Ch7 Super-position of Quasi-Parallel Plane Waves

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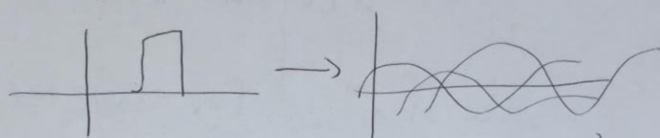
- Any Field can be described as a superposition of many plane wave fields. (pulses).



7.1 Intensity of superimposed Plane Waves

- Arbitry waveforms $= \sum$ plane waves with different $\vec{k}$, amplitudes, phases, frequencies, polarizations.

$$\vec{E}(\vec{r},t) = \sum_j \vec{E}_j \, e^{i(\vec{k}_j \cdot \vec{r} - w_j t)}$$

$$\vec{B}(\vec{r},t) = \sum_j B_j \, e^{i(\vec{k}_j \cdot \vec{r} - w_j t)}$$

$$= \sum_j \frac{\vec{k}_j \times \vec{E}_j}{w_j} \, e^{i(\vec{k}_j \cdot \vec{r} - w_j t)}$$

(2)

$$\vec{S}(\vec{r},\vec{t}) = Re\{\vec{E}(\vec{r},t)\} \times \frac{Re\{\vec{B}(\vec{r},t)\}}{\mu_0}$$

$$= \sum_{j,m} \frac{1}{W_m \mu_0} Re\left\{\vec{E}_j e^{i(\vec{k}_j\cdot\vec{r} - w_j t)}\right\}$$

$$\text{cross} \leftarrow \times Re\left\{\vec{k}_m \times \vec{E}_m e^{i(\vec{k}_m\cdot\vec{r} - W_m t)}\right\} \quad (7.3)$$

[Notice j,m for two series terms].

Simplifying Assumptions to begin with:

A1 1) Time-Average $\vec{S}$ to remove fluctuations that change on scale of optical frequencies. ?? Why

A2 2) Assume all plane waves travel $\approx$ parallel to each other.

A3 3) let $\vec{k}_m$ be real.

Use $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ (3)

on (7.3)

$$\vec{A} = Re\left\{\vec{E}_j\, e^{i(\vec{k}_j\cdot\vec{r} - w_j t)}\right\}$$

$$\vec{B} = \vec{k}_m \quad ; \quad \vec{C} = Re\left\{\vec{E}_m\, e^{i(\vec{k}_m\cdot\vec{r} - w_m t)}\right\}$$

$$\therefore \vec{S}(\vec{r},t) = \sum_{j,m} \frac{1}{W_m \mu_0}\Big[\vec{k}_m\Big(Re\left\{\vec{E}_j\, e^{i(\vec{k}_j\cdot\vec{r} - w_j t)}\right\}\cdot Re\left\{\vec{E}_m\, e^{i(\vec{k}_m\cdot\vec{r} - w_m t)}\right\}\Big)$$

$$- Re\left\{E_m\, e^{i(\vec{k}_m\cdot\vec{r} - w_m t)}\right\}\Big(Re\left\{\vec{E}_j\, e^{i(\vec{k}_j\cdot\vec{r} - w_j t)}\right\}\cdot\vec{k}_m\Big)\Big]$$

Using A2: Last term $\to 0$ because

$$\vec{k}_m \perp \vec{E}_j$$

$$\vec{S}(\vec{r},t) = \sum_{j,m}\frac{k_m}{W_m \mu_0}\left\{\frac{\vec{E}_j\, e^{i(\vec{k}_j\cdot\vec{r} - W_j t)} + E_j^*\, e^{-i(\vec{k}_j\cdot\vec{r} - w_j t)}}{2}\right\}$$

$$\cdot\left\{\frac{\vec{E}_m\, e^{i(\vec{k}_m\cdot\vec{r} - W_m t)} + E_m^*\, e^{-i(\vec{k}_m\cdot\vec{r} - w_m t)}}{2}\right\}$$

