Ch.1

- Signals are quantitive description of physical phenomenon, event or process.
- ex. 1) electrical current or voltage in circuit

  2) Audio signal

- Types of signals :
  1. continuous - time signal → $x(t)$, t: any value
  2. Discrete - time signal → $x[n]$, n: Integer
  3. Random signal like audio

// any signal will be represented in a function with either discrete or cont. variable.

- periodic or a periodic signal:

In general,

$$\underset{\substack{\uparrow\\ \text{Amp.}}}{x(t) = A} \cos(\underset{\substack{\uparrow\\ \text{ang. freq.}\\ \text{rad/sec.}}}{w_0} t + \underset{\substack{\uparrow\\ \text{phase}}}{\phi})$$

w_0 : angular frequency

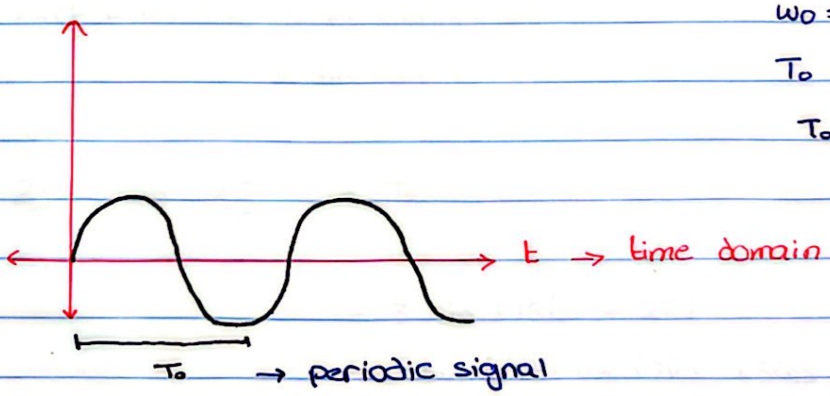
$w_0 = 2\pi f_0$

f_0 : fundamental freq.

$w_0 = \frac{2\pi}{T_0}$

T_0 : fundamental period

$T_0 = \frac{1}{f_0}$



→ periodic signal

. To check the periodicity of the signal:

$x(t + T_o) = x(t)$ يكرر نفسه بعد زمن معين

ex. Consider the following signals:

1) $x_1(t) = A \sin(2\pi f_o t + \phi)$

$x_1(t + t_o) \overset{?}{=} x_1(t)$

$\Rightarrow x_1(t + T_o) = A \sin(2\pi f_o (t + T_o) + \phi)$

$= A \sin(2\pi f_o t + 2\pi f_o T_o + \phi)$

$= A \sin(2\pi f_o t + \phi + 2\pi f_o \cdot \frac{1}{f_o})$

$= A \sin(\underbrace{2\pi f_o t + \phi}_{\alpha} + \underbrace{2\pi}_{\beta})$

. In general:

1. $\sin(\alpha \mp \beta) = \sin\alpha \cos\beta \mp \cos\alpha \sin\beta$
2. $\cos(\alpha \mp \beta) = \cos\alpha \cos\beta \pm \sin\alpha \sin\beta$

$= A \sin(2\pi f_o t + \phi) \cos(2\pi) + A\cos(2\pi \cancel{f_o} t + \phi) \sin(2\pi)$

$= A \sin(2\pi f_o t + \phi)$ → periodic signal

2) $x_2(t) = 3 \sin(15t)$

$\Rightarrow x_2(t + T_o) = 3 \sin(15(t + T_o))$

$= 3 \sin(15t + 15T_o)$

// $15 = w_o$

$15 = \frac{2\pi}{T_o} \Rightarrow \frac{2\pi}{15} = T_o$

$= 3 \sin(15t + 2\pi)$

$= 3 \sin(15t) \cos(2\pi) + 3\cancel{\cos}(15t) \sin(2\pi)$

$= 3 \sin(15t)$ → periodic signal.

3. $x_3(t) = A + B\cos(2\pi f_0 t)$

$\Rightarrow x_3(t + T_0) = A + B\cos(2\pi f_0 (t + T_0))$

$= A + B\cos(2\pi f_0 t + 2\pi f_0 T_0)$

$= A + B\cos(2\pi f_0 t + 2\pi)$

$= A + B\cos(2\pi f_0 t)$ → periodic signal.

4. $x_4(t) = \sin(\underset{\sim}{15\pi} t) + \cos(\underline{30\pi} t)$

$\Rightarrow x_4(t + T_0) \overset{?}{=} x_4(t)$ // but we have w_1, w_2

$w_1 = 2\pi f_1$ $\quad$ $w_2 = 2\pi f_2$

$w_1 = 2\pi n_1 f_0$ $\quad$ $w_2 = 2\pi n_2 f_0$

$15\pi = 2\pi n_1 f_0$ $\quad$ $30\pi = 2\pi n_2 f_0$

$\frac{15}{2} = n_1 f_0$ — 1 $\quad$ $15 = n_2 f_0$ — 2

$\frac{15}{2} \cdot \frac{1}{15} = \frac{n_1}{n_2} \Rightarrow \frac{1}{2} = \frac{n_1}{n_2}$ ⇒ rational number → periodic

. we find : $\underline{T_0} : \frac{1}{f_0}$

$\frac{15}{2} = (1) \cdot f_0 \rightarrow f_0 = \frac{15}{2}$ Hz $\Rightarrow T_0 = \frac{2}{15}$ sec.

5. $x_5(t) = \sin(15t) + \cos(30\pi t)$

$\Rightarrow \frac{15}{30\pi} = \frac{n_1}{n_2} \Rightarrow \frac{1}{2\pi} = \frac{n_1}{n_2}$ ⇒ irrational number → a periodic (not periodic)

6. $x_6(t) = \sin(\frac{5\pi}{6} t) + \cos(\frac{3\pi}{4} t) + \sin(\frac{\pi}{3} t)$

$\frac{5\pi}{6} = 2\pi f_1$ $\quad$ $\frac{3\pi}{4} = 2\pi f_2$ $\quad$ $\frac{\pi}{3} = 2\pi f_3$ $\quad$ عن طريق العامل المشترك الأكبر

$\frac{5}{12} = f_1$ $\quad$ $\frac{3}{8} = f_2$ $\quad$ $\frac{1}{6} = f_3$

$GCD\left(\frac{5}{12}, \frac{3}{8}, \frac{1}{6}\right) = \frac{1}{24} = f_0 \rightarrow$ periodic signal

// if one of them have $\pi \rightarrow$ then it is a periodic signal

. Phasor signal & spectra :

$$\bar{x}(t) = A e^{j\phi} e^{j\omega_0 t} = A e^{j(\omega_0 t + \phi)}$$

.using Euler's formula :

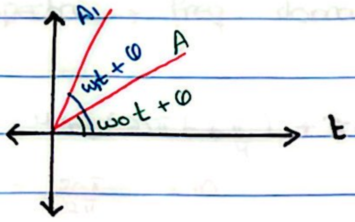
$$A e^{j(\omega_0 t + \phi)} = A[\cos(\omega_0 t + \phi) + j \sin(\omega_0 t + \phi)]$$

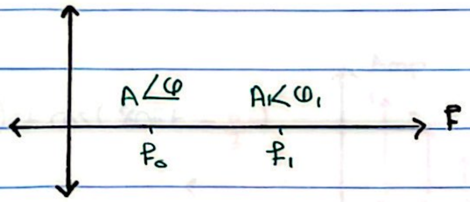
$$x(t) = Real\{\bar{x}(t)\} = A\cos(\omega_0 t + \phi)$$

// الناقل للصوت في وسط معين نعبّر عنه عن طريق الـ freq. ، يجب أن ننقل كل إشارة عبر ناقل مختلف حتى لا يحصل تشويش ← noise

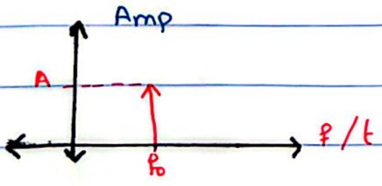
// لا نستطيع تمثيل كلّ إشارة بشكل منفصل في الـ time domain بسبب عملية الـ filtiring. فلذلك نلجأ لاستخدام الـ phasor domain.

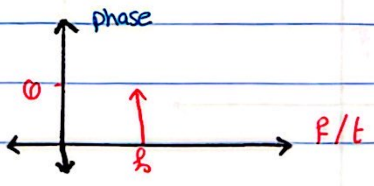
$x(t) = A\cos(\omega_0 t + \phi)$ $x_1(t) = A_1\cos(\omega_1 t + \phi_1)$





. single sided spectra:





// freq. always positive.

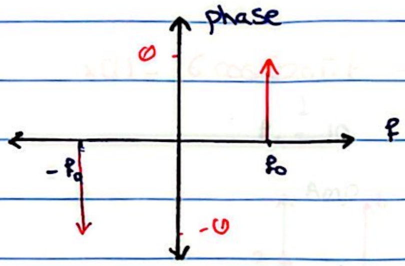
. Double sided spectra : $x(t) = A\cos(\omega_0 t + \theta)$

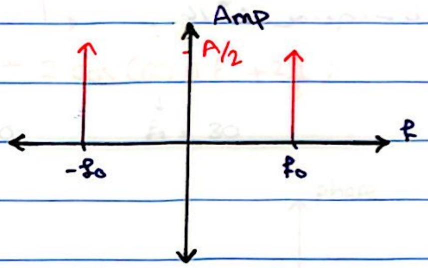
since;

$$\cos(\omega_0 t + \theta) = \frac{e^{j(\omega_0 t + \theta)} + e^{-j(\omega_0 t + \theta)}}{2}$$

$$x(t) = \frac{A}{2} e^{j(\omega_0 t + \theta)} + \frac{A}{2} e^{-j(\omega_0 t + \theta)}$$

Amp → even function.





phase → odd function.

+ → $\frac{\pi}{2}$ نطرح

− → $\frac{\pi}{2}$ نجمع

ex. Given the signal:

$x(t) = 4\cos(20\pi t + \frac{\pi}{4}) + 3\cos(60\pi t - \frac{\pi}{6}) + \sin(80\pi t + \frac{\pi}{6})$

1. sketch its single-sided amp. & phase spectra.
2. sketch its double-sided amp. & phase spectra.

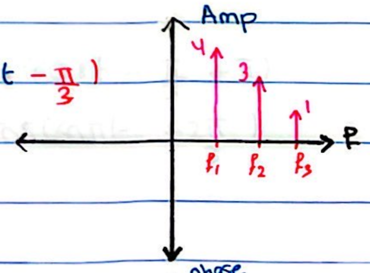
// spectra → freq. domain.

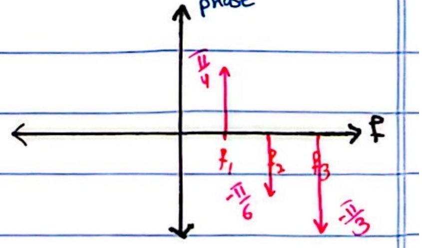
$x(t) = 4\cos(20\pi t + \frac{\pi}{4}) + 3\cos(60\pi t - \frac{\pi}{6}) + \cos(80\pi t - \frac{\pi}{3})$

1) $f_1 = \frac{20\pi}{2\pi} = 10$

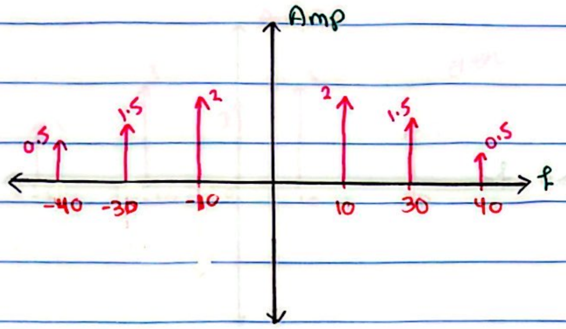
$f_2 = \frac{60\pi}{2\pi} = 30$

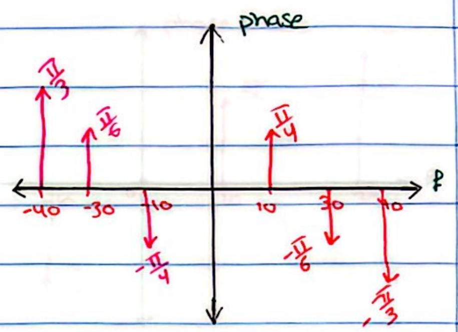
$f_3 = \frac{80\pi}{2\pi} = 40$





2)





