3.5 Change of Basis

Def. Let V be a vector space and $E=\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be an ordered basis for V.

- If $\vec{w}$ is any element in V, then $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ where $c_1, c_2, \dots, c_n$ are scalars.

  ⇒ Thus, we can associate a unique vector of scalars $\vec{c} = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$ for each $\vec{w} \in V$.

  ⇒ The vector $\vec{c}$ is called the coordinate vector of $\vec{w}$ with respect to the ordered basis E, and is denoted by $c = [w]_E$, and the c_i's are called the coordinates of $\vec{w}$ relative to E.

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* We can also compute the coordinate vector w.r.t E as follows:

$$c = [w]_E = E^{-1} w = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_n]^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where E^{-1} is the transition matrix from the basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ to the basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = E$.

Exp Let $\vec{u}_1 = (3,2)$ and $\vec{u}_2 = (1,1)$. Find the coordinates of $\vec{x} = (7,4)$ w.r.t $\vec{u}_1$ and $\vec{u}_2$.

$$c = [x]_U = U^{-1} x = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

This means $x = \begin{pmatrix} 7 \\ 4 \end{pmatrix} = (3)\begin{pmatrix} 3 \\ 2 \end{pmatrix} + (-2)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{x} = 3\vec{u}_1 - 2\vec{u}_2 \checkmark$$

Remarks :

1. The transition matrix from $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ to $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is $S = V^{-1} U$ where

$U = (u_1 \; u_2 \cdots u_n)$ and $V = (v_1 \; v_2 \cdots v_n)$

2. 〃 〃 〃 〃 $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ to $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is $S^{-1} = U^{-1} V$. That is, the transion matrix is nonsingular.

Exp. The transition matrix from the basis $\{(1,1)^T, (-1,1)^T\}$ to the basis $\{e_1, e_2\}$ is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

- 〃 〃 〃 〃 〃 〃 $\{e_1, e_2\}$ to $\{(1,1)^T, (-1,1)^T\}$ is

$$S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- 〃 〃 〃 〃 〃 〃 $\{(3,2)^T, (4,3)^T\}$ to $\{(1,1)^T, (-1,1)^T\}$ is

$$\bar{S} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{7}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Exp Let $\vec{b}_1 = (1,-1)$ and $\vec{b}_2 = (-2,3)$. Find the coordinates of $\vec{x} = (1,2)$ w.r.t $\{\vec{b}_1, \vec{b}_2\}$

$$c = [x]_B = B^{-1} x = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

That is $\vec{x} = 7\vec{b}_1 + 3\vec{b}_2$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} -2 \\ 3 \end{pmatrix} \checkmark$$

Exp Find the transition matrix to the change of basis from $V=\left\{\begin{pmatrix}5\\2\end{pmatrix},\begin{pmatrix}7\\3\end{pmatrix}\right\}$ to $U=\left\{\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$

$$S=U^{-1}V=\begin{bmatrix}3&1\\2&1\end{bmatrix}^{-1}\begin{bmatrix}5&7\\2&3\end{bmatrix}=\begin{bmatrix}3&4\\-4&-5\end{bmatrix}$$

Note that the change of basis from $\{\vec{v}_1,\vec{v}_2\}$ to $\{\vec{u}_1,\vec{u}_2\}$ can be viewed as two step process:

1. Change from $\{\vec{v}_1,\vec{v}_2\}$ to $\{\vec{e}_1,\vec{e}_2\}$ and

   in this case the transition matrix is $\underline{\underline{V}}$

2. Change from $\{\vec{e}_1,\vec{e}_2\}$ to $\{\vec{u}_1,\vec{u}_2\}$

   in this case the transition matrix is U^{-1}

$$\begin{tikzcd}
{[v_1,v_2]} \arrow[r, "V"] \arrow[dr, "U^{-1}V"'] & {[e_1,e_2]} \arrow[d, "U^{-1}"] \\
& {[u_1,u_2]}
\end{tikzcd}$$

Exp Let $E=[v_1,v_2,v_3]=[(1,1,1)^T,(2,3,2)^T,(1,5,4)^T]$ and

$F=[u_1,u_2,u_3]=[(1,1,0)^T,(1,2,0)^T,(1,2,1)^T]$.

① Find the transition matrix from E to F

$$S=F^{-1}E=\begin{bmatrix}1&1&1\\1&2&2\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&2&1\\1&3&5\\1&2&4\end{bmatrix}=\begin{bmatrix}2&-1&0\\-1&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&1\\1&3&5\\1&2&4\end{bmatrix}$$

$$=\begin{bmatrix}1&1&-3\\-1&-1&0\\1&2&4\end{bmatrix}$$

② If $\vec{x}=3\vec{v}_1+2\vec{v}_2-\vec{v}_3$ and

$\vec{y}=\vec{v}_1-3\vec{v}_2+2\vec{v}_3$

Find the coordinates of $\vec{x}$ and $\vec{y}$ w.r.t the ordered basis F.