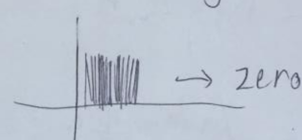
$$\vec{S}(\vec{r},t) = \sum_{j,m} \frac{k_m}{4\omega_m \mu_0} \Big\{ \vec{E}_j \cdot \vec{E}_m \, e^{i[(\vec{k}_j+\vec{k}_m)\cdot\vec{r} - (\omega_j+\omega_m)t]}$$

$$+ E_j^* \cdot E_m^* \, e^{-i[(\vec{k}_j+\vec{k}_m)\cdot\vec{r} - (\omega_j+\omega_m)t]}$$

$$+ E_j \cdot E_m^* \, e^{i[(\vec{k}_j-\vec{k}_m)\cdot\vec{r} - (\omega_j-\omega_m)t]}$$

$$+ E_j^* \cdot \vec{E}_m \, e^{-i[(\vec{k}_j-\vec{k}_m)\cdot\vec{r} - (\omega_j-\omega_m)t]} \Big\}$$

<u>Using A1:</u> all $(\omega_j+\omega_m)t$ terms oscillate quickly and time-average to zero.



We keep $(\omega_j-\omega_m)t$. It oscillates slowly. Why ??

Assume $n \approx$ constant.

$\therefore k_m/\omega_m \mu_0 \approx n \epsilon_0 c$

$$\langle \vec{S}(\vec{r},t)\rangle_{osc} = \frac{n\varepsilon_0 c}{2}\sum_{j,m} \frac{\vec{E}_j\cdot\vec{E}_m^{*}\, e^{i[(\vec{k}_j-\vec{k}_m)\cdot\vec{r}-(w_j-w_m)t]} + \vec{E}_j^{*}\cdot\vec{E}_m\, e^{-i[(\vec{k}_j-\vec{k}_m)\cdot\vec{r}-(w_n-w_m)t]}}{2}$$

$$= \frac{n\varepsilon_0 c}{2}\,\mathrm{Re}\left\{ \sum_j \vec{E}_j\, e^{i(\vec{k}_j\cdot\vec{r}-w_j t)} \sum_m \vec{E}_m^{*}\, e^{-i(\vec{k}_m\cdot\vec{r}-w_m t)} \right\}$$

$$= \frac{n\varepsilon_0 c}{2}\,\mathrm{Re}\left\{ \vec{E}(\vec{r},t)\cdot\vec{E}^{*}(\vec{r},t) \right\}$$

$$\boxed{\therefore\ \underline{I}(\vec{r},t) = \frac{n\varepsilon_0 c}{2}\,\vec{E}(\vec{r},t)\cdot\vec{E}^{*}(\vec{r},t)}$$

Parallel or nearly parallel $\vec{k}_j$.

7.2 Group vs. Phase Velocity: Sum of two plane Waves

⑥

$\vec{E}_1 = \vec{E}_0 e^{i(\vec{k}_1 \cdot \vec{r} - w_1 t)}$ $\vec{E}_2 = \vec{E}_0 e^{i(\vec{k}_2 \cdot \vec{r} - W_2 t)}$

Equal Amplitudes.

phase Velocities: $V_1 = \frac{\omega_1}{k_1}$; $V_2 = \frac{W_2}{k_2}$

$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}_1 \cdot \vec{r} - W_1 t)} + \vec{E}_0 e^{i(\vec{k}_2 \cdot \vec{r} - W_2 t)}$

$\downarrow$

interference occurs.
constructive + destructive.

$$I(\vec{r},t) = \frac{n\varepsilon_0 c}{2} \vec{E}(\vec{r},t) \cdot \vec{E}^*(\vec{r},t)$$

$$= \frac{n\varepsilon_0 c}{2} \left[\vec{E}_0 e^{i(\vec{k}_1 \cdot \vec{r} - w_1 t)} + \vec{E}_0 e^{i(\vec{k}_2 \cdot \vec{r} - W_2 t)} \right]$$

$$\cdot \left[\vec{E}_0^* e^{-i(\vec{k}_1 \cdot \vec{r} - w_1 t)} + \vec{E}_0^* e^{-i(\vec{k}_2 \cdot \vec{r} - W_2 t)} \right]$$

$$= \frac{n\varepsilon_0 c}{2}\Big[\vec{E}_0 \cdot \vec{E}_0^{*} + \vec{E}_0 \cdot \vec{E}_0^{*} e^{i(\vec{k}_1 \cdot \vec{r} - w_1 t)} e^{-i(\vec{k}_2 \cdot \vec{r} - w_2 t)}$$

$$+ \vec{E}_0 \cdot \vec{E}_0^{*} e^{i(\vec{k}_2 \cdot \vec{r} - w_2 t)} e^{-i(k_1 \cdot \vec{r} - w_1 t)}$$

