A to pole C, and this transfer cannot be effected in fewer than two moves. It follows that the sequence of moves indicated in the description of the equation above is, in fact, minimal.

24. $F_{13} = F_{12} + F_{11} = 233 + 144 = 377, \quad F_{14} = F_{13} + F_{12} = 377 + 233 = 610$

30. Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^n. \qquad \leftarrow P(n)$$

Show that P(0) is true: The left-hand side of $P(0)$ is $F_{0+2}F_0 - {F_1}^2 = 2\cdot 1 - 1^2 = 1$, and the right-hand side is $(-1)^0 = 1$ also. So $P(0)$ is true.

Show that for all integers k ≥ 0, if P(k) is true then P(k + 1) is true: Let k be any integer with $k \geq 0$, and suppose that

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^k. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$F_{k+3}F_{k+1} - F_{k+2}^2 = (-1)^{k+1}. \qquad \leftarrow P(k+1)$$

But by inductive hypothesis,

$$F_{k+1}^2 = F_{k+2}F_k - (-1)^k = F_{k+2}F_k + (-1)^{k+1}. \quad (*)$$

Hence,

$$
\begin{aligned}
& F_{k+3}F_{k+1} - F_{k+2}^2 \\
= \; & (F_{k+1} + F_{k+2})F_{k+1} - F_{k+2}^2 && \text{by definition of the Fibonacci sequence} \\
= \; & F_{k+1}^2 + F_{k+2}F_{k+1} - F_{k+2}^2 \\
= \; & F_{k+2}F_k + (-1)^{k+1} + F_{k+2}F_{k+1} - F_{k+2}^2 && \text{by substitution from equation (*)} \\
= \; & F_{k+2}(F_k + F_{k+1} - F_{k+2}) + (-1)^{k+1} && \text{by factoring out } F_{k+2} \\
= \; & F_{k+2}(F_{k+2} - F_{k+2}) + (-1)^{k+1} && \text{by definition of the Fibonacci sequence} \\
= \; & F_{k+2} \cdot 0 + (-1)^{k+1} \\
= \; & (-1)^{k+1}.
\end{aligned}
$$

36. Let $L = \lim_{n\to\infty} x_n$. By definition of $x_0, x_1, x_2, \ldots$ and by the continuity of the square root function,

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{2 + x_{n-1}} = \sqrt{2 + \lim_{n\to\infty} x_{n-1}} = \sqrt{2 + L}.$$

Hence $L^2 = 2 + L$, and so $L^2 - L - 2 = 0$. Factoring gives $(L-2)(L+1) = 0$, and so $L = 2$ or $L = -1$. But $L \geq 0$ because each $x_i \geq 0$. Thus $L = 2$.

42. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any real numbers, then $\prod_{i=1}^n (a_i b_i) = \left(\prod_{i=1}^n a_i\right)\left(\prod_{i=1}^n b_i\right)$. $\leftarrow P(n)$

Show that P(1) is true: Let a_1 and b_1 be any real numbers. By the recursive definition of product,

$$\prod_{i=1}^{1}(a_i b_i) = a_1 b_1, \prod_{i=1}^{1} a_i = a_1, \mathit{and} \prod_{i=1}^{1} b_i = b_1.$$

Show that for all integers k ≥ 1 , if P(k) is true then P(k + 1) is true: Let k be any integer with $k \geq 1$, and suppose that

If $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ are any real numbers, then $\prod_{i=1}^k (a_i b_i) = \left(\prod_{i=1}^k a_i\right)\left(\prod_{i=1}^k b_i\right)$. $\leftarrow$ $P(k)$ inductive hypothesis

We must show that

If $a_1, a_2, \ldots, a_{k+1}$ and $b_1, b_2, \ldots, b_{k+1}$ are any real numbers, then $\prod_{i=1}^{k+1}(a_ib_i) = \left(\prod_{i=1}^{k+1} a_i\right)\left(\prod_{i=1}^{k+1} b_i\right)$. $\leftarrow P(k+1)$

So suppose $a_1, a_2, \ldots, a_{k+1}$ and $b_1, b_2, \ldots, b_{k+1}$ are any real numbers. Then

$\prod_{i=1}^{k+1}(a_ib_i)$

$= \left(\prod_{i=1}^{k}(a_ib_i)\right)(a_{k+1}b_{k+1})$ by the recursive definition of product

$= \left(\left(\prod_{i=1}^{k} a_i\right)\left(\prod_{i=1}^{k} b_i\right)\right)(a_{k+1}b_{k+1})$ by substitution from the inductive hypothesis

$= \left(\left(\prod_{i=1}^{k} a_i\right)a_{k+1}\right)\left(\left(\prod_{i=1}^{k} b_i\right)b_{k+1}\right)$ by the associative and commutative laws of algebra

$= \left(\prod_{i=1}^{k+1} a_i\right)\left(\prod_{i=1}^{k+1} b_i\right)$ by the recursive definition of product.

[This is what was to be shown.]

Section 5.7

6.
$$
\begin{aligned}
d_1 &= 2\\
d_2 &= 2d_1 + 3 = 2\cdot 2 + 3 = 2^2 + 3\\
d_3 &= 2d_2 + 3 = 2(2^2+3) + 3 = 2^3 + 2\cdot 3 + 3\\
d_4 &= 2d_3 + 3 = 2(2^3 + 2\cdot 3 + 3) + 3 = 2^4 + 2^2\cdot 3 + 2\cdot 3 + 3\\
d_5 &= 2d_4 + 3 = 2(2^4 + 2^2\cdot 3 + 2\cdot 3 + 3) + 3 = 2^5 + 2^3\cdot 3 + 2^2\cdot 3 + 2\cdot 3 + 3\\
&\vdots
\end{aligned}
$$

Guess:
$$
\begin{aligned}
d_n &= 2^n + 2^{n-2}\cdot 3 + 2^{n-3}\cdot 3 + \cdots + 2^2\cdot 3 + 2\cdot 3 + 3\\
&= 2^n + 3(2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2 + 1)\\
&= 2^n + 3\left(\frac{2^{(n-2)+1} - 1}{2-1}\right) \quad \textit{[by Theorem 5.2.3]}\\
&= 2^n + 3(2^{n-1} - 1)\\
&= 2^{n-1}(2+3) - 3 = 5\cdot 2^{n-1} - 3 \quad \text{for all integers } n \ge 1
\end{aligned}
$$