$$[x]_F=Sx=\begin{bmatrix}1&1&-3\\-1&-1&0\\1&2&4\end{bmatrix}\begin{bmatrix}3\\2\\-1\end{bmatrix}=\begin{bmatrix}8\\-5\\3\end{bmatrix}$$

$$[y]_F=Sy=\begin{bmatrix}1&1&-3\\-1&-1&0\\1&2&4\end{bmatrix}\begin{bmatrix}1\\-3\\2\end{bmatrix}=\begin{bmatrix}-8\\2\\3\end{bmatrix}$$

This means:

$\vec{x}=3\vec{v}_1+2\vec{v}_2-\vec{v}_3=8\vec{u}_1-5\vec{u}_2+3\vec{u}_3$

$\vec{y}=\vec{v}_1-3\vec{v}_2+2\vec{v}_3=-8\vec{u}_1+2\vec{u}_2+3\vec{u}_3$

Exp ① Find the transition matrix representing the change of coordinates on P_3 from the ordered basis $[1, x, x^2]$ to the ordered basis $[1, 2x, 4x^2-2]$

$\Rightarrow$ Since $[1, x, x^2]$ is the standard basis for P_3, it is easier to find the transition matrix from $[1, 2x, 4x^2-2]$ to $[1, x, x^2]$:

$$1 = 1 + 0x + 0x^2$$
$$2x = 0 + 2x + 0x^2$$
$$4x^2-2 = -2 + 0x + 4x^2$$

$\Rightarrow$ Hence, the transition matrix $S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$\Rightarrow$ S^{-1} is the transition matrix from $[1, x, x^2]$ to $[1, 2x, 4x^2-2]$:

$$S^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

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② If $p(x) = a + bx + cx^2$, then find the coordinates of $p(x)$ w.r.t the basis $[1, 2x, 4x^2-2] = F$

$$[p(x)]_F = S^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + c/2 \\ b/2 \\ c/4 \end{bmatrix}$$

This means : $p(x) = a + bx + cx^2$

$$= \left(a + \frac{c}{2}\right)(1) + \left(\frac{b}{2}\right)(2x) + \left(\frac{c}{4}\right)(4x^2-2)$$

Note 1 If V is any n-dimensional vector space, then it is possible to change from one basis to another using a transition matrix $S_{n\times n}$. Moreover, $S_{n\times n}$ is nonsingular.

2 Any nonsingular matrix can be thought as a transition matrix.

Exp Let $\vec{y} = (2,1)$ and $\vec{z} = (1,4)$.

→ The vectors $\vec{y}$ and $\vec{z}$ are linearly independent

→ Hence, they form a basis for $\mathbb{R}^2$

⇒ Let $\vec{x} = (7,7) \in \mathbb{R}^2 \Rightarrow \vec{x}$ can be written as a linear combination of $\vec{y}$ and $\vec{z}$:

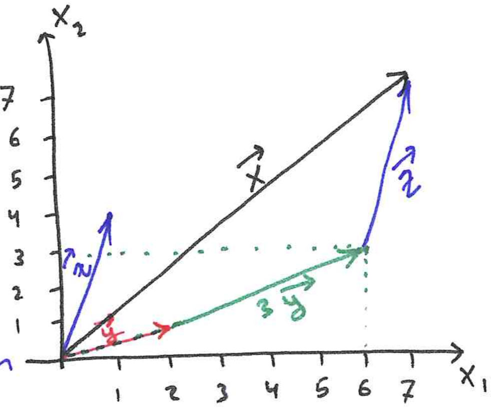
→ In fact: $\vec{x} = 3\vec{y} + \vec{z}$

→ This means that the coordinat vector of $\vec{x} = (7,7)$ w.r.t the ordered basis $[y,z]$ is $\vec{c} = (3,1)$

That is $[x]_{[y,z]} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

→ Geometrically:

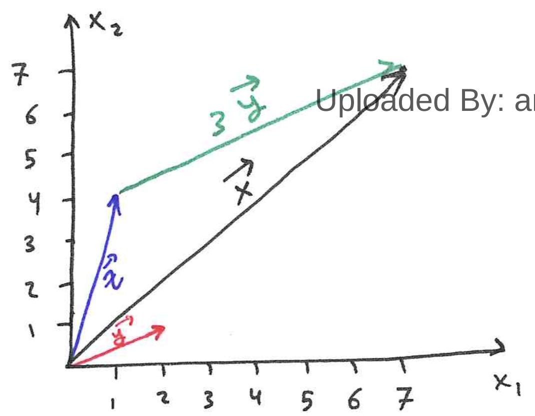
→ The coordinate vector $\vec{c} = (3,1)$ specifies how to get from the origin to the point $(7,7)$ by moving first in the direction of $\vec{y}$ and then in the direction of $\vec{z}$.



← If instead, we treat $\vec{z}$ as the first basis and $\vec{y}$ as the second basis then the coordinate vector becomes

$$c = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \frac{-1}{7} \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

That is $\vec{x} = \vec{z} + 3\vec{y}$



* Hence, ordered basis is important since the coordinate vector w.r.t the orderd basis $[y,z]$ is $(3,1)^T$ and the coordinate vector w.r.t the ordered basis $[z,y]$ is $(1,3)^T$.