14.8 Lagrange Multipliers

A Lagrange Multipliers with one constraint: "$g(x,y,z)=0$"

- Suppose that $f(x,y,z)$ and $g(x,y,z)$ are diff and $\nabla g \neq \vec{0}$ when $g(x,y,z)=0$.
- To find the extreme values (local max and local min) of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$, we find the points (x,y,z) and the Lagrange multiplier λ that simultaneously satisfy

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z)=0$$

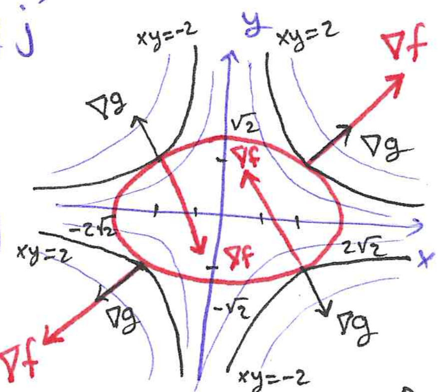
Exp Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$

- $f(x,y) = xy \Rightarrow \nabla f = y\vec{i} + x\vec{j}$

$\nabla f = \vec{i} + 2\vec{j}$

$\nabla f = \frac{1}{2}\vec{i} + \vec{j}$

$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 \Rightarrow \nabla g = \left(\frac{x}{4}\right)\vec{i} + y\vec{j}$



- $\nabla f = \lambda \nabla g$ and $g(x,y) = 0$

$y = \frac{\lambda x}{4}$, $x = \lambda y$ and $\frac{x^2}{8} + \frac{y^2}{2} = 1$

$y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2 y}{4} \Rightarrow y[4 - \lambda^2] = 0$

- Case 1: $y = 0 \Rightarrow x = 0$. But $(0,0)$ is not on the ellipse. Hence $y \neq 0$

$\boxed{\nabla f = \pm 2 \nabla g}$

- Case 2: $\lambda = \pm 2 \Rightarrow x = \pm 2y \Rightarrow \frac{4y^2}{8} + \frac{y^2}{2} = 1 \Rightarrow y = \pm 1 \Rightarrow x = \pm 2$
- Four points $(\pm 2, 1), (\pm 2, -1)$
- $\left.\begin{array}{l} f(2,1) = 2 \\ f(-2,-1) = 2 \end{array}\right\} \Rightarrow$ "Max Value" and $\left.\begin{array}{l} f(-2,1) = -2 \\ f(2,-1) = -2 \end{array}\right\}$ min value

Th suppose that $f(x,y,z)$ is diff in a region whose interior contains a smooth curve $C: \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$.
If $P_0(x_0, y_0, z_0)$ is a point on the curve C where f has a local max or local min, then $\nabla f \perp C$ at P_0.
More over, $\nabla f \cdot \vec{V} = 0$, where $\vec{V} = \frac{d\vec{r}}{dt}$.

Proof: The values of f on the curve C are $f(g(t), h(t), k(t))$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt}$$

If f has local max or local min at P_0, then $\frac{df}{dt}(P_0) = 0$
That is $\nabla f \cdot \vec{V} = 0$

Exp Find the maximum and minimum values of $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$

- $f(x,y) = 3x + 4y \Rightarrow \nabla f = 3\vec{i} + 4\vec{j}$
- $g(x,y) = x^2 + y^2 - 1 \Rightarrow \nabla g = (2x)\vec{i} + (2y)\vec{j}$
- $\nabla f = \lambda \nabla g$ and $g(x,y) = 0$