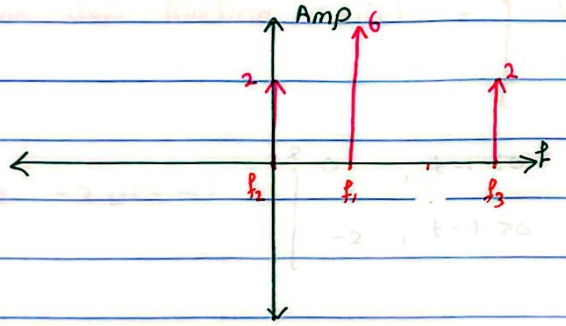
<u>ex.</u> $x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 4\sin^2(30\pi - \frac{\pi}{6})$

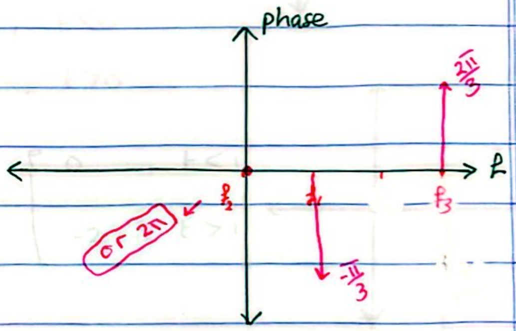
$= 6\cos(20\pi t - \frac{\pi}{3}) + 4\left[\frac{1}{2} - \frac{1}{2}\cos(2.30\pi t - \frac{\pi}{3})\right]$

↳ نتخلص من السالب

$x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 2 + 2\cos(60\pi t + \frac{2\pi}{3})$

$f_1 = 10$ $f_2 = 0$ $f_3 = 30$





// $2 = 2\angle 0$ or $2 = 2\angle 2\pi$

$x(t) = 6\cos(20\pi t \div \frac{\pi}{3}) + 2 \underset{\sim}{\overset{\swarrow}{-}} 2\cos(60\pi t \overset{\swarrow}{-} \frac{\pi}{3})$

. $\cos(\alpha \mp \pi) = \cos(\alpha)\cos(\pi) \pm \sin(\alpha)\sin(\pi)$

$\cos(\alpha \mp \pi) = -\cos\alpha$

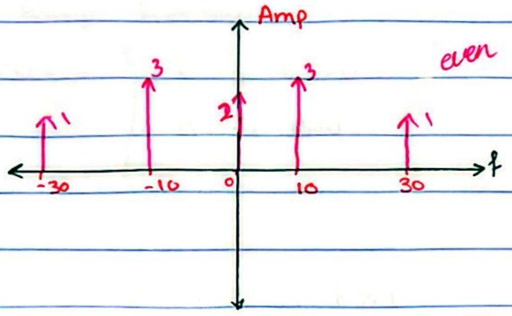
$x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 2\cos(0) + 2\cos(60\pi t - \frac{\pi}{3} + \pi)$

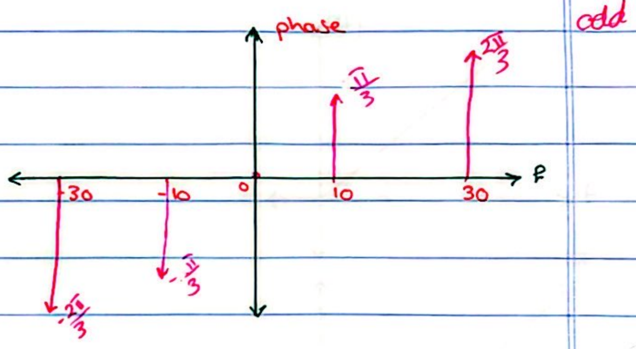
$x(t) = 6\cos(20\pi t - \frac{\pi}{3}) + 2\cos(0) + 2\cos(60\pi t + \frac{2\pi}{3})$

// $-2 = 2\angle\pi$

// $j2 = 2\angle\pi/2$

// $-j2 = 2\angle -\pi/2$

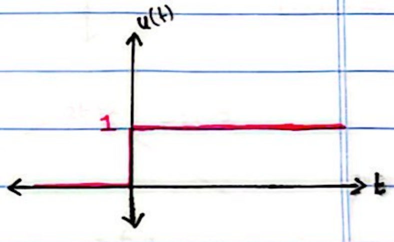




. Singularity functions : (elementary functions)

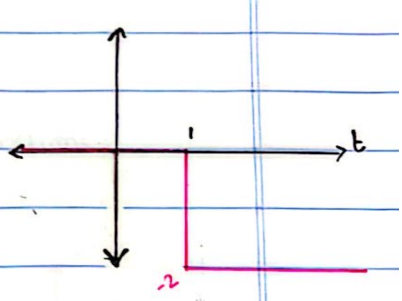
1. Step function : $Au(t)$

$$A\,u(t) = \begin{cases} 0 , & t<0 \\ A , & t>0 \end{cases}$$

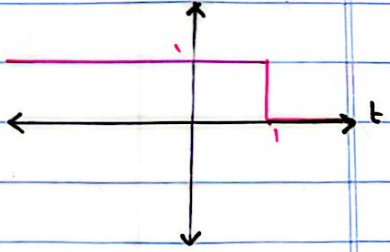


. unit step function : $u(t) = \begin{cases} 0 , & t<0 \\ 1 , & t>0 \end{cases}$

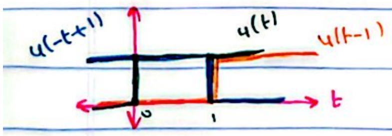
ex. $-2\,u(t-1) = \begin{cases} 0 , & t-1<0 \\ -2 , & t-1>0 \end{cases} = \begin{cases} 0 , & t<1 \\ -2 , & t>1 \end{cases}$



. $u(-t+1) = u(-(t-1))$



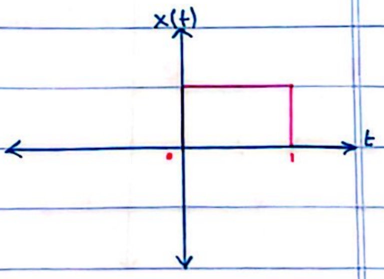
Ex. Consider the following signal, express $x(t)$ in terms of step function:



→ $x(t) = u(t) * u(-t+1)$

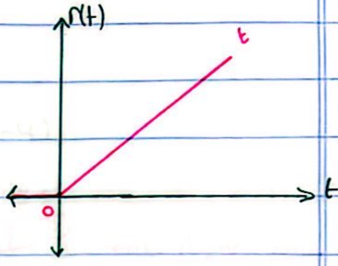
or → $x(t) = u(t) - u(t-1)$

or → $x(t) = u(-t+1) - u(-t)$

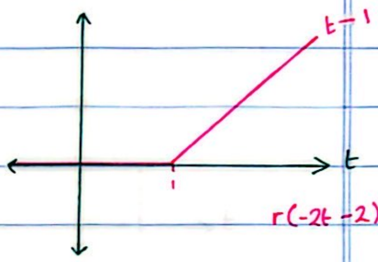


2. Ramp function :

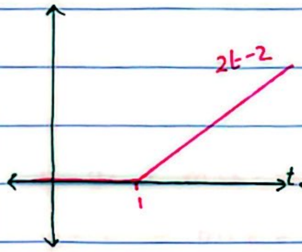
$$r(t) = \begin{cases} t , & t \geq 0 \\ 0 , & o.w \end{cases}$$



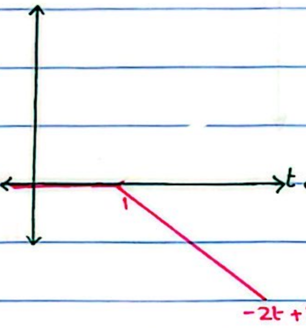
$$r(t-1) = \begin{cases} t-1 , & t \geq 1 \\ 0 , & o.w \end{cases}$$



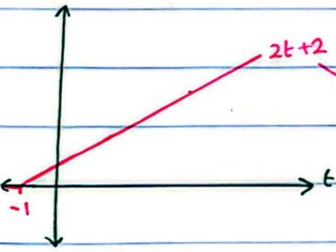
$2r(t-1)$



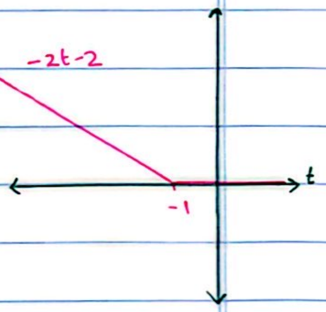
$-2r(t-1)$



$r(2t+2)$



$r(-2t-2) = r(-2(t+1))$



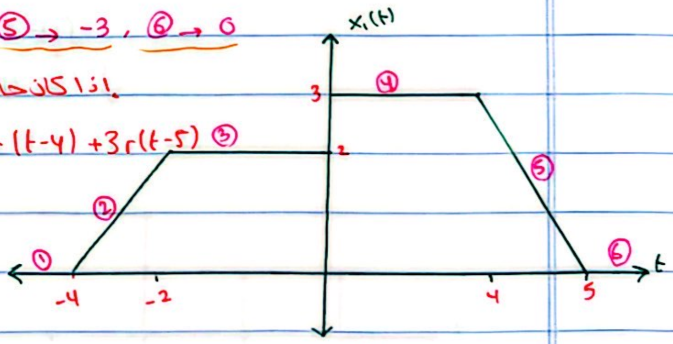
<u>ex.</u> Express the following signals in terms of singularity functions :

. slope :

① → 0 , ② → 1 , ③ → 0 , ④ → 0 , ⑤ → -3 , ⑥ → 0

. ramp. ← إذا كان حاصل طرح ميل اليمين - الشمال ≠ صفر

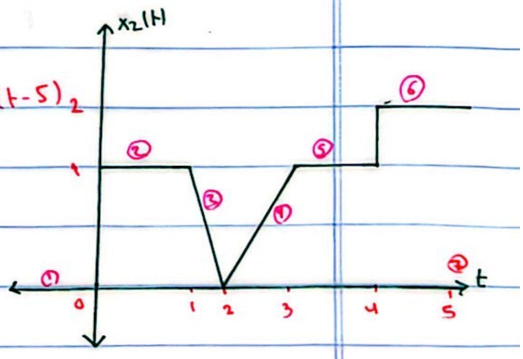
$x_1(t) = 1\,r(t+4) - 1\,r(t+2) + \underset{\underset{(3-2)}{\uparrow}}{1}\,\underset{\underset{u(t-0)}{\uparrow}}{u(t)} - 3r(t-4) + 3r(t-5)$ ③



slope:

① → 0 , ② → 0 , ③ → -1 , ④ → 1 , ⑤ → 0 , ⑥ → 0 , ⑦ → 0

$x_2(t) = u(t) - r(t-1) + 2(t-2) - r(t-3) + u(t-4) - 2u(t-5)_2$

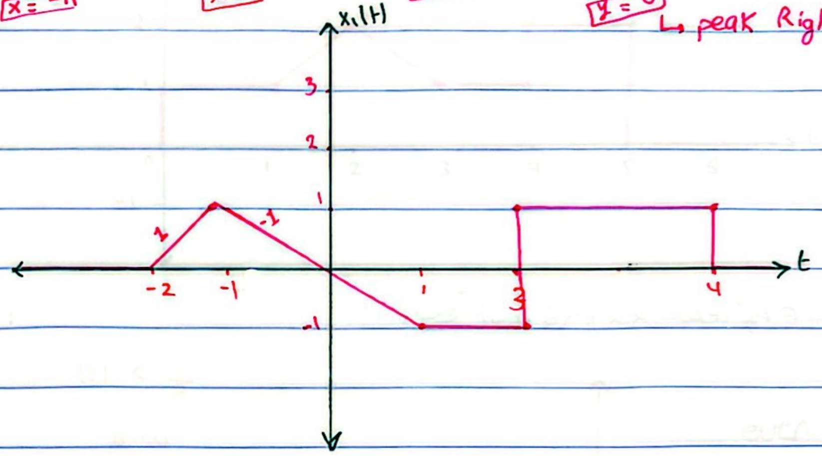


- Sketch the following signals:

// نقوم بترتيب الفنكشن بالبداية

$x_1(t) = \underset{\substack{\uparrow \\ \text{slope}}}{1} r(t+2) - \underset{\substack{\uparrow \\ x-1=-2 \\ \boxed{x=-1}}}{2r(t+1)} + \underset{\substack{\uparrow \\ 1=x+1 \\ \boxed{x=0}}}{r(t-1)} + \underset{\substack{\downarrow \\ 2=y-1 \\ \boxed{y=1}}}{2u(t-3)} - \underset{\substack{\downarrow \\ -1=y-1 \\ \boxed{y=0}}}{u(t-4)}$