9.
$$
\begin{aligned}
g_1 &= 1\\
g_2 &= \frac{g_1}{g_1+2} = \frac{1}{1+2}\\
g_3 &= \frac{g_2}{g_2+2} = \frac{\frac{1}{1+2}}{\frac{1}{1+2}+2} = \frac{1}{1+2(1+2)} = \frac{1}{1+2+2^2}\\
g_4 &= \frac{g_3}{g_3+2} = \frac{\frac{1}{1+2+2^2}}{\frac{1}{1+2+2^2}+2} = \frac{1}{1+2(1+2+2^2)} = \frac{1}{1+2+2^2+2^3}\\
g_5 &= \frac{g_4}{g_4+2} = \frac{\frac{1}{1+2+2^2+2^3}}{\frac{1}{1+2+2^2+2^3}+2} = \frac{1}{1+2(1+2+2^2+2^3)} = \frac{1}{1+2+2^2+2^3+2^4}\\
&\vdots
\end{aligned}
$$

Guess:
$$
\begin{aligned}
g_n &= \frac{1}{1+2+2^2+2^3+\cdots+2^{n-1}}\\
&= \frac{1}{2^n - 1} \quad \textit{[by Theorem 5.2.3)]} \quad \text{for all integers } n \ge 1
\end{aligned}
$$

15.
$$\begin{aligned} y_1 &= 1 \\ y_2 &= y_1 + 2^2 = 1 + 2^2 \\ y_3 &= y_2 + 3^2 = (1 + 2^2) + 3^2 = 1 + 2^2 + 3^2 \\ y_4 &= y_3 + 4^2 = (1 + 2^2 + 3^2) + 4^2 = 1^2 + 2^2 + 3^2 + 4^2 \\ &\ \vdots \end{aligned}$$

Guess:
$$y_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \text{ by exercise 10 of Section 5.2}$$

21. Proof (by mathematical induction): Let r be a fixed constant and $a_0, a_1, a_2, \ldots$. a sequence that satisfies the recurrence relation $a_k = ra_{k-1}$ for all integers $k \geq 1$ and the initial condition $a_0 = a$. Let the property $P(n)$ be the equation $a_n = ar^n$.

$$a_n = ar^n.$$

Show that $P(0)$ is true: The right-hand side of $P(0)$ is $ar^0 = a \cdot 1 = a$, which is the left-hand side of $P(0)$. So $P(0)$ is true.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose that

$$a_k = ar^k. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$a_{k+1} = ar^{k+1}. \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= ra_k && \text{by definition of } a_0, a_1, a_2, \ldots \\ &= r(ar^k) && \text{by inductive hypothesis} \\ &= ar^{k+1} && \text{by the laws of exponents,} \end{aligned}$$

which is the right-hand side of $P(k+1)$, *[as to be shown]*.

27. *a.* Let the original balance in the account be A dollars, and let A_n be the amount owed in month n assuming the balance is not reduced by making payments during the year. The annual interest rate is 18%, and so the monthly interest rate is (18/12)% = 1.5% = 0.015. The sequence $A_0, A_1, A_2, \ldots$ satisfies the recurrence relation

$$A_k = A_{k-1} + 0.015A_{k-1} = 1.015A_{k-1}.$$

Thus
$$\begin{aligned} A_1 &= 1.015A_0 = 1.015A \\ A_2 &= 1.015A_1 = 1.015(1.015A) = (1.015)^2 A \\ &\ \vdots \\ A_{12} &= 1.015A_{11} = 1.015(1.015)^{11}A = (1.015)^{12}A. \end{aligned}$$

So the amount owed at the end of the year is $(1.015)^{12}A$. It follows that the APR is

$$\frac{(1.015)^{12}A - A}{A} = \frac{A((1.015)^{12} - 1)}{A} = (1.015)^{12} - 1 \cong 19.6\%.$$

Note: Because $A_k = 1.015A_{k-1}$ for each integer $k \geq 1$, we could have immediately concluded that the sequence is geometric and, therefore, satisfies the equation $A_n = A_0(1.015)^n = A(1.015)^n$.

b. Because the person pays \$150 per month to pay off the loan, the balance at the end of month k is $B_k = 1.015B_{k-1} - 150$. We use iteration to find an explicit formula for $B_0, B_1, B_2, \ldots$.

$$\begin{aligned}
B_0 &= 3000\\
B_1 &= (1.015)B_0 - 150 = 1.015(3000) - 150\\
B_2 &= (1.015)B_1 - 150 = (1.015)[1.015(3000) - 150] - 150\\
&= 3000(1.015)^2 - 150(1.015) - 150\\
B_3 &= (1.015)B_2 - 150 = (1.015)[3000(1.015)^2 - 150(1.015) - 150] - 150\\
&= 3000(1.015)^3 - 150(1.015)^2 - 150(1.015) - 150\\
B_4 &= (1.015)B_3 - 150\\
&= (1.015)[3000(1.015)^3 - 150(1.015)^2 - 150(1.015) - 150] - 150\\
&= 3000(1.015)^4 - 150(1.015)^3 - 150(1.015)^2 - 150(1.015) - 150\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\text{Guess: } B_n &= 3000(1.015)^n + [150(1.015)^{n-1} - 150(1.015)^{n-2} + \ldots\\
&\qquad -150(1.015)^2 - 150(1.015) - 150]\\
&= 3000(1.015)^n - 150[(1.015)^{n-1} + (1.015)^{n-2} + \cdots + (1.015)^2 + 1.015 + 1]\\
&= 3000(1.015)^n - 150\left(\frac{(1.015)^n - 1}{1.015 - 1}\right)\\
&= (1.015)^n(3000) - \frac{150}{0.015}((1.015)^n - 1)\\
&= (1.015)^n(3000) - 10000((1.015)^n - 1)\\
&= (1.015)^n(3000 - 10000) + 10000\\
&= (-7000)(1.015)^n + 10000
\end{aligned}$$

So it appears that $B_n = (-7000)(1.015)^n + 10000$. We use mathematical induction to confirm this guess.

