

Partial fraction

① cover Method

Exp ⁽¹²⁾

$$\int \frac{2x+1}{x^2-7x+12} dx$$

$$A = \frac{2(\boxed{4})+1}{\boxed{4}-3} = \frac{9}{1} = 9$$

$$B = \frac{2(\boxed{3})+1}{\boxed{3}-4} = \frac{7}{-1} = -7$$

$$\int \frac{2x+1}{(x-4)(x-3)} dx$$

linear factor "distinct"

$$\int \left(\frac{A}{x-4} + \frac{B}{x-3} \right) dx$$

$$\int \left(\frac{9}{x-4} - \frac{7}{x-3} \right) dx$$

$$9 \ln|x-4| - 7 \ln|x-3| + C$$

Q. $\int \frac{f(x)}{g(x)} dx$ f and g are poly.

• If $\deg f \geq \deg g$ we use long Division

$$g(x) \overline{) f(x)}$$

$$\int \frac{7}{(x-1)(x+1)^2} dx = \int \left(\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right) dx$$

$\uparrow$
 $(x+1)(x+1)$

$$\int \frac{x}{x^3(x^2+1)^2} = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} \right)$$

Partial fraction decomposition of $\frac{x}{x^3(x^2+1)^2}$. The denominator is $x^3(x^2+1)^2$. The decomposition includes terms for the linear factor x and the quadratic factor x^2+1 raised to the power of 2.

$$\int \frac{x}{(x^2+1)(x^3+2)} dx = \int \left(\frac{Ax+B}{x^2+1} + \frac{Cx^2+Dx+E}{x^3+2} \right) dx$$

Partial fraction decomposition of $\frac{x}{(x^2+1)(x^3+2)}$. The denominator consists of a quadratic factor x^2+1 and a cubic factor x^3+2 .

$$\int \frac{x^2}{(x-1)^3(x+1)^2} dx = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} \right) dx$$

Partial fraction decomposition of $\frac{x^2}{(x-1)^3(x+1)^2}$. The denominator consists of a linear factor $x-1$ raised to the power of 3 and a quadratic factor x^2+1 raised to the power of 2.

Exp 18

$$\int_{-1}^0 \frac{x^3}{x^2-2x+1} dx$$

Integration problem 18: Evaluate the definite integral of $\frac{x^3}{x^2-2x+1}$ from -1 to 0 .

$$= \int \left[(x+2) + \frac{3x-2}{x^2-2x+1} \right] dx$$

Result of long division: $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{x^2-2x+1}$.

Long Division

$$\begin{array}{r} x+2 \\ x^2-2x+1 \overline{) x^3 } \\ \underline{-(x^3 - 2x^2 + x)} \\ 3x^2 - x - 2 \end{array}$$

Long division of x^3 by x^2-2x+1 to find the quotient $x+2$ and remainder $3x-2$.

$$= \int_{-1}^0 \left((x+2) + \frac{3x-2}{x^2-2x+1} \right) dx$$

$$= \int_{-1}^0 (x+2) dx + \int_{-1}^0 \frac{3x-2}{(x-1)^2} dx$$

$$= \left. \frac{x^2}{2} + 2x \right|_{-1}^0 + \int_{-1}^0 \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} \right) dx$$

$$= (0+0) - \left(\frac{1}{2} - 2 \right) + \int_{-1}^0 \left(\frac{3}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$= \frac{3}{2} + 3 \ln|x-1| + \int_{-1}^0 \frac{dx}{(x-1)^2}$$

$$= \frac{3}{2} + 3 \ln 1 - 3 \ln 2 + \int_{-2}^{-1} \frac{du}{u^2}$$

$$= \frac{3}{2} - 3 \ln 2 + \int_{-2}^{-1} u^{-2} du$$

$$= \frac{3}{2} - 3 \ln 2 + \left. \frac{u^{-1}}{-1} \right|_{-2}^{-1}$$

$$= \frac{3}{2} - 3 \ln 2 + \frac{1}{u} \Big|_{-2}^{-1}$$

$$= \dots = 2 - 3 \ln 2$$

$$\begin{array}{r} -x^3 + 2x^2 + x \\ \underline{-2x^2 + 4x + 2} \\ 3x - 2 \end{array}$$

$$\begin{aligned} \frac{3x-2}{(x-1)^2} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} \\ &= \frac{A(x-1)}{(x-1)^2} + \frac{B}{(x-1)^2} \\ &= \frac{A(x-1) + B}{(x-1)^2} \end{aligned}$$

$$3x-2 = A(x-1) + B$$

مقایسه ضرایب.

$$3 = A, \quad -2 = -A + B$$

$$= -3 + B$$

$$1 = B$$

آوردن مقادیر

$$x=1 \Rightarrow 3-2 = A(0) + B$$

$$\underline{1 = B}$$

$$x=0 \Rightarrow 0-2 = A(0-1) + B$$

$$-2 = -A + B$$

$$-2 = -A + 1$$

$$\underline{3 = A}$$

$$(33) \int \frac{y^2 + 2y + 1}{(y^2 + 1)^2} dy = \int \left(\frac{Ay + B}{y^2 + 1} + \frac{Cy + D}{(y^2 + 1)^2} \right) dy$$

$$\frac{y^2 + 2y + 1}{(y^2 + 1)^2} = \frac{Ay + B}{y^2 + 1} + \frac{Cy + D}{(y^2 + 1)^2}$$