↳ peak Right - peak left



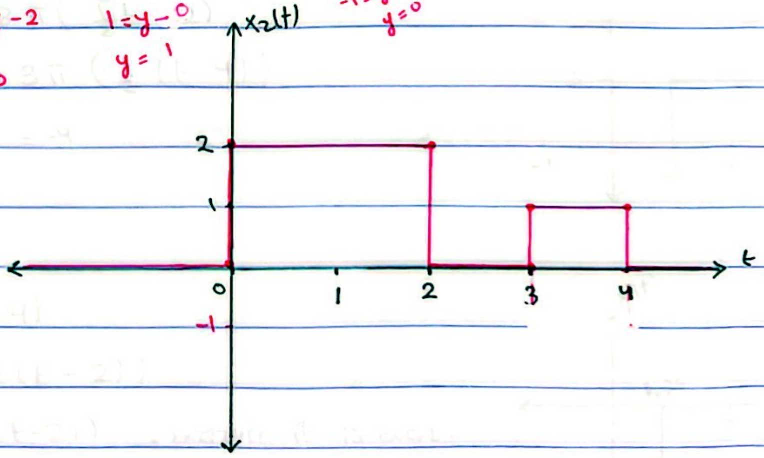
$y - y_0 = m(x - x_0)$

$y - 1 = m(t+1)$

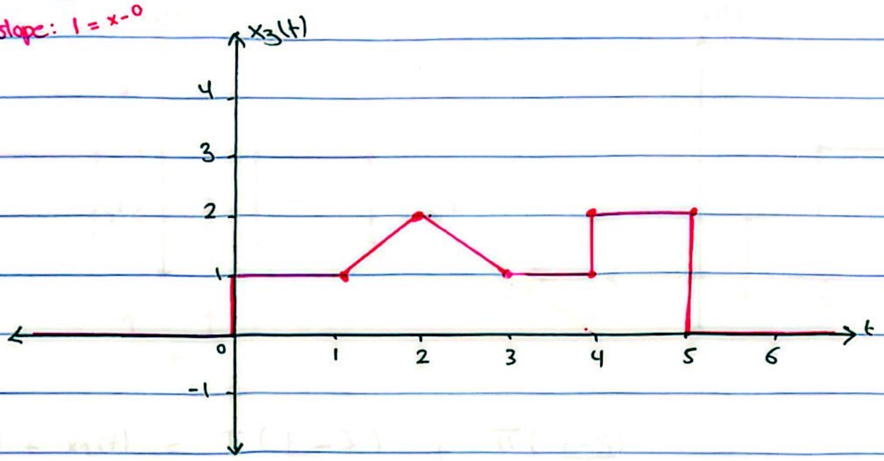
$y - 1 = -t - 1$

$\boxed{y = -t}$ معادلة الخط المستقيم ← يوضح العكس

$x_2(t) = \underset{\substack{\downarrow \\ 2=y-0 \\ y=2}}{2u(t)} - \underset{\substack{\downarrow \\ -2=y-2 \\ y=0}}{2u(t-2)} + \underset{\substack{\downarrow \\ 1=y-0 \\ y=1}}{u(t-3)} - \underset{\substack{\downarrow \\ -1=y-1 \\ y=0}}{u(t-4)}$



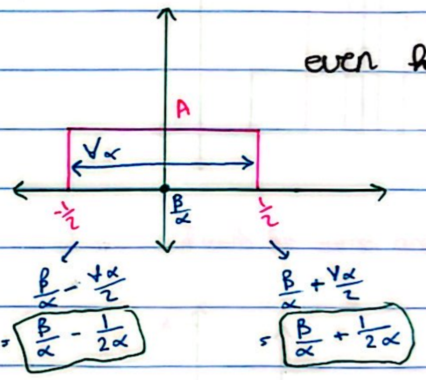
$$x_3(t) = u(t) + \underset{\text{slope: } 1 = x-0}{r(t-1)} - 2\overset{-2 = x-1}{r(t-2)} + \overset{1 = x+1}{r(t-3)} + \overset{1 = y-1}{u(t-4)} - 2\overset{-2 = y-2}{u(t-5)}$$



3. Pulse function :

// نفتحه ثم نغلقه بعد فترة من الزمن

$$A\,\Pi(t) = \begin{cases} A , & |t| < \frac{1}{2} \\ 0 , & \text{o.w} \end{cases}$$



even function.

In general :

$A\,\Pi(\alpha t + \beta)$; α & β are >0

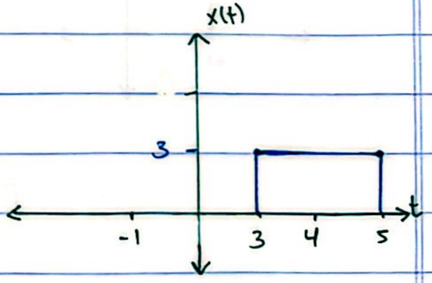
$$\underset{\text{peak}}{A}\,\Pi(\underset{\text{scaling}}{\alpha}(t + \underset{\text{centre}}{\frac{\beta}{\alpha}}))$$

ex. sketch $x(t) = 3\,\Pi(\frac{1}{2}t - 2)$

$= 3\,\Pi(\frac{1}{2}(t-4))$

$A = 3$, $\alpha = \frac{1}{2}$, $\frac{\beta}{\alpha} = -4$

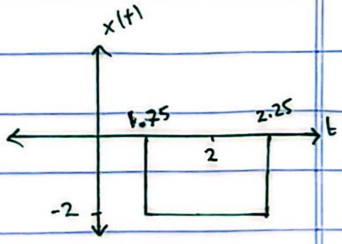
$\frac{1}{\alpha} = 2 \rightarrow \frac{1}{2\alpha} = 1$



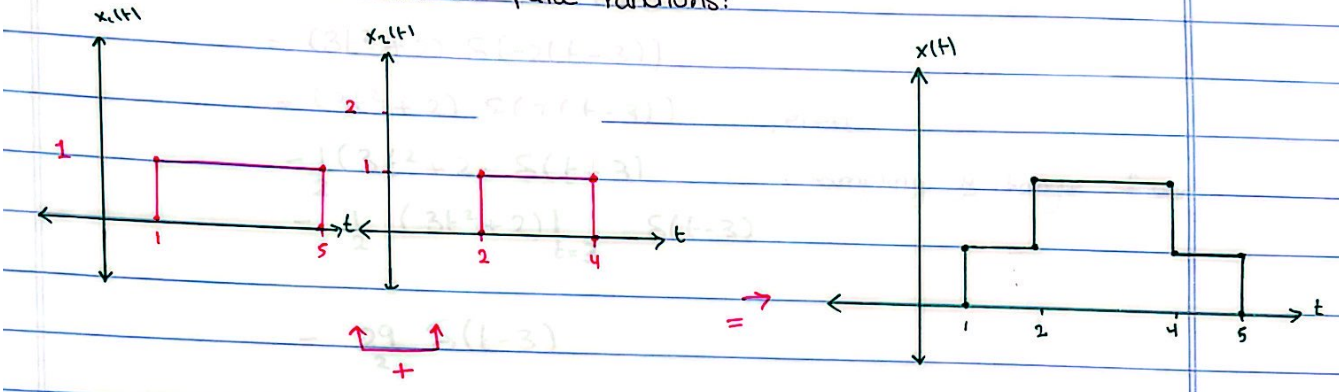
$x(t) = -2\,\Pi(-2t+4)$

$= -2\,\Pi(-2(t-2))$

$= -2\,\Pi(2(t-2))$ → because it is even.

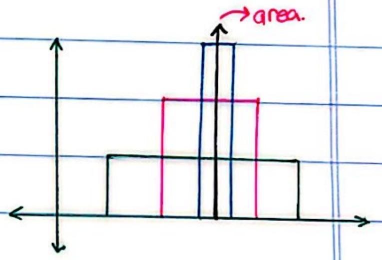


ex. Express x(t) in terms of pulse functions:



$x(t) = x_1(t) + x_2(t) = \Pi\left(\frac{t-3}{4}\right) + \Pi\left(\frac{t-3}{2}\right)$

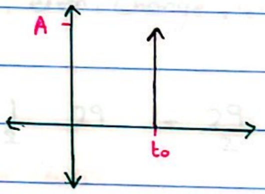
4. Impulse function



$$A\,\delta(t) = \begin{cases} A & , t=0 \\ 0 & , \text{o.w} \end{cases}$$

$$A\,\delta(t-t_0) = \begin{cases} A & , t=t_0 \\ 0 & , \text{o.w} \end{cases}$$

// with the same area.



// if A = 1 → unit impulse function.

. **Properties** of delta function:

1. $\delta(at) = \frac{1}{|a|}\,\delta(t)$ [change of variables]

2. $\delta(-t) = \delta(t)$ [even function]

ex. Evaluate : $\delta(-3t) = \delta(3t) = \frac{1}{3}\,\delta(t)$

3. Sampling Theorem.

$x(t)\,.\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

ex. Evaluate : $(3t^2+2)\;\delta(-2t+6)$

$= (3t^2+2)\;\delta(-2(t-3))$

$= (3t^2+2)\;\delta(2(t-3))$; even

$= \frac{1}{2}(3t^2+2)\;\delta(t-3)$; sampling & change of var.

$= \frac{1}{2}\,(3t^2+2)\Big|_{t=3}\cdot\delta(t-3)$

$= \frac{29}{2}\,\delta(t-3)$

4. Sifting Theorem:

$$\int_{t_1}^{t_2} x(t)\,\delta(t-t_0)\,dt = \begin{cases} x(t_0) & ,\ t_1<t_0<t_2 \\ 0 & ,\ o.w \end{cases}$$

ex. Evaluate : $\int_{-2}^{4}(3t^2+2)\;\delta(-2t+6)\;dt$

$= \int_{-2}^{4}(3t^2+2)\;\delta(2(t-3))\;dt$; even, change, sifting, sampling

$= \int_{-2}^{4}(3t^2+2)\cdot\frac{1}{2}\;\delta(t-\underset{\downarrow}{3})\;dt = \frac{1}{2}\cdot 29 = \frac{29}{2}$

$-2<3<4 \rightarrow$ not zero

5. Derivative : $\int_{t_1}^{t_2} x(t)\;\delta^{(n)}(t-t_0)\;dt = \begin{cases} (-1)^n\,\frac{d^n}{dt^n}\,x(t)\Big|_{t=t_0} & ,\ t_1<t_0<t_2 \\ 0 & ,\ o.w \end{cases}$

ex. Evaluate : $\int_{-2}^{3}(3t^2+3t)\;\overset{\cdot\rightarrow(1)}{\delta}(-2t+2)\;dt$

$= \int_{-2}^{3}(3t^2+3t)\cdot\frac{1}{2}\cdot\dot{\delta}(t-1)\;dt$; even, sampling, change, sifting, derivativ

$= \frac{-1}{2}\;\frac{d}{dt}\;(3t^2+3t)\Big|_{t=1} = \frac{-1}{2}\;(6t+3)\Big|_{t=1} = -\frac{9}{2}$

$-2\overset{\downarrow}{<}1<3 \rightarrow$ not zero.

ex. $\int_{-2}^{2} \cos(20\pi t)\ \delta'(t+1)\ dt = (-1) \cdot -\sin(20\pi t) \cdot 20\pi \Big|_{t=-1} = 0$

//we need to calculate the answer in radian not degree..