Proof (by mathematical induction): : Let $B_0, B_1, B_2, \ldots$. be a sequence that satisfies the recurrence relation $B_k = (1.015)B_{k-1} - 150$ for all integers $k \geq 1$, with initial condition $B_0 = 3000$, and let the property $P(n)$ be the equation

$$B_n = (-7000)(1.015)^n + 10000. \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: The right-hand side of $P(0)$ is $(-7000)(1.015)^0 + 10000 = 3000$, which equals B_0, the left-hand side of $P(0)$. So $P(0)$ is true.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$, and suppose that

$$B_k = (-7000)(1.015)^k + 10000. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$B_{k+1} = (-7000)(1.015)^{k+1} + 10000. \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
B_{k+1} &= (1.015)B_k - 150 && \text{by definition of } B_0, B_1, B_2, \ldots\\
&= (1.015)[(-7000)(1.015)^k + 10000] - 150 && \text{by substitution from the inductive hypothesis}\\
&= (-7000)(1.015)^{k+1} + 10150 - 150\\
&= (-7000)(1.015)^{k+1} + 10000 && \text{by the laws of algebra,}
\end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as to be shown]*.

c. By part (b), $B_n = (-7000)(1.015)^n + 10000$, and so we need to find the value of n for which

$$(-7000)(1.015)^n + 10000 = 0.$$

But this equation holds

$$\begin{aligned}
&\Leftrightarrow \quad 7000(1.015)^n = 10000 \\
&\Leftrightarrow \quad (1.015)^n = \frac{10000}{7000} = \frac{10}{7} \\
&\Leftrightarrow \quad \log_{10}(1.015)^n = \log_{10}\left(\frac{10}{7}\right) \quad \text{by a property of logarithms} \\
&\Leftrightarrow \quad n\log_{10}(1.015) = \log_{10}\left(\frac{10}{7}\right) \quad \text{by a property of logarithms} \\
&\Leftrightarrow \quad n = \frac{\log_{10}(10/7)}{\log_{10}(1.015)} \cong 24.
\end{aligned}$$

So $n \cong 24$ months $= 2$ years. It will require approximately 2 years to pay off the balance, assuming that payments of \$150 are made each month and the balance is not increased by any additional purchases.

d. Assuming that the person makes no additional purchases and pays \$150 each month, the person will have made 24 payments of \$150 each, for a total of \$3600 to pay off the initial balance of \$3000.

33. Proof (by mathematical induction): : Let $f_1, f_2, f_3, \ldots$. be a sequence that satisfies the recurrence relation $f_k = f_{k-1} + 2^k$ for all integers $k \geq 2$, with initial condition $f_1 = 1$, and let the property $P(n)$ be the equation

$$f_n = 2^{n+1} - 3. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: The right-hand side of $P(1)$ is $2^{1+1} - 3 = 2^2 - 3 = 1$, which equals f_1, the left-hand side of $P(1)$. So $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$f_k = 2^{k+1} - 3. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$f_{k+1} = 2^{(k+1)+1} - 3$$

or, equivalently,

$$f_{k+1} = 2^{k+2} - 3. \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
f_{k+1} &= f_k + 2^{k+1} && \text{by definition of } f_1, f_2, f_3, \ldots \\
&= 2^{k+1} - 3 + 2^{k+1} && \text{by inductive hypothesis} \\
&= 2 \cdot 2^{k+1} - 3 \\
&= 2^{k+2} - 3 && \text{by the laws of algebra,}
\end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as to be shown]*.

36. Proof (by mathematical induction): Let $p_1, p_2, p_3, \ldots$ be a sequence that satisfies the recurrence relation $p_k = p_{k-1} + 2 \cdot 3^k$ for all integers $k \geq 2$, with initial condition $p_1 = 2$, and let the property $P(n)$ be the equation

$$p_n = 3^{n+1} - 7. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: The right-hand side of $P(1)$ is $3^{1+1} - 7 = 3^2 - 7 = 9 - 7 = 2$, which equals p_1, the left-hand side of $P(1)$. So $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$p_k = 3^{k+1} - 7. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$p_{k+1} = 3^{(k+1)+1} - 7,$$

or, equivalently, that

$$p_{k+1} = 3^{k+2} - 7. \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned} p_{k+1} &= p_k + 2 \cdot 3^{k+1} && \text{by definition of } p_1, p_2, p_3, \ldots \\ &= (3^{k+1} - 7) + 2 \cdot 3^{k+1} && \text{by inductive hypothesis} \\ &= 3^{k+1}(1+2) - 7 \\ &= 3 \cdot 3^{k+1} - 7 \\ &= 3^{k+2} - 7 && \text{by the laws of algebra,} \end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as to be shown]*.

42. Proof (by mathematical induction): Let $t_1, t_2, t_3, \ldots$ be a sequence that satisfies the recurrence relation $t_k = 2t_{k-1} + 2$ for all integers $k \geq 2$, with initial condition $t_1 = 2$, and let the property $P(n)$ be the equation $t_n = 2^{n+1} - 2$.

$$t_n = 2^{n+1} - 2. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: The right-hand side of $P(1)$ is $2^2 - 2 = 2$, which equals t_1, the left-hand side of $P(1)$. So $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$t_k = 2^{k+1} - 2. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$t_{k+1} = 2^{(k+1)+1} - 2$$

or, equivalently, that

$$t_{k+1} = 2^{k+2} - 2. \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned} t_{k+1} &= 2t_k + 2 && \text{by definition of } t_1, t_2, t_3, \ldots \\ &= 2(2^{k+1} - 2) + 2 && \text{by inductive hypothesis} \\ &= 2^{k+2} - 4 + 2 \\ &= 2^{k+2} - 2 && \text{by the laws of algebra,} \end{aligned}$$

which is the right-hand side of $P(k+1)$ *[as to be shown]*.

