"State Space" Modeling

$$u(t) - kx(t) - b\dot{x}(t) = m\ddot{x}(t)$$

$x(t) = x_1$
$\dot{x}(t) = x_2 = \dot{x}_1$
$u(t) = u_1$
$\dot{x}_1 = ??$
$\dot{x}_2 = ??$

$$u_1 - kx_1 - bx_2 = m\dot{x}_2$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}u_1 - \frac{k}{m}x_1 - \frac{b}{m}x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k/m & -b/m\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}[u_1]$$

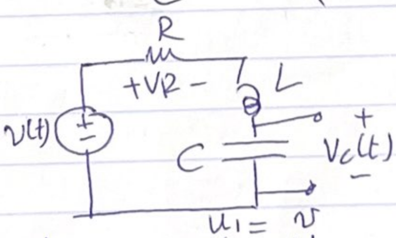
$y = x_1 = x(t)$

$$\dot{X} = AX + Bu$$
$$y = CX + Du$$

$$y = 1\,x_1 + 0\,x_2 + 0\,u_1$$
$$= \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{X} + \underbrace{[0]}_{D}\,\underset{u}{u_1}$$

$$\boxed{\begin{aligned}\dot{X} &= AX + Bu\\ y &= CX + Du\end{aligned}}$$

Example:



$V_L = L\frac{di}{dt}$

$i = C\frac{dV_C}{dt}$

$\frac{dV_C}{dt} = \frac{1}{C}i$

$u_1 = v$
$x_1 = V_C$
$x_2 = i$
$y_1 = V_C = x_1$
$y_2 = Ri = Rx_2$

$V = iR + V_L + V_C$
$V = iR + L\frac{di}{dt} + V_C$
$\frac{di}{dt} = \frac{1}{L}v + -\frac{R}{L}i - \frac{1}{L}V_C$

$$\dot{x}_2 = \frac{1}{L}u - \frac{R}{L}x_2 - \frac{1}{L}x_1$$
$$\dot{x}_1 = \frac{1}{C}x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{L} & -R/L\\ 0 & 1/C\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1/L\\ 0\end{bmatrix}u$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{[0]}_{D}u$$

State Space Versus Transfer function

$\dot{X} = AX + Bu$

$y = CX + Du$

$G_p(s) = \dfrac{Y(s)}{U(s)}$

$sX(s) = A\,X(s) + B\,u(s) \rightarrow X(s)[Is - A] = B\,U(s)$

$Y(s) = C\,X(s) + D\,U(s)$

$X(s) = [Is - A]^{-1} B\,U(s)$

$Y(s) = C\,[Is - A]^{-1} B\,U(s) + D\,U(s)$

$$\boxed{\frac{Y(s)}{U(s)} = C[Is - A]^{-1} B + D = G_p(s)}$$

$A = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = [0]$

$G_p(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ k/m & s + b/m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} + [0]$

$E = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow E^{-1} = \dfrac{1}{\det(E)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$G_p(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \dfrac{1}{s(s + b/m) + k/m} \begin{bmatrix} s + b/m & 1 \\ -k/m & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \dfrac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} \frac{1}{m} \\ s/m \end{bmatrix}$

$G_p(s) = \dfrac{1}{s^2 + \frac{b}{m}s + k/m} \cdot \left[\dfrac{1}{m} + 0\right] = \dfrac{1}{ms^2 + bs + k}$

Ex:

$x_1 = y$; $x_2 = \dot{y} = \dot{x}_1$; $u_1 = u$

$u_{1e} = 0$; $x_{1e} = ?\,4$ $\quad x_{2e} = ??\,0$

$\ddot{y} = \overset{2y\,+}{\cancel{\pm}}\, y^2\dot{y} + \dot{y}u - 8$

$\dot{x}_2 = \underbrace{2x_1 + x_1^2 x_2 + x_2 u_1 - 8}_{f(x_1, x_2, u)}$; linearize

$\dot{x}_1 = 0$

$\dot{x}_2 = 0$

$\dot{x}_1 = x_2$

$0 = x_{2e}$; $0 = 2x_{1e} + \cancel{x_{1e}^2 x_{2e}} + \cancel{x_{2e} u_{1e}} - 8 \Rightarrow x_{1e} = 4$

linearize $\rightarrow x_1 = x_{1e} + \delta x_1$; $x_2 = x_{2e} + \delta x_2$; $u_1 = u_{1e} + \delta u_1$

$\delta\dot{x}_1 = \delta x_2$

$\delta\dot{x}_2 = \cancel{f(x_{1e}, x_{2e}, u_{1e})} + (2 + \cancel{x_{2e}\, 2x_{1e}})\delta x_1 + (x_{1e}^2 + \cancel{u_{1e}})\delta x_2 + \cancel{x_{2e}} \cdot \delta u_1 \Rightarrow \delta\dot{x}_2 = 2\delta x_1 + 16\delta x_2$