. Show that $\int_{-\infty}^{\infty} x(t)\ \delta(t-t_0)\, dt = x(t_0)$

By using Mathematical analysis:

In general; $\frac{du(t)}{dt} = \delta(t)$

$\frac{d}{dt} r(t) = u(t)$, $\frac{d^2}{dt^2} r(t) = \delta(t)$

Proof: Assume $\frac{d}{dt} u(t-t_0) = \delta(t-t_0)$

$\int_{-\infty}^{\infty} x(t)\ du(t-t_0)$, by parts

let $v = x(t)$ $dr = du(t-t_0)$

$dv = \frac{d}{dt}\, x(t)$ $r = u(t-t_0)$

$x(t)\ u(t-t_0) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x'(t)\ u(t-t_0)\, dt$

$= x(\infty)\ u(\infty) - x(-\infty)\ u(-\infty) - \int_{t_0}^{\infty} x(t)\ dt$

$= x(\infty)\ \overset{1}{\cancel{u(\infty)}} - x(-\infty)\ \overset{0}{\cancel{u(-\infty)}} - x(\infty) + x(t_0)$

$= \cancel{x(\infty)} - \cancel{x(\infty)} + x(t_0) = \boxed{x(t_0)}$

. Evaluate the following Integrals:

1) $\int_{5}^{10} \cos(2\pi t)\, \delta(t-2)\, dt = 0 \quad (2 \notin 5-10)$

2) $\int_{0}^{5} \cos(2\pi t)\, \delta(t-2)\, dt = \cos(4\pi) = 1$

3) $\int_{-\infty}^{\infty} \left[e^{-3t} + \cos(2\pi t)\right] \delta(t)\, dt = 1 + \cos(0) = 1$

4) $\int_{-\infty}^{\infty} \left[e^{-3t} + \cos(2\pi t)\right] \dot{\delta}(t)\, dt = (-1)^1 \left[-3e^{-3t} - 2\pi \sin(2\pi t)\right]\Big|_{t=0} = 3$

⑤ $\int_{-\infty}^{\infty} e^{3t}\, \ddot{\delta}(t-2)\, dt = \frac{d^2}{dt^2}(e^{3t})\Big|_{t=2} = (3e^{3t})' = 9e^{3t}\Big|_{t=2} = 9e^{6}$

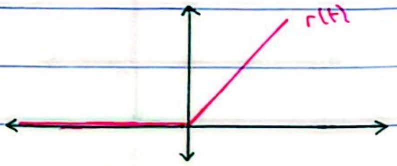
. Sketch the following signals:

$x_1(t) = \sum_{n=0}^{\infty} x_a(t-2n)$ [plot for $0 \leq t \leq 6$]

↓ periodic

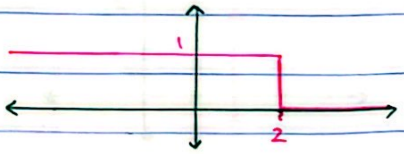
where $x_a(t) = r(t)\, u(-t+2)$

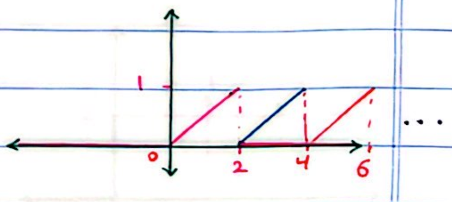
$\Rightarrow x_1(t) = \sum_{n=0}^{\infty} r(t-2n)\, u(2+2n-t)$



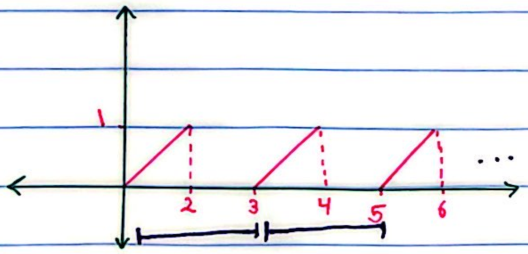
when; $n=0$:

$x_1(t)\Big|_{n=0} = r(t)\, u(2-t)$





. if it were $\underline{3n}$:



period → 3n

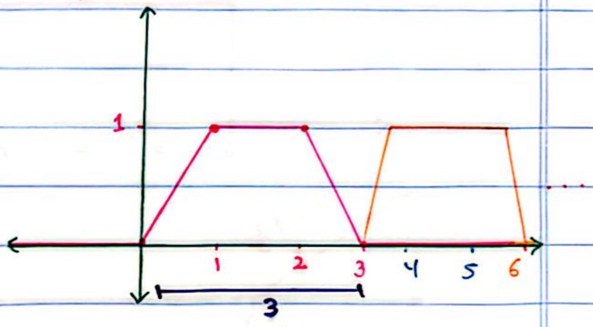
- $x_2(t) = \sum_{n=0}^{\infty} x_b(t-3n)$ [plot $0 \leq t \leq 6$]

where, $x_b(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$

$$\Rightarrow x_2(t) = \sum_{n=0}^{\infty} r(t-3n) - r(t-3n-1) - r(t-3n-2) + r(t-3n-3)$$

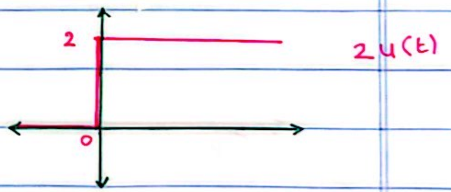
when; $n=0$

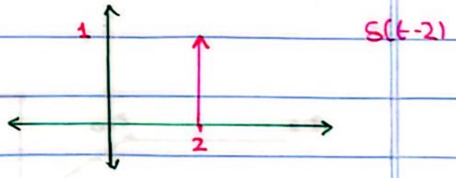
$$x_2(t)\Big|_{n=0} = \underset{\substack{\downarrow \\ 1 = x-0 \\ x=1}}{r(t)} - \underset{\substack{\downarrow \\ -1 = x-1 \\ x=0}}{r(t-1)} - \underset{\substack{-1 = x-0 \\ x=-1}}{r(t-2)} + \underset{\substack{1 = x+1 \\ x=0}}{r(t-3)}$$

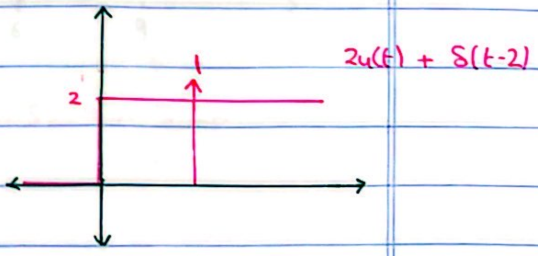


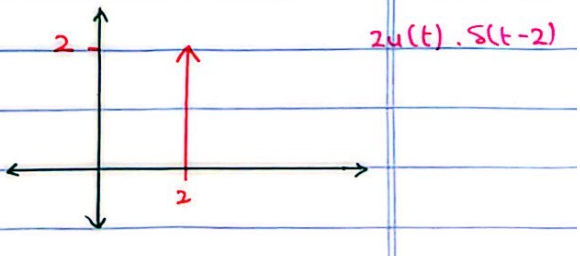
- $x_3(t) = 2u(t) + \delta(t-2)$

- $x_4(t) = 2u(t) \,.\, \delta(t-2) = 2u(2)\,\delta(t-2)$







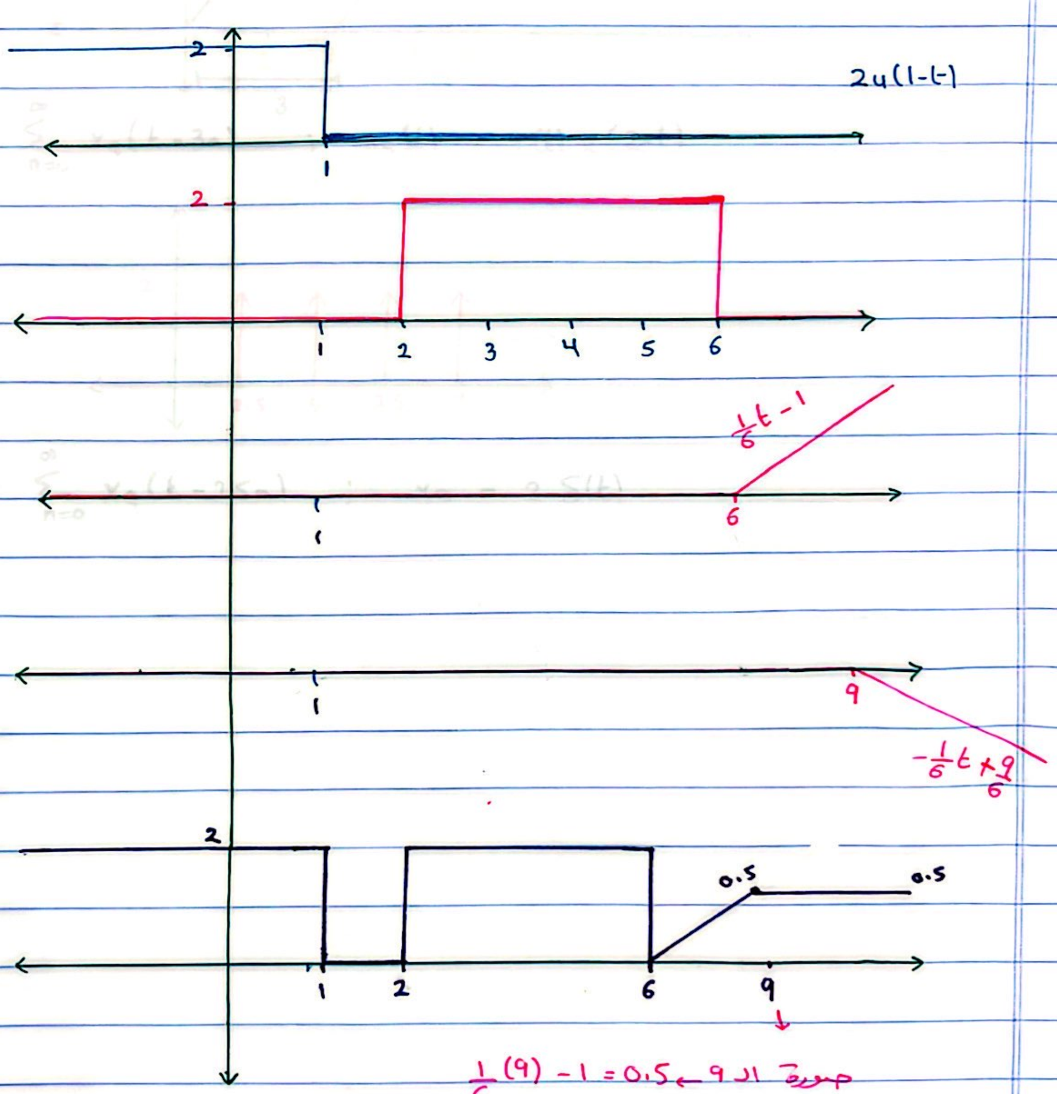


. Sketch the following signal using elementary functions:

$$x(t) = 2\Pi\left(\frac{t-4}{2}\right) + r\left(\frac{t-6}{6}\right) - r\left(\frac{t-9}{6}\right) + 2u(1-t)$$

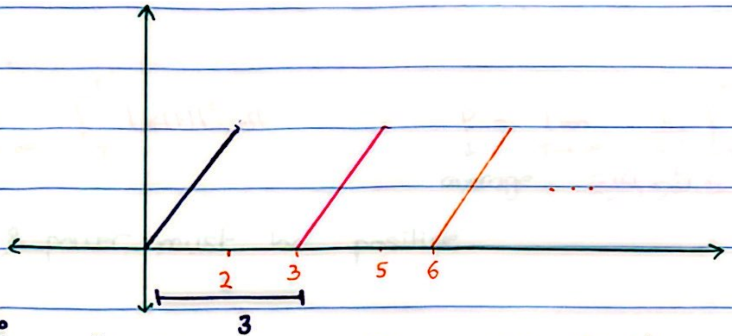
$$x(t) = 2u(1-t) + 2\Pi\left(\frac{t-4}{4}\right) + r\left(\frac{t-6}{6}\right) - r\left(\frac{t-9}{6}\right)$$

$$x(t) = 2u(1-t) + 2\Pi\left(\frac{1}{4}(t-4)\right) + r\left(\frac{1}{6}(t-6)\right) - r\left(\frac{1}{6}(t-9)\right)$$

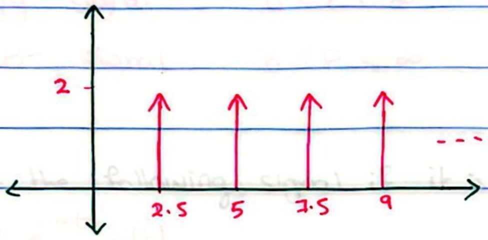


$\frac{1}{6}(9)-1=0.5 \leftarrow$ تعويض ال 9

- Express the signals shown below in terms of singularity functions



$x_1(t) = \sum_{n=0}^{\infty} x_a(t-3n) \quad ; \quad x_a(t) = r(t)\, u(2-t)$



$x_2(t) = \sum_{n=0}^{\infty} x_a(t-2.5n) \quad ; \quad x_a = 2\,\delta(t)$

• Energy signal & power signal :

• In general :

$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt$ $\qquad$ $\underset{\downarrow}{P} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$

average. نقسم على الفترة كاملة

// energy & power must be positive.