48. *a.*

$$\begin{aligned} w_1 &= 1 \\ w_2 &= 2 \\ w_3 &= w_1 + 3 = 1 + 3 \\ w_4 &= w_2 + 4 = 2 + 4 \\ w_5 &= w_3 + 5 = 1 + 3 + 5 \\ w_6 &= w_4 + 6 = 2 + 4 + 6 \\ w_7 &= w_5 + 7 = 1 + 3 + 5 + 7 \\ &\vdots \end{aligned}$$

Guess:

$$w_n = \begin{cases} 1 + 3 + 5 + \cdots + n & \text{if } n \text{ is odd} \\ 2 + 4 + 6 + \cdots + n & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} \left(\dfrac{n+1}{2}\right)^2 & \text{if } n \text{ is odd} \\ 2\left(1 + 2 + 3 + \cdots + \dfrac{n}{2}\right) & \text{if } n \text{ is even} \end{cases} \quad \text{by exercise 5 of Section 5.2}$$

$$= \begin{cases} \left(\dfrac{n+1}{2}\right)^2 & \text{if } n \text{ is odd} \\ 2\left(\dfrac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\right) & \text{if } n \text{ is even} \end{cases} \quad \text{by Theorem 5.2.2}$$

$$= \begin{cases} \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd} \\ \dfrac{n(n+2)}{4} & \text{if } n \text{ is even} \end{cases} \quad \text{by the laws of algebra.}$$

b. Proof (by strong mathematical induction): Let $w_1, w_2, w_3, \ldots$ be a sequence that satisfies the recurrence relation $w_k = w_{k-2} + k$ for all integers $k \geq 3$, with initial conditions $w_1 = 1$ and $w_2 = 2$, and let the property $P(n)$ be the equation

$$w_n = \begin{cases} \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd} \\ \dfrac{n(n+2)}{4} & \text{if } n \text{ is even} \end{cases} \quad \text{for all integers } n \geq 1.$$

Show that P(1) and P(2) are true: For $n = 1$ and $n = 2$ the right-hand sides of $P(n)$ are

$$\frac{(1+1)^2}{4} = 1 \quad \text{and} \quad \frac{2(2+2)}{4} = 2,$$

which equal w_1 and w_2 respectively. So $P(1)$ and $P(2)$ are true.

Show that if $k \geq 2$ and $P(i)$ is true for all integers i from 1 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 2$, and suppose that

$$w_i = \begin{cases} \dfrac{(i+1)^2}{4} & \text{if } i \text{ is odd} \\ \dfrac{i(i+2)}{4} & \text{if } i \text{ is even} \end{cases} \quad \text{for all integers } i \text{ with } 1 \leq i < k. \quad \leftarrow \text{inductive hypothesis}$$

We must show that

$$w_{k+1} = \begin{cases} \dfrac{(k+2)^2}{4} & \text{if } k+1 \text{ is odd} \\ \dfrac{(k+1)(k+3)}{4} & \text{if } k+1 \text{ is even} \end{cases}$$

But

$$
\begin{aligned}
w_{k+1} &= w_{k-1} + (k+1) && \text{by definition of } w_1, w_2, w_3, \ldots \\
&= \begin{cases} \dfrac{((k-1)+1)^2}{4} + (k+1) & \text{if } k-1 \text{ is odd} \\ \dfrac{(k-1)[(k-1)+2]}{4} + (k+1) & \text{if } k-1 \text{ is even} \end{cases} && \text{by inductive hypothesis} \\
&= \begin{cases} \dfrac{k^2}{4} + \dfrac{4(k+1)}{4} & \text{if } k+1 \text{ is odd} \\ \dfrac{(k-1)(k+1)}{4} + \dfrac{4(k+1)}{4} & \text{if } k+1 \text{ is even} \end{cases} && \text{because } k-1 \text{ and } k+1 \text{ have the same parity} \\
&= \begin{cases} \dfrac{k^2+4k+4}{4} & \text{if } k+1 \text{ is odd} \\ \dfrac{k^2+4k+3}{4} & \text{if } k+1 \text{ is even} \end{cases} \\
&= \begin{cases} \dfrac{(k+2)^2}{4} & \text{if } k+1 \text{ is odd} \\ \dfrac{(k+1)(k+3)}{4} & \text{if } k+1 \text{ is even} \end{cases} && \text{by the laws of algebra.}
\end{aligned}
$$

[This is what was to be shown.]

51. The sequence does not satisfy the formula. By definition of $a_1, a_2, a_3, \ldots$, $a_1 = 0$, $a_2 = (a_1+1)^2 = 1^2 = 1$, $a_3 = (a_2+1)^2 = (1+1)^2 = 4$, $a_4 = (a_3+1)^2 = (4+1)^2 = 25$. But according to the formula $a_4 = (4-1)^2 = 9 \neq 25$.

54. *a.*

$Y_1 = E + c + mY_0$

$Y_2 = E + c + mY_1 = E + c + m(E + c + mY_0) = (E+c) + m(E+c) + m^2Y_0$

$Y_3 = E+c+mY_2 = E+c+m((E+c)+m(E+c)+m^2Y_0) = (E+c)+m(E+c)+m^2(E+c)+m^3Y_0$

$Y_4 = E + c + mY_3 = E + c + m((E+c) + m(E+c) + m^2(E+c) + m^3Y_0)$

$\quad = (E+c) + m(E+c) + m^2(E+c) + m^3(E+c) + m^4Y_0$

.

.

.

Guess: $Y_n = (E+c) + m(E+c) + m^2(E+c) + \cdots + m^{n-1}(E+c) + m^nY_0$

$\qquad = (E+c)[1 + m + m^2 + \cdots + m^{n-1}] + m^nY_0$

$\qquad = (E+c)\left(\dfrac{m^n-1}{m-1}\right) + m^nY_0$, for all integers $n \geq 1$.

b. Suppose $0 < m < 1$. Then

$$\begin{aligned}
\lim_{n\to\infty} Y_n &= \lim_{n\to\infty}\left((E+c)\left(\frac{m^n-1}{m-1}\right)+m^nY_0\right)\\
&= (E+c)\left(\frac{\lim_{n\to\infty} m^n-1}{m-1}\right)+\lim_{n\to\infty} m^nY_0\\
&= (E+c)\left(\frac{0-1}{m-1}\right)+0\cdot Y_0 \qquad \text{because when } 0<m<1, \text{ then } \lim_{n\to\infty} m^n = 0\\
&= \frac{E+c}{1-m}.
\end{aligned}$$

Section 5.8

3. *b*.

$$\left\{\begin{array}{l} a_0 = C\cdot 2^0 + D = C + D = 0 \\ a_1 = C\cdot 2^1 + D = 2C + D = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D = -C \\ 2C + (-C) = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D = -C \\ C = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C = 2 \\ D = -2 \end{array}\right\}$$