$3 = 2x\lambda$, $4 = 2\lambda y$ and $x^2 + y^2 = 1$

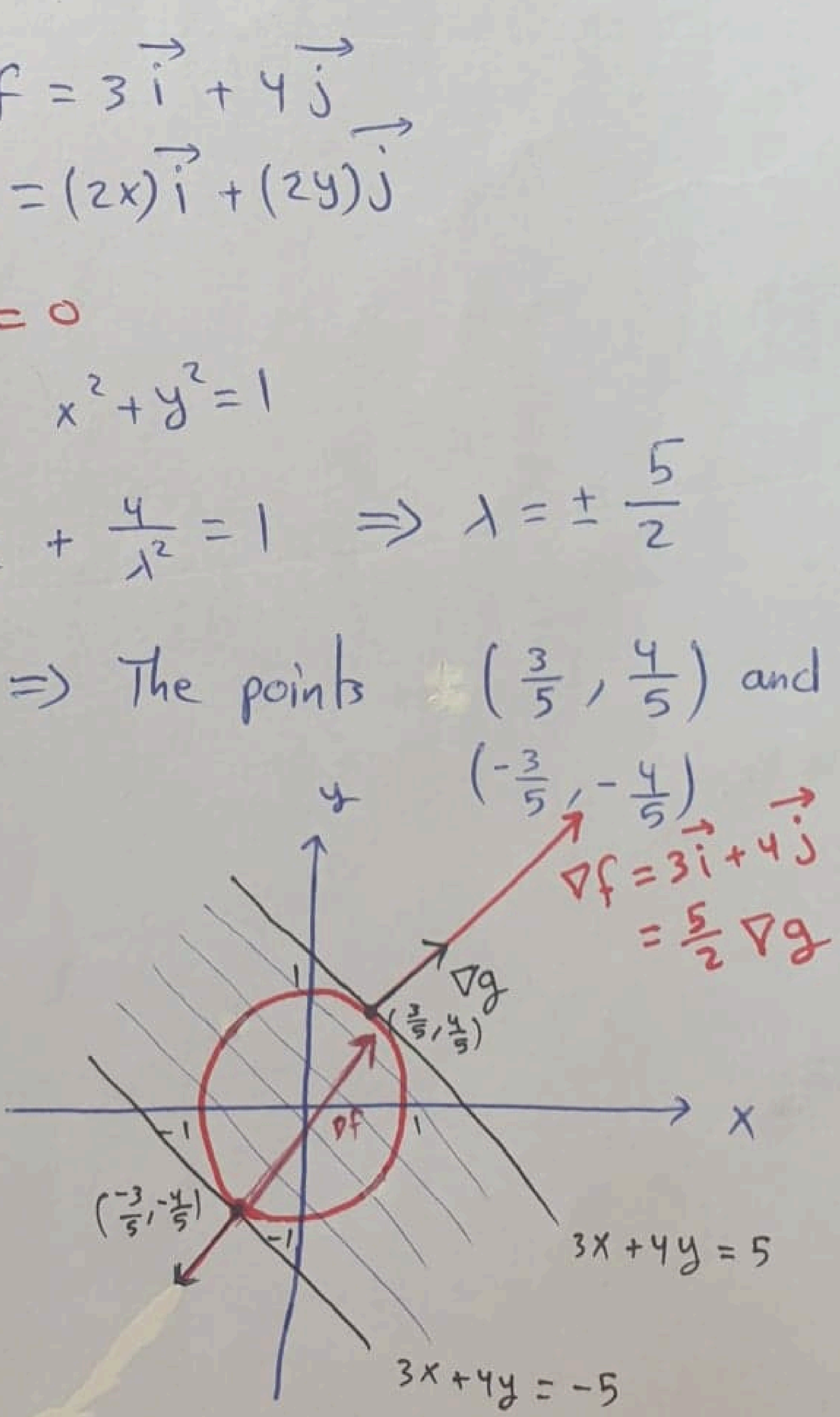
$x = \frac{3}{2\lambda}$, $y = \frac{2}{\lambda} \Rightarrow \frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \Rightarrow \lambda = \pm\frac{5}{2}$

$\Rightarrow x = \pm\frac{3}{5}$, $y = \pm\frac{4}{5} \Rightarrow$ The points $(\frac{3}{5}, \frac{4}{5})$ and $(-\frac{3}{5}, -\frac{4}{5})$

- $f(\frac{3}{5}, \frac{4}{5}) = 5$ Max value
- $f(-\frac{3}{5}, -\frac{4}{5}) = -5$ Min value

$\nabla f = 3\vec{i} + 4\vec{j}$

$\nabla g = \frac{6}{5}\vec{i} + \frac{8}{5}\vec{j}$



Exp Find the point on the plane $x+2y+3z=13$ closest to the point $(1,1,1)$.

- $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ "$f = d^2$"
  $\nabla f = 2(x-1)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k}$
- $g(x,y,z) = x+2y+3z-13 \Rightarrow \nabla g = \vec{i} + 2\vec{j} + 3\vec{k}$
- $\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$

$2(x-1)=\lambda$, $y-1=\lambda$, $2(z-1)=3\lambda$ and $x+2y+3z=13$

$2(x-1) = y-1 \Rightarrow x = \frac{y+1}{2}$ $\qquad 2(z-1) = 3[y-1] \Rightarrow z = \frac{3y-1}{2}$

Hence, $\frac{y+1}{2} + 2y + \frac{3}{2}(3y-1) = 13 \Rightarrow y = 2 \Rightarrow x = \frac{3}{2} \Rightarrow z = \frac{5}{2}$

- The point is $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$

B Lagrange Multipliers with two constraints:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \text{ and } g_1(x,y,z)=0 \text{ and } g_2(x,y,z)=0$$

Exp Find the extreme values of $f(x,y,z) = xy + z^2$ on the circle in which the plane $y-x=0$ intersects the sphere $x^2+y^2+z^2=4$.

- $f(x,y,z) = xy+z^2 \Rightarrow \nabla f = y\vec{i} + x\vec{j} + (2z)\vec{k}$
  $g_1(x,y,z) = y-x \Rightarrow \nabla g_1 = -\vec{i} + \vec{j}$
  $g_2(x,y,z) = x^2+y^2+z^2-4 \Rightarrow \nabla g_2 = (2x)\vec{i} + (2y)\vec{j} + (2z)\vec{k}$
- $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$, $y = x$, $x^2+y^2+z^2=4$

$y = -\lambda + 2\mu x$, $x = \lambda + 2y\mu$, $\boxed{z = z\mu} \Rightarrow z(1-\mu) = 0$

- Cas 1: $z=0 \Rightarrow x^2+y^2=4 \Rightarrow 2x^2=4 \Rightarrow x = \pm\sqrt{2} \Rightarrow y = \pm\sqrt{2}$
- Cas 2: $\mu = 1 \Rightarrow \left.\begin{array}{l} y = -\lambda + 2x \\ x = \lambda + 2y \end{array}\right\} \Rightarrow x+y = 2(x+y) \Rightarrow x+y=0$
  $\Rightarrow 2x = 0 \Rightarrow x = 0 \Rightarrow y = 0$
  $\Rightarrow z^2 = 4 \Rightarrow z = \pm 2$

- $f(\pm\sqrt{2}, \pm\sqrt{2}, 0) = 2 \Rightarrow f$ has min value at $(\pm\sqrt{2}, \pm\sqrt{2}, 0)$
- $f(0,0,\pm 2) = 4 \Rightarrow f$ has max value at $(0,0,\pm 2)$