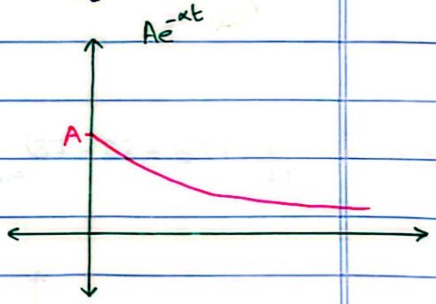
• Signal classes :

1. x(t) energy signal $\quad 0 < E < \infty \quad \& \; P = 0$

2. x(t) power signal $\quad 0 < P < \infty \quad \& \; E = \infty$

• Ex. Check if the following signal if it is power / energy signal:

$x(t) = A e^{-\alpha t} u(t)$



$\Rightarrow E = \lim_{T \to \infty} \int_{-T}^{T} |Ae^{-\alpha t}|^2 \, (u(t))^2$

$= \lim_{T \to \infty} \int_{-T}^{0} (0)^2 \, dt + \lim_{T \to \infty} \int_{0}^{T} A^2 e^{-2\alpha t} \, dt$

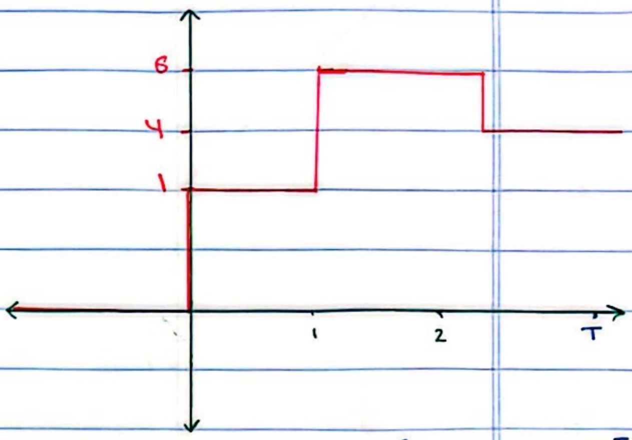
$= A^2 \lim_{T \to \infty} \left. \frac{e^{-2\alpha}}{-2\alpha} \right|_{0}^{T} = -A^2 \left(0 - \frac{1}{2\alpha} \right) = \frac{A^2}{2\alpha}$

$\Rightarrow P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \cdot \frac{A^2}{2\alpha} = 0 \Rightarrow$ energy signal.

Ex. Which of the following signals are power & which are energy signals

1. $x_1(t) = u(t) + 5u(t-1) - 2u(t-2)$

$\downarrow$ $5 = y - 1$, $y = 6$ $\qquad$ $\downarrow$ $-2 = y - 6$, $4 = y$



$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{0} 0^2 dt + \lim_{T\to\infty} \int_{0}^{1} 1^2 dt + \lim_{T\to\infty} \int_{1}^{2} (6)^2 dt + \lim_{T\to\infty} \int_{2}^{T} (4)^2 dt$$

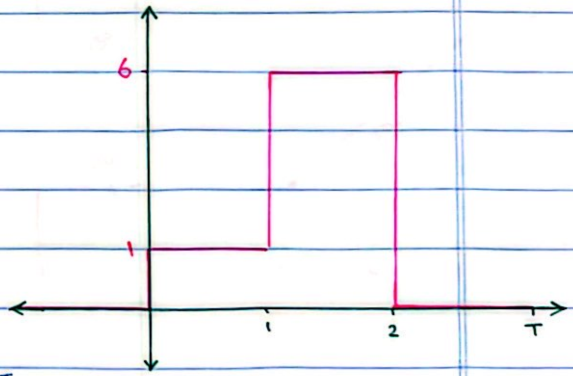
$$= \lim_{T\to\infty} 0 + 1 + 36 + 16(T-2) = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \left[\int_{0}^{1} 1^2 dt + \int_{1}^{2} (6)^2 dt + \int_{2}^{T} (4)^2 dt \right] = \lim_{T\to\infty} (37 - 32 + 16T) \cdot \frac{1}{2T} = \underline{8 < \infty}$$

$\Rightarrow$ power signal.

2. $x_2(t) = u(t) + 5u(t-1) - 6u(t-2)$

$\downarrow$ $5 = y - 1$, $y = 6$ $\qquad$ $\downarrow$ $-6 = y - 6$, $y = 0$



$$E = \lim_{T\to\infty} \left[\int_{-T}^{0} (0)^2 dt + \int_{0}^{1} (1)^2 dt + \int_{1}^{2} (6)^2 dt + \int_{2}^{T} (0)^2 dt \right] = 37$$

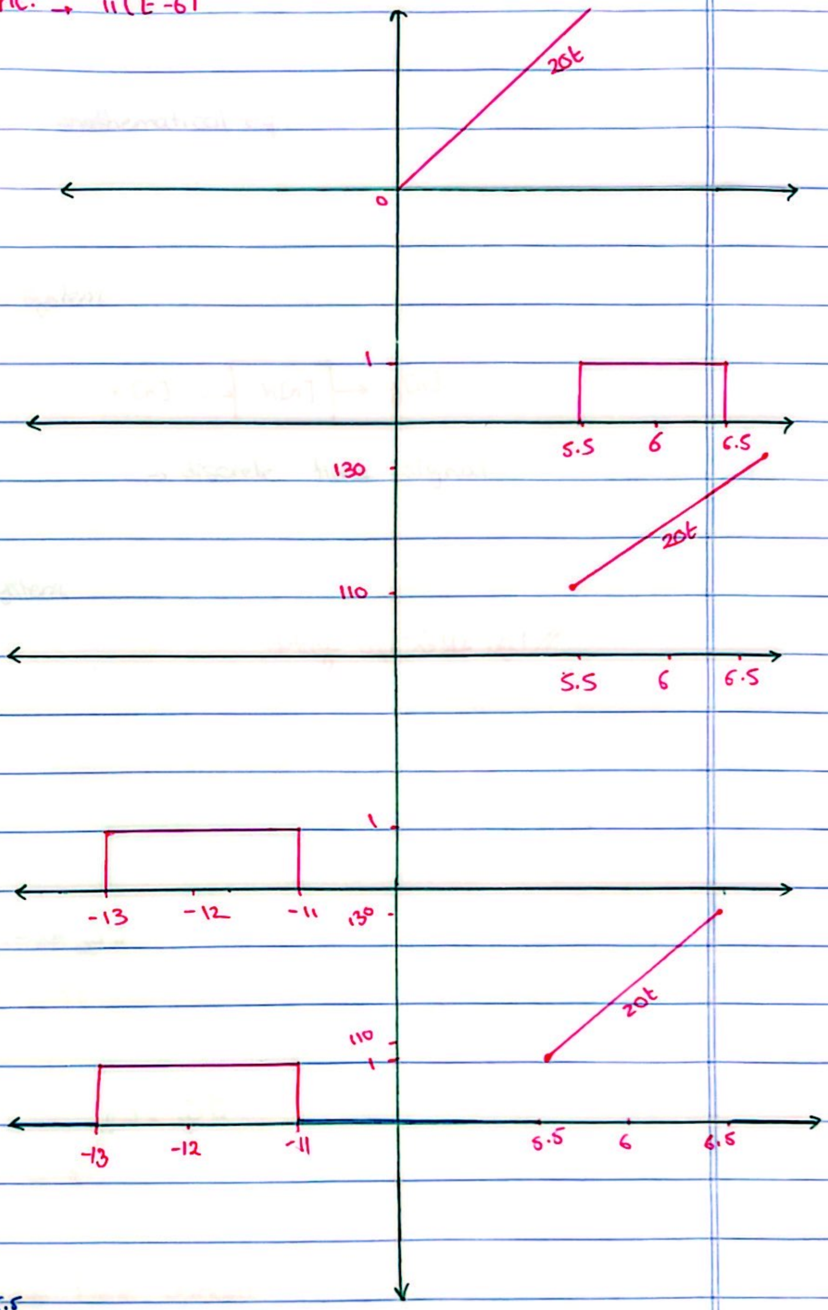
$$P = \lim_{T\to-\infty} \frac{37}{2T} = 0$$

$\Rightarrow$ energy signal

energy signal ← bounded ← إذا كان الاقتران يبدأ بصفر وينتهي بصفر //

3. $x_3(t) = 20\, r(t)\, \underset{\downarrow}{\Pi(t+6)} + \Pi(0.5t+6)$

even func. → $\Pi(t-6)$



⇒ bounded or time limitid

⇒ energy signal.

$$E = \lim_{T\to\infty} \left[\int_{-13}^{-11} (1)^2 dt + \int_{5.5}^{6.5} (20t)^2 dt \right. = 14.44\, K \text{ jouls.}$$

Ch. 2 : Systems :

$$x(t) \rightarrow \boxed{Sys.} \rightarrow y(t)$$

// we connect x & y using mathematical eq.

. Properities of systems :

1. Continuous & discrete time system

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$ $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

→ continuous time signal → discrete time signal

2. time invariant & variant system

//لها علاقة بال delay.

Ex.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$y(t) = x(t^2)$

1. Delay time. تأثير خارجي

$y_1(t-t_0) = x_1((t-t_0)^2)$ — ①

2. Delay function time تأثير داخلي

$y_2(t-t_0) = x_2(t^2-t_0)$ — ②

⇒ Eq.① ≠ Eq.② ⟶ time variant

3. causal & non causal system

$y(t) = x(t^2)$

- $y(2) = x(4)$ → non-causal ⇒ $t > 1$
  $\overset{\uparrow}{\text{present}}$ $\overset{\uparrow}{\text{future}}$

// value in output must be greater than value in input → causal (or equal)

- $y(0.5) = x(0.25)$ → causal ⇒ $t < 1$

Ex. $3y(t) + \int_{-\infty}^{2t} y(\lambda)\, d\lambda = x(t)$

$3y(t) + v(2t) - v(-\infty) = x(t)$

$\downarrow$

$2t > t$ → causal system. (يعتمد على قيمة سابقة في الماضي)

- $3y(t) + \int_{-\infty}^{t} y(\lambda)\, d\lambda = x(2t)$ → non causal. (يعتمد على قيمة مستقبلية)

- $y(t) = 10x(t+2) + 5$ → $t < t+2$ → non causal.

4. Instantaneous (memory less) & dynamic (memory)

Ex. $y(t+2) + 3y(t+2) = x(t+2)$ → inst.

// all of them must be zero order & same input → inst.

// integral or derivative → dynamic

Ex. $y(t) = x(t^2)$ → dynamic

- $\frac{dy(t)}{dt} + 3y(t) = x(t)$ → dynamic

5. linear & non linear

Ex. $\frac{dy(t)}{dt} + y(t) = x(t)$, check the linearity

. Scaling:

$$\frac{d}{dt}\alpha_1 y_1(t) + \alpha_1 y_1(t) = \alpha_1 x_1(t) \quad -①$$

$$\frac{d}{dt}\alpha_2 y_2(t) + \alpha_2 y_2(t) = \alpha_2 x_2(t) \quad -②$$

. Adding:

$$\frac{d}{dt}(\alpha_1 y_1(t) + \alpha_2 y_2(t)) + \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad -③$$

Assume $\alpha_3 y_3(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$

$\alpha_3 x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

$$\frac{d}{dt}\alpha_3 y_3(t) + \alpha_3 y_3(t) = \alpha_3 x_3(t) \qquad \alpha_3 \neq \alpha_1 + \alpha_2$$

$$\frac{d}{dt}(\alpha_1 y_1(t) + \alpha_2 y_2(t)) + \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad -④$$

. Eq. ③ = Eq. ④ → linear (التأثير الخارجي على النظام مهمل)

Ex. $\frac{d}{dt} y(t) + y(t) + 5 = x(t)$, check linearity.

. scaling : $\frac{d}{dt}\alpha_1 y_1(t) + \alpha_1 y_1(t) + 5\alpha_1 = \alpha_1 x_1(t) \quad -①$

$\frac{d}{dt}\alpha_2 y_2(t) + \alpha_2 y_2(t) + 5\alpha_2 = \alpha_2 x_2(t) \quad -②$

. Adding : $\frac{d}{dt}(\alpha_1 y_1(t) + \alpha_2 y_2(t)) + \alpha_1 y_1(t) + \alpha_2 y_2(t) + 5(\alpha_1 + \alpha_2) = \alpha_1 \alpha_1(t) + \alpha_2 x_2(t)$

. Assume : $\alpha_3 y_3(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$

$\alpha_3 x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$; $\alpha_3 \neq \alpha_1 + \alpha_2$

$$\frac{d}{dt}\alpha_3 y_3(t) + \alpha_3 y_3(t) + 5\alpha_3 = \alpha_3 x_3(t)$$

$$\frac{d}{dt}\left[\alpha_1 y_1(t) + \alpha_2 y_2(t)\right] + \alpha_1 y_1(t) + \alpha_2 y_2(t) + \underline{5\alpha_3} = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad - ④$$

⇒ non linear. (Eq. ③ ≠ Eq. ④)

<u>Ex.</u> $\frac{d}{dt}y(t) \cdot y(t) + y(t) = x(t)$; check linearity.

$$\alpha_1 y_1(t) \frac{d}{dt}\alpha_1 y_1 + \alpha_1 y_1(t) = \alpha_1 x_1(t) \quad - ①$$

$$\alpha_2 y_2(t) \frac{d}{dt}\alpha_2 y_2(t) + \alpha_2 y_2(t) = \alpha_2 x_2(t) \quad - ②$$

$$\alpha_1 y_1(t) \frac{d}{dt}\alpha_1 y_1(t) + \alpha_2 y_2(t) \frac{d}{dt}\alpha_2 y_2(t) + \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad - ③$$

. Assume : $\alpha_3 y_3(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$

$\alpha_3 x_3(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$

$$\alpha_3 y_3(t) \frac{d}{dt}\alpha_3 y_3(t) + \alpha_3 y_3(t) = \alpha_3 x_3(t)$$

$$\underline{(\alpha_1 y_1(t) + \alpha_2 y_2(t))} \frac{d}{dt}(\alpha_1 y_1(t) + \alpha_2 y_2(t)) + \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad - ④$$

⇒ non Linear . (Eq. ③ ≠ Eq. ④)

$$\Rightarrow \alpha_1 y_1(t) \frac{d}{dt}\alpha_1 y_1(t) + \alpha_1 y_1(t) \frac{d}{dt}\alpha_2 y_2(t) + \underline{\alpha_2 y_2(t) \frac{d}{dt}\alpha_2 y_2(t)} + \underline{\alpha_2 y_2(t) \frac{d}{dt}\alpha_1 y_1(t)}$$
$$+ \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad - ④$$

Ex. Which one of the following signals is linear, causal, TI, dynamic.