$a_2 = C\cdot 2^2 + D = 2\cdot 2^2 + (-2) = 6$

6. <u>Proof</u>: Given that $b_n = C\cdot 3^n + D(-2)^n$, then for any choice of C and D and integer $k \geq 2$,

$$b_k = C\cdot 3^k + D\cdot(-2)^k,\quad b_{k-1} = C\cdot 3^{k-1} + D\cdot(-2)^{k-1},\quad \text{and}\quad b_{k-2} = C\cdot 3^{k-2} + D\cdot(-2)^{k-2}.$$

Hence,

$$\begin{aligned}
b_{k-1} + 6b_{k-2} &= \left(C\cdot 3^{k-1} + D(-2)^{k-1}\right) + 6\left(C\cdot 3^{k-2} + D\cdot(-2)^{k-2}\right)\\
&= C\cdot\left(3^{k-1} + 6\cdot 3^{k-2}\right) + D\cdot\left((-2)^{k-1} + 6\cdot(-2)^{k-2}\right)\\
&= C\cdot 3^{k-2}(3+6) + D\cdot(-2)^{k-2}(-2+6)\\
&= C\cdot 3^{k-2}\cdot 3^2 + D\cdot(-2)^{k-2}2^2\\
&= C\cdot 3^k + D\cdot(-2)^k\\
&= b_k.
\end{aligned}$$

9. *a*. If for all integers $k \geq 2$, $t^k = 7t^{k-1} - 10t^{k-2}$ and $t \neq 0$, then $t^2 = 7t - 10$ and so $t^2 - 7t + 10 = 0$. But $t^2 - 7t + 10 = (t-2)(t-5)$. Thus $t = 2$ or $t = 5$.

b. It follows from part (a) and the distinct roots theorem that for some constants C and D, the terms of the sequence $b_0, b_1, b_2, \ldots$ satisfy the equation

$$b_n = C\cdot 2^n + D\cdot 5^n \quad \text{for all integers } n \geq 0.$$

Since $b_0 = 2$ and $b_1 = 2$, then

$$\left\{\begin{array}{l} b_0 = C\cdot 2^0 + D\cdot 5^0 = C + D = 2 \\ b_1 = C\cdot 2^1 + D\cdot 5^1 = 2C + 5D = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D = 2 - C \\ 2C + 5(2-C) = 2 \end{array}\right\}$$

$$\Leftrightarrow \left\{\begin{array}{l} D = 2 - C \\ C = 8/3 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D = 2 - (8/3) = -(2/3) \\ C = 8/3 \end{array}\right\}$$

Thus $b_n = \dfrac{8}{3}\cdot 2^n - \dfrac{2}{3}\cdot 5^n$ for all integers $n \geq 0$.

12. The characteristic equation is $t^2 - 9 = 0$. Since $t^2 - 9 = (t-3)(t+3)$, the roots are $t = 3$ and $t = -3$. By the distinct roots theorem, there exist constants C and D such that

$$e_n = C\cdot 3^n + D\cdot(-3)^n \quad \text{for all integers } n \geq 0.$$

Since $e_0 = 0$ and $e_1 = 2$, then

$$\left\{\begin{array}{l} e_0 = C\cdot 3^0 + D\cdot(-3)^0 = C + D = 0 \\ e_1 = C\cdot 3^1 + D\cdot(-3)^1 = 3C - 3D = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D = -C \\ 3C - 3(-C) = 2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} D = -1/3 \\ C = 1/3 \end{array}\right\}$$

Thus $e_n = \frac{1}{3}\cdot 3^n - \frac{1}{3}(-3)^n = 3^{n-1} + (-3)^{n-1} = 3^{n-1}(1+(-1)^{n-1}) = \begin{cases} 2\cdot 3^{n-1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ for all integers $n \geq 0$.

15. The characteristic equation is $t^2 - 6t + 9 = 0$. Since $t^2 - 6t + 9 = (t-3)^2$, there is only one root, $t = 3$. By the single root theorem, there exist constants C and D such that

$$t_n = C\cdot 3^n + D\cdot n\cdot 3^n \quad \text{for all integers } n \geq 0.$$

Since $t_0 = 1$ and $t_1 = 3$, then

$$\left\{\begin{array}{l} t_0 = C\cdot 3^0 + D\cdot 0\cdot 3^0 = C = 1 \\ t_1 = C\cdot 3^1 + D\cdot 1\cdot 3^1 = 3C + 3D = 3 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C = 1 \\ C + D = 1 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} C = 1 \\ D = 0 \end{array}\right\}$$

Thus $t_n = 1\cdot 3^n + 0\cdot n\cdot 3^n = 3^n$ for all integers $n \geq 0$.

18. Proof: Suppose that $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ are sequences such that for all integers $k \geq 2$,

$$s_k = 5s_{k-1} - 4s_{k-2} \quad \text{and} \quad t_k = 5t_{k-1} - 4t_{k-2}.$$

Then for all integers $k \geq 2$,

$$\begin{aligned} 5(2s_{k-1} + 3t_{k-1}) - 4(2s_{k-2} + 3t_{k-2}) &= (5\cdot 2s_{k-1} - 4\cdot 2s_{k-2}) + (5\cdot 3t_{k-1} - 4\cdot 3t_{k-2}) \\ &= 2(5s_{k-1} - 4s_{k-2}) + 3(5t_{k-1} - 4t_{k-2}) \\ &= 2s_k + 3t_k. \end{aligned}$$

[This is what was to be shown.]

21. Let $a_0, a_1, a_2, \ldots$ be any sequence that satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$ for some real numbers A and B with $B \neq 0$ and for all integers $k \geq 2$. Furthermore, suppose that the equation $t^2 - At - B = 0$ has a single real root r. First note that $r \neq 0$ because otherwise we would have $0^2 - A\cdot 0 - B = 0$, which would imply that $B = 0$ and contradict the hypothesis. Second, note that the following system of equations with unknowns C and D has a unique solution.

$$\begin{aligned} a_0 &= Cr^0 + 0\cdot Dr^0 = 1\cdot C + 0\cdot D \\ a_1 &= Cr^1 + 1\cdot Dr^1 = C\cdot r + D\cdot r \end{aligned}$$

One way to reach this conclusion is to observe that the determinant of the system is $1\cdot r - r\cdot 0 = r \neq 0$. Another way to reach the conclusion is to write the system as

$$\begin{aligned} a_0 &= C \\ a_1 &= Cr + Dr \end{aligned}$$

and let $C = a_0$ and $D = (a_1 - a_0 r)/r$. It is clear by substitution that these values of C and D satisfy the system. Conversely, if any numbers C and D satisfy the system, then $C = a_0$, and substituting C into the second equation and solving for D yields $D = (a_1 - Cr)/r$.