$$+ \vec{E}_0 \cdot \vec{E}_0^{*}\Big]$$

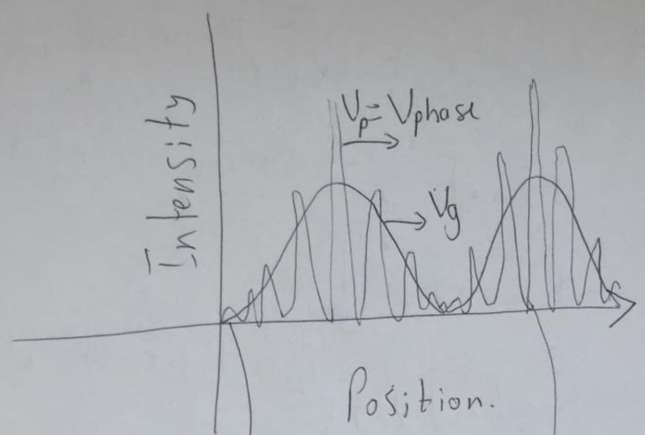
$$= \frac{n\varepsilon_0 c}{2} \vec{E}_0 \cdot \vec{E}_0^{*}\Big[2 + e^{i[(\vec{k}_2 - \vec{k}_1) \cdot \vec{r} - (W_2 - W_1)t]} + e^{-i[(\vec{k}_2 - \vec{k}_1) \cdot \vec{r} - (W_2 - W_1)t]}\Big]$$

$$= n\varepsilon_0 c\, \vec{E}_0 \cdot \vec{E}_0^{*}\Big[1 + \cos[(\vec{k}_2 - \vec{k}_1) \cdot \vec{r} - (W_2 - W_1)t]\Big]$$

$$= n\varepsilon_0 c\, \vec{E}_0 \cdot \vec{E}_0^{*}\Big[1 + \cos(\Delta \vec{k} \cdot \vec{r} - \Delta W t)\Big]$$

$$\Delta \vec{k} \equiv \vec{k}_2 - \vec{k}_1$$

$$\Delta W \equiv W_2 - W_1$$



averaged over rapid oscillations.

keeping rapid oscillations.

$$V_p = \frac{\bar{W}}{\bar{k}} \quad ; \quad V_g \equiv \frac{\Delta W}{\Delta k} \cong \left.\frac{dW}{dk}\right|_{\bar{W}}$$

phase velocity

group velocity.
(envelope velocity)

P 7.3 HW

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Example 7.1

$$n_{plasma}(w) = \sqrt{1 - w_p^2/w^2} \quad (<1) \; (\text{let } w > w_p)$$

Find V_p, V_g for superposition of two plane waves in a plasma.

$$V_p = \frac{\bar{w}}{\bar{k}} = \frac{w_1 + w_2/2}{k_1 + k_2/2}$$

$$= \frac{w_1 + w_2}{\frac{n_p(w_1)\, w_1}{c} + \frac{n_p(w_2)\, w_2}{c}}$$

if $w_1 \approx w_2 \rightarrow n_p(w_1) \cong n_p(w_2)$

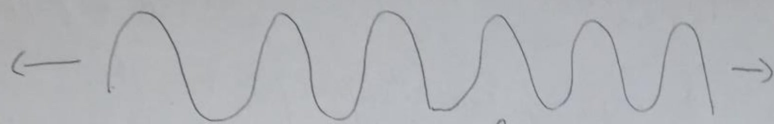
$$V_p = \frac{c}{\underbrace{n_p(w)}_{<1}} \quad \therefore \quad \boxed{V_p > c}$$

$$V_g = \frac{\Delta w}{\Delta k} \cong \frac{dw}{dk} = \left[\frac{dk}{dw}\right]^{-1} = \left[\frac{d}{dw} \frac{w\, n_p(w)}{c}\right]^{-1}$$

$$= \left[\frac{d}{dw} \frac{w\sqrt{1 - w_p^2/w^2}}{c}\right]^{-1} = n_p(w)\, c$$

$$\left(1 - \frac{w_p^2}{w^2} + \frac{w_p^2}{w^2}\right) \Big/ \sqrt{1 - \frac{w_p^2}{w^2}}$$

7.3 Frequency Spectrum of Light



Plane wave with one frequency ω has infinite length and infinite duration.

A waveform that does not repeat, like a pulse is given by:

$$\vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

Spectrum

gives amplitude and phase of each plane wave component in the waveform.