1. $y(t) = x(t-2) + x(2-t)$

- linearity: $\alpha_1 y_1 = \alpha_1 x_1(t-2) + \alpha_1 . x_1(2-t)$

$\alpha_2 y_2 = \alpha_2 x_2(t-2) + \alpha_2 x_2(2-t)$

$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 x_1(t-2) + \alpha_2 x_2(t-2) + \alpha_1 x_1(2-t) + \alpha_2 x_2(t+2)$ – ①

Assume: $\alpha_3 y_3 = \alpha_1 y_1 + \alpha_2 y_2$

$\alpha_3 x_3 = \alpha_1 x_1 + \alpha_2 x_2$

$\alpha_3 y_3(t) = \alpha_3 x_3(t-2) + \alpha_3 x_3(2-t)$

$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 x_1(t-2) + \alpha_2 x_2(t-2) + \alpha_1 x_1(2-t) + \alpha_2 x_2(2-t)$ → ②

⇒ linear (Eq. ① = Eq. ②)

- causal: to check the causality:

$\underset{\substack{\uparrow\\ \text{past}}}{}$ $y(0) = \underset{\substack{\uparrow\\ \text{past}}}{x(-2)} + \underset{\substack{\uparrow\\ \text{future}}}{x(2)}$ → non causal system.

- TI: . delay time: $y_1(t-t_0) = x_1(t-2-t_0) + \underline{x_1(2-(t-t_0))}$ – ①

. delay function time: $y_2(t-t_0) = x_2(t-2-t_0) + x_2(2-t-t_0)$ – ②

⇒ time variant (Eq. ① ≠ Eq. ②)

// اذا كانت ال t مضروبة بـ - دائماً ← time variant

- dynamic:

t, t-2, 2-t ⇒ dynamic //depends on time future, past, present.

2. $y(t) = \cos(3t)\ x(t)$

- linearity :

$\alpha_1 y_1 = \cos(3t)\ \alpha_1 x_1(t)$

$\alpha_2 y_2 = \cos(3t)\ \alpha_2 x_2(t)$

$\alpha_1 y_1(t) + \alpha_2 y_2(t) = \cos(3t)\ (\alpha_1 x_1(t) + x_2(t)\alpha_2)$ — ①

. Assume: $\alpha_3 y_3 = \alpha_1 y_1 + \alpha_2 y_2$

$\alpha_3 x_3 = \alpha_1 x_1 + \alpha_2 x_2$

$\alpha_3 y_3 = \cos(3t)\ \alpha_3 x_3$

$\alpha_1 y_1 + \alpha_2 y_2 = \cos(3t)\,(\alpha_1 x_1 + \alpha_2 x_2)$ — ②

$\Rightarrow$ Eq. ① = Eq. ② $\Rightarrow$ linear

- causality : (just input & output values)

t , t → causal.

$y(0) = \cos(0)\ x(0)$

$\uparrow \qquad\qquad \uparrow$

- TI : $y_1(t-t_0) = \cos(3(t-t_0))\ x_1(t-t_0)$

$y_2(t-t_0) = \cos(3t-t_0)\ x_2(t-t_0)$

$\Rightarrow$ time variant

- dynamic → t , t → instantaneous. system.

3. $y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau$

4. $y(t) = x\left(\frac{t}{3}\right)$

5. $y(t) = \begin{cases} 0 & , t<0 \\ x(t) + x(t-2) & , t \geq 0 \end{cases}$

6. $y(t) = \begin{cases} 0 & , x(t) < 0 \\ x(t) + x(t-2) & , x(t) \geq 0 \end{cases}$

3. $y(t) = \int_{-\infty}^{2t} x(\tau)\, d\tau$

- linearity:

$$\alpha_1 y_1(t) = \int_{-\infty}^{2t} \alpha_1 x_1(\tau)\, d\tau$$

$$\alpha_2 y_2(t) = \int_{-\infty}^{2t} \alpha_2 x_2(\tau)\, d\tau$$

$$\alpha_1 y_1 + \alpha_2 y_2 = \int_{-\infty}^{2t} (\alpha_1 x_1 + \alpha_2 x_2)\, d\tau \quad - ①$$

Assume: $\alpha_3 y_3 = \alpha_1 y_1 + \alpha_2 y_2$

$\alpha_3 x_3 = \alpha_1 x_1 + \alpha_2 x_2$

$$\alpha_3 y_3 = \int_{-\infty}^{2t} \alpha_3 x_3(\tau)\, d\tau$$

$$\alpha_1 y_1 + \alpha_2 y_2 = \int_{-\infty}^{2t} (\alpha_1 x_1 + \alpha_2 x_2)\, d\tau \quad - ②$$

$\Rightarrow$ linear.

- causality: $y(t) = v(2t) - v(-\infty) \Rightarrow$ non causal

- TI:
  - $y_1(t-t_0) = \int_{-\infty}^{2(t-t_0)} x_1(\tau)\, d\tau$
  - $y_2(t-t_0) = \int_{-\infty}^{2t-t_0} x_2(\tau)\, d\tau$

$\Rightarrow$ time variant

- dynamic: integral $\rightarrow$ dynamic.

- Impulse response for LTI system.

for LTI :

$y(t) = x(t) * h(t)$

- For impulse response :

$x(t) = \delta(t)$

$y(t) = x(t) * h(t)$

$y(t) = \delta(t) * h(t) = h(t)$

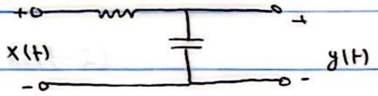
↓ تعبّر عن طبيعة النظام

In general:

$x(t) = \delta(t-t_0)$

$y(t) = \delta(t-t_0) * h(t) = h(t-t_0)$

Ex. $\tau_0 \frac{d}{dt} y(t) + y(t) = x(t)$



- to evaluate impulse response :

① impulse response : $\delta(t) = x(t)$

$y(t) = h(t)$

- $\tau_0 \frac{d}{dt} h(t) + h(t) = \delta(t)$

② Assume $t > 0$, $h(t)$ ✓ , $t < 0$, $h(t) = 0$

Assume $\frac{d^n h(t)}{dt^n} = \lambda^n \Rightarrow \frac{d}{dt} h(t) = \lambda$ & $h(t) = \lambda^0 = 1$

$\Rightarrow \tau_0 \lambda + 1 = 0 \Rightarrow \lambda = \frac{-1}{\tau_0}$

③ $h(t) = A e^{\lambda t} u(t) = A e^{-\frac{t}{\tau_0}} u(t)$

④ $T_0 \frac{d}{dt} h(t) = T_0 g(t)\, \delta(t) + T_0 g'(t)\, u(t)$

$\downarrow$ sampling theorem.

$= T_0\, g(0)\, \delta(t) + T_0\, g'(t)\, u(t)$

$\Rightarrow \underbrace{T_0\, g(0)}\, \delta(t) + T_0\, g'(t)\, u(t) + g(t)\, u(t) = \delta(t)$

$T_0\, g(0) = 1$

$g(0) = A e^{-0} = A$

$\Rightarrow A = \frac{1}{T_0}$

$\Rightarrow g(t) \Rightarrow h(t) = \frac{1}{T_0} e^{-\frac{t}{T_0}} u(t)$ (solution)

Ex. Evaluate the impulse response for the following system.

$3 \frac{d}{dt} y(t) + y(t) = x(t-2)$

① impulse response : $x(t-2) = \delta(t-2)$

$y(t) = h(t)$

② $3 \frac{d}{dt} h(t) + h(t) = \delta(t-2)$; $h(t) = h_1(t-2)$

③ $3 \frac{d}{dt} h_1(t) + h_1(t) = \delta(t)$ $\Rightarrow$ because it's LTI system.

④ Assume $\frac{d^n}{dt} h_1(t) = \lambda^n \rightarrow \frac{d}{dt} h_1(t) = \lambda$, & $h_1(t) = \lambda^0 = 1$

$3\lambda + 1 = 0$

$\lambda = \frac{-1}{3}$

⑤ $h_1(t) = A e^{\lambda t} u(t) = \underbrace{A e^{-\frac{t}{3}}}_{g(t)} u(t)$

$\Rightarrow g(0) = A$

$\Rightarrow A = \frac{1}{3}$

⑥ $\Rightarrow h_1(t) = \frac{1}{3} e^{-\frac{t}{3}} u(t)$

② $h(t) = h_1(t-2) = \frac{1}{3} e^{-\frac{(t-2)}{3}} u(t-2)$

- Superposition Integrals in terms of step response:

(LTI)

LTI: $y(t) = x(t) * h(t)$

$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$, by parts

$u = x(t-\lambda)$ $\qquad dv = h(\lambda)\, d\lambda$

$du = -x'(t-\lambda)\, d\lambda$ $\qquad v = \int_{-\infty}^{t} h(\lambda) d\lambda = a(t)$

$= \overset{0}{\cancel{x(t-\lambda)\, a(t) \Big|_{-\infty}^{\infty}}} + \int_{-\infty}^{\infty} x'(t-\lambda)\, a(t)\, d\lambda$

$y(t) = \int_{-\infty}^{\infty} x'(t-\lambda)\, a(\lambda)\, d\lambda$ → Duhamel's Theorem

- Step response → $x(t) = u(t)$

$y(t) = \int_{-\infty}^{\infty} \delta(t-\lambda)\, a(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \delta(\lambda - t)\, a(\lambda)\, d\lambda = a(t)$; sifting theo, even.

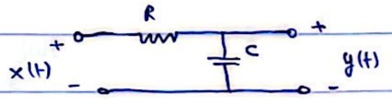
- $y_s(t) = \int_{-\infty}^{\infty} h(\lambda)\, d\lambda$

- In general : Impulse resp. $x(t) = \delta(t) \rightarrow y(t) = h(t)$

  step resp. $x(t) = u(t) \rightarrow y_s(t) = \int_{-\infty}^{\infty} h(\lambda)\, d\lambda$

  ramp resp. $x(t) = r(t) \rightarrow y_r(t) = \int_{-\infty}^{\infty} y_s(\lambda)\, d\lambda$

Ex. Consider the following RC circuit shown below:



Which has the following DFE :

$$RC \frac{d}{dt} y(t) + y(t) = x(t)$$

evaluate the response of the system :

$$x_\Delta(t) = \delta(t) - 2u(t-1) + r(t-2)$$

Note: $y_\Delta(t) = h(t) - 2y_s(t-1) + y_r(t-2)$

. For impulse resp.

$$RC \frac{d}{dt} h(t) + h(t) = \delta(t)$$

$$RC\lambda + 1 = 0 \Rightarrow \lambda = \frac{-1}{RC}$$

$$h(t) = \underbrace{A e^{-t/RC}}_{g(t)} u(t)$$

$$g(0) = A \Rightarrow A = \frac{1}{RC}$$

$$\Rightarrow h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

. For step resp.

$$y_s(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-\lambda/RC} u(\lambda)\, d\lambda = \int_{-\infty}^{t} \frac{1}{RC} e^{-\lambda/RC} d\lambda = -\frac{1}{(RC)} \left[RC e^{-\lambda/RC}\right]\Big|_{-\infty}^{t}$$

$$y_s(t) = [1 - e^{-t/RC}]u(t) \quad \Rightarrow 2y_s(t-1) = [2 - 2e^{-(t-1)/RC}]u(t-1)$$

$$y_r(t) = \int_{-\infty}^{t} [1 - e^{-\lambda/RC}]\, d\lambda = t\,u(t) - RC[1 - e^{-t/RC}]u(t)$$

$$\Rightarrow y_r(t-2) = (t-2)u(t-2) - RC[1 - e^{-(t-2)/RC}]u(t-2)$$

$\therefore y_\Delta(t) = h(t) - 2y_s(t-1) + y_r(t-2)$; using superposition.