Proof of the exercise statement by strong mathematical induction: Let $a_0, a_1, a_2, \ldots$ be any sequence that satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$ for some real numbers A and B with $B \neq 0$ and for all integers $k \geq 2$. Furthermore, suppose that the equation $t^2 - At - B = 0$ has a single real root r. Let the property $P(n)$ be the equation

$$a_n = Cr^n + nDr^n \qquad \leftarrow P(n)$$

where C and D are the unique real numbers such that $a_0 = Cr^0 + 0\cdot Dr^0$ and $a_1 = Cr^1 + 1\cdot Dr^1$.

Show that $P(0)$ and $P(1)$ are true: : The fact $P(0)$ and $P(1)$ are true is automatic because C and D are exactly those numbers for which $a_0 = Cr^0 + 0 \cdot Dr^0$ and $a_1 = C \cdot r^1 + 1 \cdot Dr^1$.

Show that if $k \geq 1$ and $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$ and suppose that

$$a_i = Cr^i + iDr^i \quad \text{for all integers } i \text{ with } 1 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$a_{k+1} = Cr^{k+1} + (k+1)Dr^{k+1}.$$

Now by the inductive hypothesis,

$$a_k = Cr^k + kDr^k \quad \text{and} \quad a_{k-1} = Cr^{k-1} + (k-1)Dr^{k-1}.$$

So

$$\begin{aligned} a_{k+1} &= Aa_k + Ba_{k-1} && \text{by definition of } a_0, a_1, a_2, \ldots \\ &= A(Cr^k + kDr^k) + B(Cr^{k-1} + (k-1)Dr^{k-1}) \\ & && \text{by inductive hypothesis} \\ &= C(Ar^k + Br^{k-1}) + D(Akr^k + B(k-1)r^{k-1}) \\ & && \text{by algebra} \\ &= Cr^{k+1} + Dkr^{k+1} && \text{by Lemma 5.8.4.} \end{aligned}$$

[This is what was to be shown.]

24. *a.* If $\dfrac{\phi}{1} = \dfrac{1}{\phi - 1}$, then $\phi(\phi - 1) = 1$, or, equivalently, $\phi^2 - \phi - 1 = 0$ and so ϕ satisfies the equation $t^2 - t - 1 = 0$.

b. By the quadratic formula, the solutions to $t^2 - t - 1 = 0$ are

$$t = \frac{1 \pm \sqrt{1+4}}{2} = \begin{cases} \dfrac{1+\sqrt{5}}{2} \\ \dfrac{1-\sqrt{5}}{2}. \end{cases}$$

Let

$$\phi_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \phi_2 = \frac{1-\sqrt{5}}{2}.$$

c. $F_n = \dfrac{1}{\sqrt{5}} \cdot \phi_1^{n+1} - \dfrac{1}{\sqrt{5}} \cdot \phi_2^{n+1} = \dfrac{1}{\sqrt{5}}(\phi_1^{n+1} - \phi_2^{n+1})$

This equation is an alternative way to write equation (5.8.8).

Section 5.9

3. *b.* (1) MI is in the MIU-system by I.

(2) MII is in the MIU-system by (1) and $II(b)$.

(3) $MIIII$ is in the MIU-system by (2) and $II(b)$.

(4) $MIIIIIIII$ is in the MIU-system by (3) and $II(b)$.

(5) $MUIIIII$ is in the MIU-system by (4) and $II(c)$.

(6) $MUIIU$ is in the MIU-system by (5) and $II(c)$.

6. Proof (by structural induction): Let the property be the following sentence: The string begins with an a.

   Show that each object in the BASE for S satisfies the property: The only object in the base is a, and the string a begins with an a.

   Show that for each rule in the RECURSION for S, if the rule is applied to objects in S that satisfy the property, then the objects defined by the rule also satisfy the property:

   The recursion for S consists of two rules, denoted II(a) and II(b).

   In case rule II(a) is applied to a string s in S that begins with a a, the result is the string sa, which begins with the same character as s, namely a.

   Similarly, in case rule II(b) is applied to a string s that begins with a a, the result is the string sb, which also begins with an a.

   Thus, when each rule in the RECURSION is applied to strings in S that begin with an a, the results are also strings that begin with an a.

   Because no objects other than those obtained through the BASE and RECURSION conditions are contained in S, every string in S begins with an a.

9. Proof (by structural induction): Let the property be the following sentence: The string represents an odd integer.

   Show that each object in the BASE for S satisfies the property: The objects in the base are 1, 3, 5, 7, and 9. All of these strings represent odd integers.

   Show that for each rule in the RECURSION for S, if the rule is applied to objects in S that satisfy the property, then the objects defined by the rule also satisfy the property:

   The recursion for S consists of five rules, denoted II(a)–II(e).

   Suppose s and t are strings in S that represent odd integers.

   Then the right-most character for each of s and t is 1, 3, 5, 7, or 9.

   In case rule II(a) is applied to s and t, the result is the string st, which has the same right-most character as t. So st represents an odd integer.

   In case rules II(b)–II(e) are applied to s, the results are $2s$, $4s$, $6s$, or $8s$. All of these strings have the same right-most character as s, and, therefore, they all represent odd integers.

   Thus when each rule in the RECURSION is applied to strings in S that represent odd integers, the result is also a string that represents an odd integer.

   Because no objects other than those obtained through the BASE and RECURSION conditions are contained in S, all the strings in S represent odd integers.

12. The string MU is not in the system because the number of I's in MU is 0, which is divisible by 3, and for all strings in the MIU-system, the number of I's in the string is not divisible by 3.

    Proof (by structural induction): Let the property be the following sentence: The number of I's in the string is not divisible by 3.

    Show that each object in the BASE for the MIU-system satisfies the property: The only object in the base is MI, which has one I, and the number 1 is not divisible by 3.