$$= \frac{(Ay + B)(y^2 + 1)}{(y^2 + 1)^2} + \frac{Cy + D}{(y^2 + 1)^2}$$

$$= \frac{(Ay + B)(y^2 + 1) + Cy + D}{(y^2 + 1)^2}$$

$$y^2 + 2y + 1 = (Ay + B)(y^2 + 1) + Cy + D$$

$y=0 \Rightarrow$

$$1 = B + D$$

①

Compare coefficients

$$2y + 2 = (Ay + B)(2y) + (y^2 + 1)A + C$$

$y=0$

$$2 = 0 + A + C$$

$$2 = A + C$$

$$D = 0$$

$$2 = A + C$$

$$2y + 2 = 2Ay^2 + 2By + Ay^2 + A + C$$

$$C = 2$$

$$2 = 4Ay + 2B + 2Ay$$

$$\Rightarrow B = 1$$

$$y=0 \Rightarrow 2 = 0 + 2B + 0$$

$$0 = 4A + 0 + 2A$$

$$0 = 6A \Rightarrow A = 0$$

$$\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2} dy = \int \frac{1}{y^2 + 1} dy + \int \frac{2y}{(y^2 + 1)^2} dy$$

$$u = y^2 + 1$$

$$du = 2y dy$$

$$= \tan^{-1} y + \int \frac{du}{u^2}$$

$$= \tan^{-1} y + \int u^{-2} du$$

$$= \tan^{-1} y + \frac{u^{-1}}{-1} + C$$

$$= \tan^{-1} y - \frac{1}{u} + C$$

$$= \tan^{-1} y - \frac{1}{y^2 + 1} + C$$

30

$$\int \frac{x^2 + x}{x^4 - 3x^2 - 4} dx = \int \frac{x^2 + x}{(x^2 - 4)(x^2 + 1)} dx$$

↗ ↘
irreducible

$$= \int \frac{x^2 + x}{(x-2)(x+2)(x^2+1)} dx$$

$$= \int \left(\frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1} \right) dx$$

$$\frac{x^2 + x}{(x-2)(x+2)(x^2+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x+2)(x^2+1)}{(x-2)(x+2)(x^2+1)} + \frac{B(x-2)(x^2+1)}{(x+2)(x-2)(x^2+1)} + \frac{(Cx+D)(x-2)(x+2)}{(x^2+1)(x-2)(x+2)}$$

$$x^2 + x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x-2)(x+2)$$

$$x=2 \Rightarrow 4+2 = A(4)(5)$$

$$6 = 20A$$

$$A = \frac{3}{10}$$

$$6 = 20H$$

$$A = \frac{3}{10}$$

$$x = -2 \Rightarrow 4 - 2 = 0 + B(-4)(5) + 0$$

$$2 = -20B$$

$$B = -\frac{1}{10}$$

$$x = 0 \Rightarrow 0 + 0 = 2A + 2B - 4D$$

$$0 = 2 \cdot \frac{3}{10} - 2 \left(-\frac{1}{10}\right) - 4D \Rightarrow 4D = \frac{8}{10} \Rightarrow D = \frac{1}{5}$$

$$x = 1 \Rightarrow 1 + 1 = A(3)(2) + B(-1)(2) + \left(C + \frac{1}{5}\right)(-3)$$

$\uparrow$ $\frac{3}{10}$ $\uparrow$ $-\frac{1}{10}$

$$C = -\frac{1}{5}$$

$$\int \frac{x^2 + x}{x^4 - 3x^2 - 4} dx = \int \frac{\frac{3}{10}}{x-2} dx - \int \frac{\frac{1}{10}}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{1}{5}}{x^2 + 1} dx$$

$$= \frac{3}{10} \ln|x-2| - \frac{1}{10} \ln|x+2| + \left(-\frac{1}{5} \cdot \frac{1}{2} \int \frac{2x}{x^2+1} dx + \frac{1}{5} \int \frac{dx}{x^2+1}\right)$$

$$= \frac{3}{10} \ln|x-2| - \frac{1}{10} \ln|x+2| - \frac{1}{5} \cdot \frac{1}{2} \ln(x^2+1) + \frac{1}{5} \left(\tan^{-1} x + C \right)$$

34

$$\int \frac{x^4}{x^2 - 1} dx$$

المقسوم عليه

$$\begin{array}{r} x^2 + 1 \\ x^4 \\ \hline -x^4 + x^2 + \end{array}$$

المقسوم عليه

$$\begin{array}{r} x^2 \\ x^2 \\ \hline 0 \end{array}$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\frac{du}{2} = x dx$$

$$\int \left(x^2 + 1 + \frac{1}{x^2 - 1} \right) dx$$

$$\int \left(x^2 + 1 + \frac{1}{x^2 - 1} \right) dx$$

$$\begin{array}{r} x^2 \\ -x^2 + 1 \\ \hline 1 \end{array}$$

بقسمة

$$\int (x^2 + 1) dx + \int \frac{1}{(x-1)(x+1)} dx$$

$$\frac{x^3}{3} + x + \int \left(\frac{A}{x-1} + \frac{B}{x+1} \right) dx$$

$\uparrow$ $\uparrow$
 $+1$ -1

$$A = \frac{1}{\boxed{1} + 1} = \frac{1}{2}$$

$$B = \frac{1}{\boxed{-1} - 1} = \frac{-1}{2}$$

$$\frac{x^3}{3} + x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$$