- Frequency response (LTI) system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad x(t) = e^{j\omega t}$$

$$x(t) * h(t) = y(t) \qquad (LTI)$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} e^{j\omega(t-\lambda)}\, h(\lambda)\, d\lambda$$

$$= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega\lambda}\, h(\lambda)\, d\lambda}_{H(\omega)} \quad ;$$

Ex. Find the freq. resp. of RC circuit where ;

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

- Freq. Resp. : $H(\omega) = \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega\lambda} d\lambda = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-\lambda/RC} u(\lambda) e^{-j\omega\lambda} d\lambda$

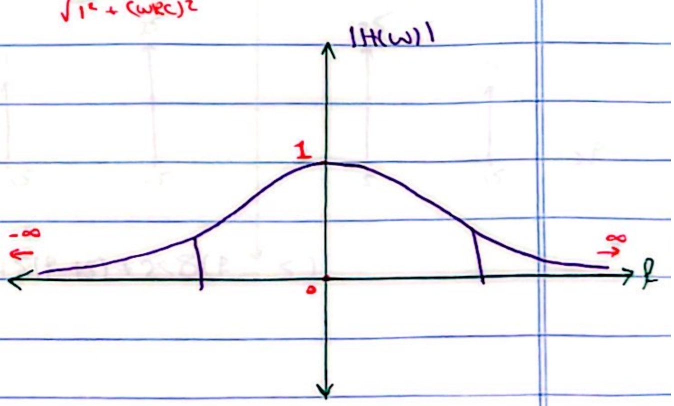
$$= \int_{0}^{\infty} \frac{1}{RC} e^{-\lambda/RC} e^{-j\omega\lambda} d\lambda$$

$$= \int_{0}^{} \frac{1}{RC} e^{-(\frac{1}{RC}+j\omega)\lambda} d\lambda = \frac{1}{1+j\omega RC}$$

- $H(\omega) = \dfrac{1}{1+j\omega RC} \Rightarrow |H(\omega)| \angle \theta_{H(\omega)} = |H(\omega)|\, e^{j\theta_{H(\omega)}}$

$$H(\omega) = \frac{1 \angle 0^\circ}{\sqrt{1^2 + (\omega RC)^2} \angle \tan^{-1}(\omega RC)} = \frac{1}{\sqrt{1^2 + (\omega RC)^2}} \angle -\tan^{-1}(\omega RC)$$

- to plot $|H(\omega)|$ & $\angle \theta_{H(\omega)}$

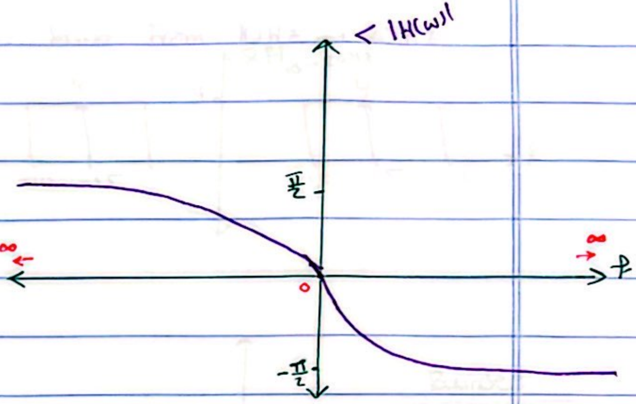


resp. (مثل الفلتر) ← تُعطي معلومات واضحة عن الـ

$< -\tan(wRC)$

when $\omega \to -\infty$

$< -\tan(-\infty) = \tan(\infty) \to \frac{\pi}{2}$



when $\omega \to \infty$

$< -\tan(\infty) \to -\frac{\pi}{2}$

// we can understand the real behaviour of the circuit.

· Energy & power signal :

The power spectral density can be obtained from:

$$P = \int_{-\infty}^{\infty} S(f)\, df$$

where;

P → total power , S(f) → power spectral density (PSD)

$$E = \int_{-\infty}^{\infty} G(f)\, df \quad ; \quad G(f) = |H(\omega)|^2$$

where;

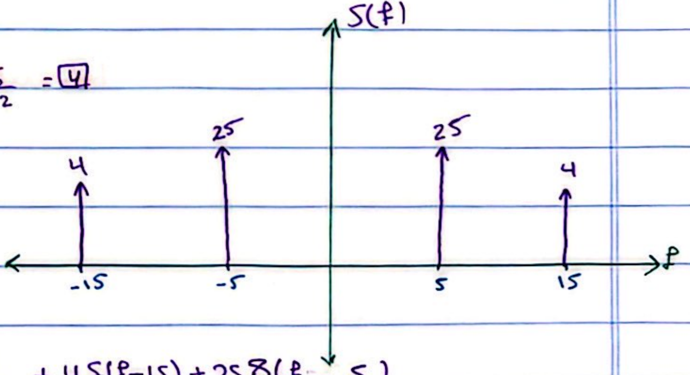
E → total energy , G(f) → energy spectral density

Ex. Consider the signal: $x(t) = 10\cos(10\pi t + \frac{\pi}{7}) + 4\sin(20\pi t + \frac{\pi}{8})$

① plot & evaluate it's (PSD)

power → $\frac{A^2}{2} = \frac{100}{2.2} = \boxed{25}$, $\frac{16}{2.2} = \boxed{4}$

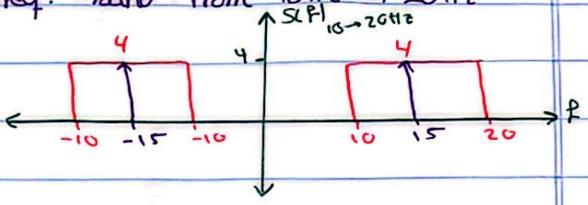
//double sided



$S(f) = 4\delta(f+15) + 25\delta(f+5) + 4\delta(f-15) + 25\delta(f-5)$

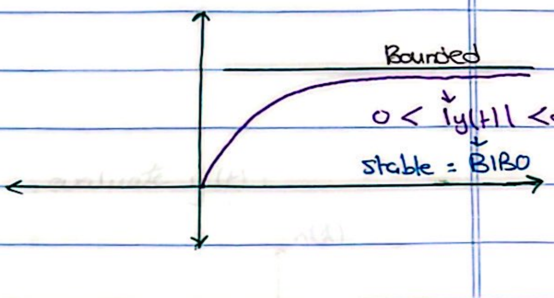
$P_{total} = 4 + 25 + 25 + 4 = \boxed{58\ W}$

② Compute the power lying within a freq. band from 10 Hz → 20 Hz



$P_{\underset{10\to 20\,Hz}{total}} = 4 + 4 = 8\,W$

. Stability of Linear System.



(LTI)

$$\text{LTI}: \; y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

bounded out.

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda \right| \leq \int_{-\infty}^{\infty} |x(t-\lambda)|\, |h(\lambda)|\, d\lambda$$

bounded دائماً // always bounded inp.

$.x(t) \longrightarrow \; 0 < |x(t-\lambda)| < \infty$

$0 < M < \infty \qquad \leq M \underline{\int_{-\infty}^{\infty} |h(\lambda)|\, d\lambda}$

ل جميعه بيعتمد على ال response. حتى يكون bounded

Ex. For the following response system, check the stability,

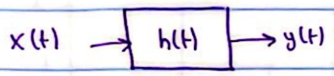
$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\int_{-\infty}^{\infty} |h(\lambda)|\, d\lambda = \int_{-\infty}^{\infty} \frac{1}{RC} e^{-\lambda/RC} u(\lambda)\, d\lambda = \int_{0}^{\infty} \frac{1}{RC} e^{-\lambda/RC}\, d\lambda = 1 < \infty$$

$\Rightarrow$ BIBO = stable

Bounded input → $x(t) = \delta(t)$ → bounded

. Ideal case: LTI: Linear time invariant system:



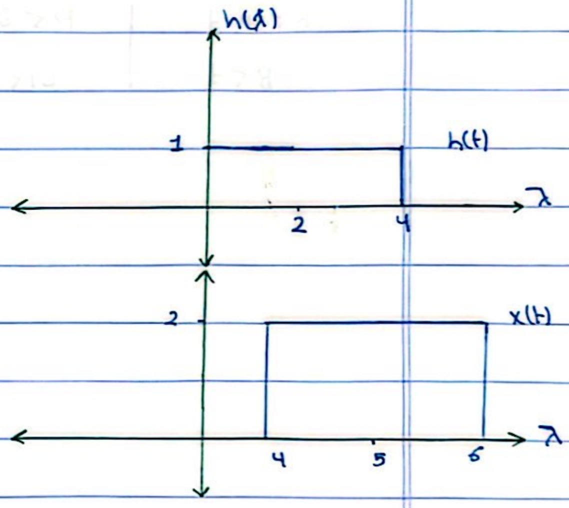
$$y(t) = \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda \quad \text{or} \quad y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

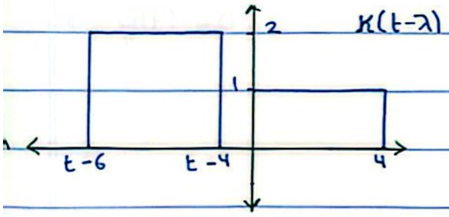
// $x(t) * h(t) = y(t)$

Ex. $x(t) = 2\Pi\left(\frac{t-5}{2}\right)$, $h(t) = \Pi\left(\frac{t-2}{4}\right)$, evaluate $y(t)$:

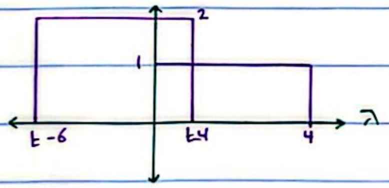
$y(t) = x(t) * h(t)$

$\Rightarrow$ ① $x(t-\lambda)$, $h(\lambda)$





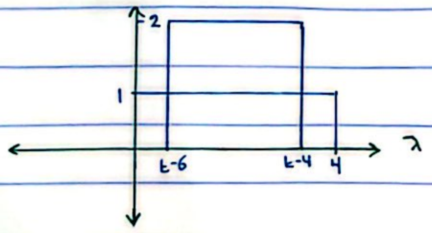
$\boxed{1}$ $t-4 \leq 0 \Rightarrow t<4 \Rightarrow y(t)=0$



$\boxed{2}$ $t-4>0$, $t-6<0$

$t>4$ $t<6$ $\Rightarrow 4<t<6$

$y(t) = \int_{0}^{t-4} 2\cdot 1\, d\lambda = 2(t-4)$

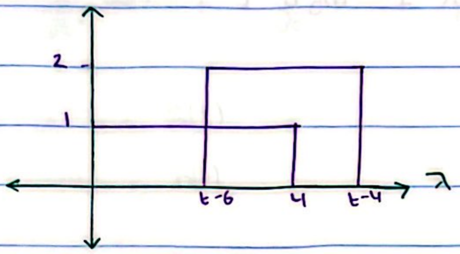


$t-4<4$, $t-4>0$ | $t-6>0$, $t-6<4$

$t<8$, $t>4$ | $t>6$, $t<10$

$6<t<8$

$\boxed{3}$ $y(t) = \int_{t-6}^{t-4} 2.1\, d\lambda = 4$

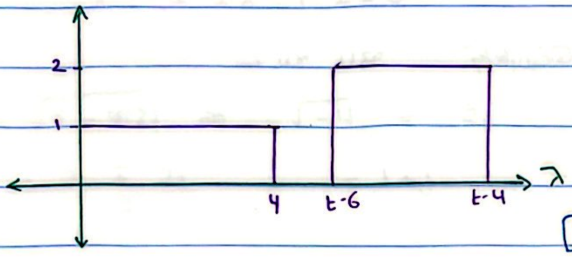


$t-6 < 4 \quad , \quad t-6 > 0 \quad \Big| \quad t-4 > 4$

$t < 10 \quad , \quad t > 6 \quad \Big| \quad t > 8$

$8 < t < 10$

④ $y(t) = \int_{t-6}^{4} 2.1 \, d\lambda = 2[-t+10]$



$t-6 > 4 \quad \Big| \quad t-4 > 4$

$t > 10 \quad \Big| \quad t > 8$

$\boxed{t > 10}$

⑤ $y(t) = 0$

// Second Worksheet.