    Show that for each rule in the RECURSION for the MIU-system , if the rule is applied to objects in the system that satisfy the property, then the objects defined by the rule also satisfy the property:

The recursion for the MIU-system consists of four rules, denoted II(a)–(d). Let s be a string, let n be the number of I's in s, and suppose $3 \nmid n$. Consider the effect of acting upon s by each recursion rule in turn.

In case rule II(a) is applied to s, s has the form xI, where x is a string. The result is the string xIU. This string has the same number of I's as xI, namely n, and n is not divisible by 3.

In case rule II(b) is applied to s, s has the form Mx, where x is a string. The result is the string Mxx. This string has twice the number of I's as Mx. Because n is the number of I's in Mx and $3 \nmid n$, we have $n = 3k + 1$ or $n = 3k + 2$ for some integer k. In case $n = 3k + 1$, the number of I's in Mxx is $2(3k + 1) = 3(2k) + 2$, which is not divisible by 3. In case $n = 3k + 2$, the number of I's in Mxx is $2(3k + 2) = 6k + 4 = 3(2k + 1) + 1$, which is not divisible by 3 either.

In case rule II(c) is applied to s, s has the form $xIIIy$, where x and y are strings. The result is the string xUy. This string has three fewer I's than the number of I's in s. Because n is the number of I's in $xIIIy$ and $3 \nmid n$, we have that $3 \nmid (n - 3)$ either *[for if* $3 \mid (n - 3)$ *then* $n - 3 = 3k$, *for some integer* k. *Hence* $n = 3k + 3 = 3(k + 1)$, *and so* n *would be divisible by 3, which it is not].* Thus the number of I's in xUy is not divisible by 3.

In case rule II(d) is applied to s, s has the form $xUUy$, where x and y are strings. The result is the string xUy. This string has the same number of I's as $xUUy$, namely n, and n is not divisible by 3.

By the restriction for the MIU-system, no strings other than those derived from the base and the recursion are in the system. Therefore, for all strings in the MIU-system, the number of I's in the string is not divisible by 3.

18. Let S be the set of all strings of a's and $b's$ that contain exactly one a. The following is a recursive definition of S.

    I. BASE: $a \in S$

    II. RECURSION: If $s \in S$, then

    a. $bs \in S$ *b.* $sb \in S$

    III. RESTRICTION: There are no elements of S other than those obtained from I and II.

21. *b.* $A(2,1) = A(1, A(2,0)) = A(1, A(1,1)) = A(1,3)$ *[by part (a)]* $= A(0, A(1,2)) = A(0,4)$ *[by Example 5.9.9]* $= 4 + 1 = 5$

Review Guide: Chapter 5

Sequences and Summations

- What is a method that can sometimes find an explicit formula for a sequence whose first few terms are given (provided a nice explicit formula exists!)? *(p. 229)*
- What is the expanded form for a sum that is given in summation notation? *(p. 231)*
- What is the summation notation for a sum that is given in expanded form? *(p. 231)*
- How do you evaluate $a_1 + a_2 + a_3 + \cdots a_n$ when n is small? *(p. 232)*
- What does it mean to "separate off the final term of a summation"? *(p. 232)*
- What is the product notation? *(p. 233)*
- What are some properties of summations and products? *(p. 234)*
- How do you transform a summation by making a change of variable? *(p. 235)*
- What is factorial notation? *(p. 237)*
- What is the n choose r notation? *(p. 238)*
- What is an algorithm for converting from base 10 to base 2? *(p. 241)*
- What does it mean for a sum to be written in closed form? *(p. 251)*

Mathematical Induction

- What do you show in the basis step and what do you show in the inductive step when you use (ordinary) mathematical induction to prove that a property involving an integer n is true for all integers greater than or equal to some initial integer? *(p. 247)*
- What is the inductive hypothesis in a proof by (ordinary) mathematical induction? *(p. 247)*
- Are you able to use (ordinary) mathematical induction to construct proofs involving various kinds of statements such as formulas, divisibility properties, inequalities, and other situations? *(pp. 247, 249, 253, 259, 263, 265,)*
- Are you able to apply the formula for the sum of the first n positive integers? *(p. 251)*
- Are you able to apply the formula for the sum of the successive powers of a number, starting with the zeroth power? *(p. 255)*

Strong Mathematical Induction and The Well-Ordering Principle for the Integers

- What do you show in the basis step and what do you show in the inductive step when you use strong mathematical induction to prove that a property involving an integer n is true for all integers greater than or equal to some initial integer? *(p. 268)*
- What is the inductive hypothesis in a proof by strong mathematical induction? *(p. 268)*
- Are you able to use strong mathematical induction to construct proofs of various statements? *(pp. 269-274)*
- What is the well-ordering principle for the integers? *(p. 275)*
- Are you able to use the well-ordering principle for the integers to prove statements, such as the existence part of the quotient-remainder theorem? *(p. 276)*
- How are ordinary mathematical induction, strong mathematical induction, and the well-ordering principle for the integers related? *(p. 275 and exercises 31 and 32 on p. 279)*

Algorithm Correctness

- What are the pre-condition and the post-condition for an algorithm? *(p. 280)*
- What does it mean for a loop to be correct with respect to its pre- and post-conditions? *(p. 281)*

- What is a loop invariant? *(p. 282)*
- How do you use the loop invariant theorem to prove that a loop is correct with respect to its pre- and post-conditions? *(pp. 283-288)*

Recursion

- What is an explicit formula for a sequence? *(p. 290)*
- What does it mean to define a sequence recursively? *(p. 290)*
- What is a recurrence relation with initial conditions? *(p. 290)*
- How do you compute terms of a recursively defined sequence? *(p. 290)*
- Can different sequences satisfy the same recurrence relation? *(p. 291)*
- What is the "recursive paradigm"? *(p. 293)*
- How do you develop a recurrence relation for the tower of Hanoi sequence? *(p. 294)*
- How do you develop a recurrence relations for the Fibonacci sequence? *(p. 297)*
- How do you develop recurrence relations for sequences that involve compound interest? *(pp. 298-299)*
- How do you mathematical induction to prove properties of summations? *(p. 301)*

Solving Recurrence Relations

- What is the method of iteration for solving a recurrence relation? *(p. 305)*
- What is an arithmetic sequence? *(p. 307)*
- What is a geometric sequence? *(p. 308)*
- How do you use the formula for the sum of the first n integers and the formula for the sum of the first n powers of a real number r to simplify the answers you obtain when you solve recurrence relations? *(pp. 309-310)*
- How is mathematical induction used to check that the solution to a recurrence relation is correct? *(p. 312-314)*
- What is a second-order linear homogeneous recurrence relation with constant coefficients? *(p. 317)*
- What is the characteristic equation for a second-order linear homogeneous recurrence relation with constant coefficients? *(p. 319)*
- What is the distinct-roots theorem? If the characteristic equation of a relation has two distinct roots, how do you solve the relation? *(p. 321)*
- What is the single-root theorem? If the characteristic equation of a relation has a single root, how do you solve the relation? *(p. 325)*

General Recursive Definitions

- When a set is defined recursively, what are the three parts of the definition? *(p. 328)*
- Given a recursive definition for a set, how can you tell that a given element is in the set? *(p. 328-329)*
- What is structural induction? *(p. 331)*
- Given a recursive definition for a set, is there a way to tell that a given element is not in the set? *(solution for exercise 14a on p. A-49)*
- What is a recursive function? *(p. 332)*

Formats for Proving Formulas by Mathematical Induction

When using mathematical induction to prove a formula, students are sometimes tempted to present their proofs in a way that assumes what is to be proved. There are several formats you can use, besides the one shown most frequently in the textbook, to avoid this fallacy. A crucial point is this:

> If you are hoping to prove that an equation is true but you haven't yet done so, either preface it with the words "We must show that" or put a question mark above the equal sign.

Format 1 (the format used most often in the textbook for the inductive step): Start with the left-hand side (LHS) of the equation to be proved and successively transform it using definitions, known facts from basic algebra, and (for the inductive step) the inductive hypothesis until you obtain the right-hand side (RHS) of the equation.

Format 2 (the format used most often in the textbook for the basis step): Transform the LHS and the RHS of the equation to be proved *independently*, one after the other, until both sides are shown to equal the same expression. Because two quantities equal to the same quantity are equal to each other, you can conclude that the two sides of the equation are equal to each other.

Format 3: This format is just like Format 2 except that the computations are done in parallel. But in order to avoid the fallacy of assuming what is to be proved, do NOT put an equal sign between the two sides of the equation until the very last step. Separate the two sides of the equation with a vertical line.

Format 4: This format is just like Format 3 except that the two sides of the equation are separated by an equal sign with a question mark on top: $\stackrel{?}{=}$

Format 5: Start by writing something like "We must show that" and the equation you want to prove true. In successive steps, indicate that this equation is true if, and only if, ($\Leftrightarrow$) various other equations are true. But be sure that both the directions of your "if and only if" claims are correct. In other words, be sure that the $\Leftarrow$ direction is just as true as the $\Rightarrow$ direction. If you finally get down to an equation that is known to be true, then because each subsequent equation is true *if, and only if,* the previous equation is true, you will have shown that the original equation is true.

Example: Let the property $P(n)$ be the equation

$$\boxed{1 + 3 + 5 + \cdots + (2n - 1) = n^2} \leftarrow P(n).$$

Proof that $P(1)$ is true:

Solution (Format 2):
When $n = 1$, the LHS of $P(1)$ equals 1, and the RHS equals 1^2 which also equals 1. So $P(1)$ is true.

Proof that for all integers $k \geq 1$, if $P(k)$ is true then $P(k + 1)$ is true:

Solution (Format 2):
Suppose that k is any integer with $k \geq 1$ such that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k + 1)$ is true, where $P(k + 1)$ is the equation $1 + 3 + 5 + \cdots + (2k + 1) = (k + 1)^2$.

Now the LHS of $P(k+1)$ is

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &= 1+3+5+\cdots+(2k-1)+(2k+1) \quad \text{by making the next-to-last term explicit} \\ &= k^2+(2k+1) \quad \text{by inductive hypothesis.} \end{aligned}$$

And the RHS of $P(k+1)$ is

$$(k+1)^2 = k^2+2k+1 \quad \text{by basic algebra.}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus and thus $P(k+1)$ is true *[as was to be shown]*.

Solution (Format 3):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
Consider the left-hand and right-hand sides of $P(k+1)$:

$1+3+5+\cdots+(2k+1)$	$(k+1)^2$
$= 1+3+5+\cdots+(2k-1)+(2k+1)$ by making the next-to-last term explicit	
$= k^2+(2k+1)$ by inductive hypothesis	
$= k^2+2k+1$ by basic algebra	$= k^2+2k+1$ by basic algebra

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus and thus $P(k+1)$ is true *[as was to be shown]*.

Solution (Format 4):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
Consider the left-hand and right-hand sides of $P(k+1)$:

$$\begin{aligned} 1+3+5+\cdots+(2k+1) &\stackrel{?}{=} (k+1)^2 \\ 1+3+5+\cdots+(2k-1)+(2k+1) &\stackrel{?}{=} k^2+2k+1 \\ \text{by making the next-to-last term explicit} &\qquad \text{by basic algebra} \\ k^2+(2k+1) &\stackrel{?}{=} k^2+2k+1 \\ \text{by inductive hypothesis} & \\ k^2+2k+1 &= k^2+2k+1 \\ \text{by basic algebra} & \end{aligned}$$

So the left-hand and right-hand sides of $P(k+1)$ equal the same quantity, and thus $P(k+1)$ is true *[as was to be shown]*.

Solution (Format 5):
Suppose that k is any integer with $k \geq 1$ such that $1+3+5+\cdots+(2k-1) = k^2$. *[This is the inductive hypothesis, $P(k)$.]* We must show that $P(k+1)$ is true, where $P(k+1)$ is the equation $1+3+5+\cdots+(2k+1) = (k+1)^2$.
But $P(k+1)$ is true if, and only if, ($\Leftrightarrow$)

$$\begin{aligned} & 1+3+5+\cdots+(2k-1)+(2k+1) = (k+1)^2 \quad \text{by making the next-to-last term explicit} \\ \Leftrightarrow\ & k^2+(2k+1) = (k+1)^2 \quad \text{by inductive hypothesis} \\ \Leftrightarrow\ & k^2+2k+1 = (k+1)^2 \end{aligned}$$

which is true by basic algebra. Thus $P(k+1)$ is true *[as was to be shown]*.