$\boxed{1}$ $\frac{d^2}{dt^2} y(t) + 2 \frac{d}{dt} y(t) + 2y(t) = 18 \ddot{x}(t-2)$

$\Rightarrow x(t) = \delta(t)$

$y(t) = h(t)$

$\Rightarrow \frac{d^2}{dt^2} h(t) + 2 \frac{d}{dt} h(t) + 2 h(t) = 18 \ddot{\delta}(t)$

$\lambda^2 + 2\lambda + 2 = 0$

$\rightarrow$ we use القانون العام

$\Rightarrow \sqrt{4 - 4(2)} \Rightarrow \sqrt{-4} = 2j$

$\lambda_{1,2} = -\frac{2 \mp 2j}{2} = -1+j \; , \; -1-j$

$\lambda_1 = -1+j \; , \; \lambda_2 = -1-j$

$* \; h(t) = g(t) \, u(t) = e^{-t}(A\cos t + B\sin t] \, u(t) + \underline{C} \, \delta(t)$

$g(t) = e^{\alpha t} \left(A \cos(\omega_d t) + B \sin(\omega_d t) \right)$

$g(t) = e^{-t} (A \cos t + B \sin t)$

$g(0^+) = A$

$g'(t) = e^{-t} (-A \sin t + B \cos t) + (A \cos t + B \sin t) \cdot -e^{-t}$

$g'(0^+) = B - A$

$* \; h'(t) = g(t) \, \delta(t) + g'(t) \, u(t) = \underline{g(0)} \, \delta(t) + g'(t) \, u(t) + \underline{C} \, \dot{\delta}(t)$

$h''(t) = g(0) \, \delta'(t) + g'(t) \, \delta(t) + g''(t) \, u(t)$

$* \; h''(t) = \underline{g(0)} \, \delta'(t) + \underline{g'(0)} \, \delta(t) + g''(t) \, u(t) + \underline{C} \, \ddot{\delta}(t)$

$\delta : \; 2C + 2A + B - A = 0 \Rightarrow B = 0$

$\dot{\delta} : \; A + 2C = 0 \Rightarrow A = -36$

$\ddot{\delta} : \; C = 18$

$\cdot \; h(t) = e^{-t} (-36 \cos(t)) \, u(t) + 18 \, \delta(t)$

$y(t) = -36e^{-(t-2)}\cos(t-2)\,u(t-2) + 18\delta(t-2)$

$\boxed{2}$ $\frac{d^2}{dt^2}y(t) + 6\frac{dy(t)}{dt} + 5y(t) = 18\ddot{x}(t-4)$

$x(t) = \delta(t)$, $h(t) = y(t)$

$\Rightarrow \frac{d^2}{dt^2}h(t) + 6\frac{d}{dt}h(t) + 5h(t) = 18\ddot{\delta}(t)$

$\lambda^2 + 6\lambda + 5 = 0$

$(\lambda + 5)(\lambda + 1) = 0$

$\lambda_1 = -5$, $\lambda_2 = -1$

$g(t) = A_1 e^{+\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{-t} + A_2 e^{-5t}$

$g(0^+) = A_1 + A_2$

$g'(t) = -A_1 e^{-t} - 5A_2 e^{-5t}$

$g'(0^+) = -A_1 - 5A_2$

$h(t) = g(t)u(t) + \underline{B}\,\delta(t)$

$h'(t) = \underline{g(0)}\,\delta(t) + g'(t)u(t) + \underline{B}\,\dot{\delta}(t)$

$h''(t) = \underline{g(0)}\,\dot{\delta}(t) + \underline{g'(0)}\,\delta(t) + g''(t)u(t) + \underline{B}\,\ddot{\delta}(t)$

$\ddot{\delta}$: $B = 18$

$\dot{\delta}$: $A_1 + A_2 + 6B = 0$

δ : $-A_1 - 5A_2 + 6(A_1 + A_2) + 5B = 0$

$-A_1 - 5A_2 + 6A_1 + 6A_2 + 90 = 0$

$5A_1 + A_2 = -90$ $-①$

$-A_1 \mp A_2 = +108$ $-②$

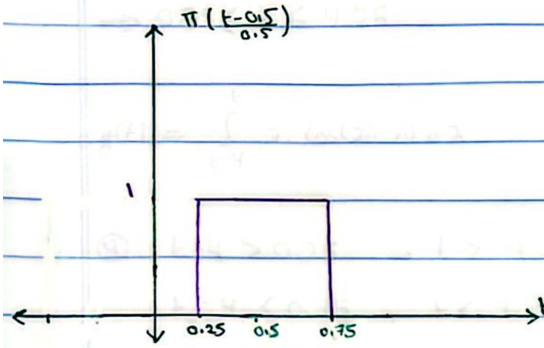
$4A_1 = 18 \Rightarrow A_1 = 4.5$, $A_2 = -112.5$

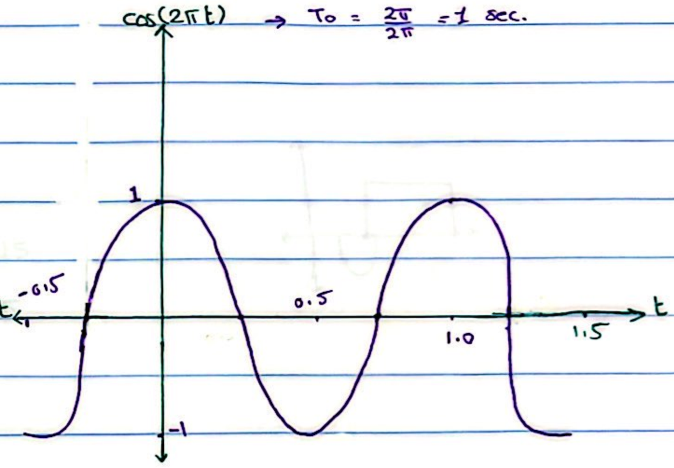
$\cdot\, y(t) = (4.5e^{-t+4} - 112.5e^{-5t+20})\,u(t-4) + 18\delta(t-4)$

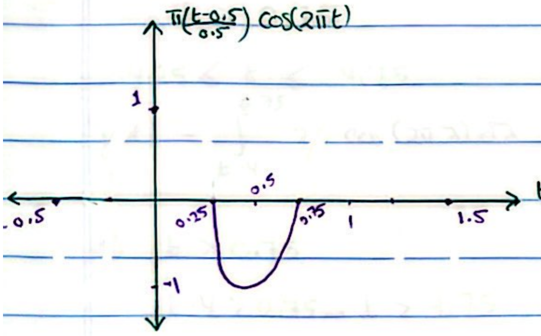
③ Using convolution integral, $h(t) = 2\Pi\left(\frac{t-2}{4}\right)$, $x(t) = \cos(2\pi t) . \Pi\left(\frac{t-0.5}{0.5}\right)$

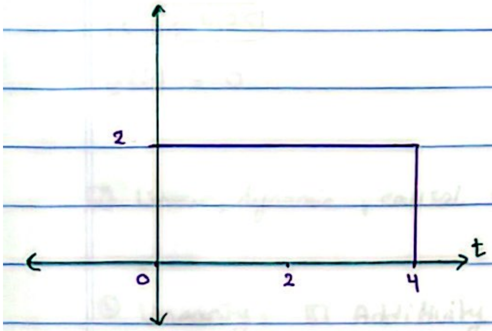
$\Rightarrow$ LTI : $y(t) = \int_{-\infty}^{\infty} h(t-\lambda)\, x(\lambda)\, d\lambda \quad = x(t) * h(t)$

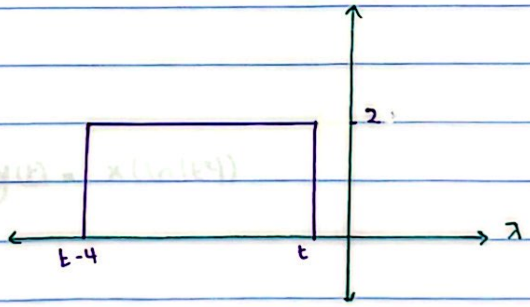
① $\cos(2\pi t) . \Pi\left(\frac{t-0.5}{0.5}\right)$



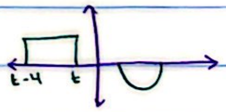




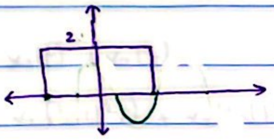




① $\boxed{t \leqslant 0.25}$



$t-4 < 0.25 \Rightarrow t < 4.25 \quad \Rightarrow y(t) = 0$



② $t-4 < 0.25 \rightarrow t < 4.25$

$t < 0.75$

$t > 0.25$

$0.25 \leqslant t \leqslant 0.75 \rightarrow y(t) = \int_{0.25}^{t} 2 . \cos(2\pi\lambda)\, d\lambda$

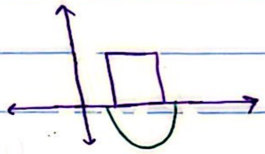
③ $t-4 > 0.25 \rightarrow t > 4.25$

$t-4 < 0.75 \rightarrow t < 4.75$

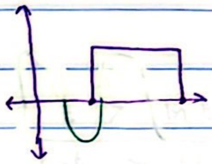
$t < 0.75$

$t > 0.25$

$\Rightarrow 0.75 \leq t \leq 4.25$



$$y(t) = \int_{t-4}^{t} 2.\cos(2\pi\lambda)\, d\lambda$$



④ $t-4 > 0.25 \rightarrow t > 4.25$

$t-4 < 0.75 \rightarrow t < 4.75$

$t > 0.75$

$4.25 \leq t \leq 4.75$

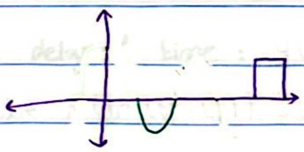
$$y(t) = \int_{t-4}^{0.75} 2.\cos(2\pi\lambda)\, d\lambda$$

⑤ $t > 0.75$

$t-4 > 0.75 \rightarrow t > 4.75$

$\boxed{t \geq 4.75}$

$y(t) = 0$



$\boxed{3}$ Linear, dynamic, causal : $y(t) = x(\ln(t^2))$

① Linearity: $\boxed{I}$ Additivity :

$x_1(\ln(t^2)) \Rightarrow y_1(t) = x_1(\ln(t^2))$

$x_2(\ln(t^2)) \Rightarrow y_2(t) = x_2(\ln(t^2))$

$x(t) = x_1(t) + x_2(t) \Rightarrow y(t) = x(t) = x_1(t) + x_2(t)$

$= x_1(\ln(t^2)) + x_2(\ln(t^2))$

$= y_1(t) + y_2(t)$ ✓

$\boxed{2}$ proportionality:

$x(\ln(t^2)) \longrightarrow y(t) = x(\ln(t^2))$

$\alpha\, x(\ln(t^2)) \longrightarrow \alpha y(t) \Rightarrow \alpha\, x(\ln(t^2))$

$= \alpha\, y(t)$ ✓

since, ① & ② have been proved → it is <u>linear</u>

② causality:

$y(t) = x(\ln(t^2))$

// counter example, take $t = 0.1$

$y(0.1) = x(\ln(0.1)^2)$

$y(0.1) = x(-4.6)$

⇒ it always will be greater or equal than $x(t)$

⇒ <u>causal</u>

$t > \ln t^2$

$\boxed{e^t > t^2}$

③ dynamic:

$$y(t) = x(\ln(t^2))$$

$$\downarrow \qquad \downarrow$$

$$t \qquad t^2$$

⇒ <u>dynamic</u>

④ TI: delay of time : $y(t-t_0) = x(\ln(t-t_0)^2)$ — ①

$y(t-t_0) = x(\ln(t^2-t_0))$ — ②

since ① & ② are not equal, then; it is

<u>time variant</u>

$\boxed{4}$ $y(t) = x(\ln|t|)$

// same as prev. → <u>linear</u>, <u>causal</u>, but <u>instantaneous</u>, and <u>time invariant</u>

$\boxed{5}$ $y(t) = x(e^{3t})$

– Linearity : ① Additivity:

$x_1(e^{3t}) \longrightarrow y_1(t) = x_1(e^{3t})$

$x_2(e^{3t}) \longrightarrow y_2(t) = x_2(e^{3t})$

$x(t) = x_1(e^{3t}) + x_2(e^{3t}) \longrightarrow y(t) = x(t) = x_1(e^{3t}) + x_2(e^{3t})$

$= y_1(t) + y_2(t)$ ✓

② proportionality:

$\alpha\, x(e^{3t}) \longrightarrow \alpha\, y(t) \Rightarrow \alpha\, x(e^{3t})$

$= \alpha\, y(t)$ ✓

since, ① & ② have been proved, it is linear

- causality:

$y(t) = x(e^{3t})$

// lets take $t=0$ as a counter example: $t < e^{3t}$

$y(0) = x(1)$

$0 < 1$ ×

it is non causal

* Simulink modeling:

ex. $\int y''' - \int 8y'' + \int 4y' - \int 4x' = \int \underbrace{2x-2y}$ ← lowest order

$y'' - 8y' + 4y - 4x = \int q_0$

$q_0 = 2x-2y$

$\int y'' - \int 8y' = \underbrace{\int q_0 - 4y + 4x}$

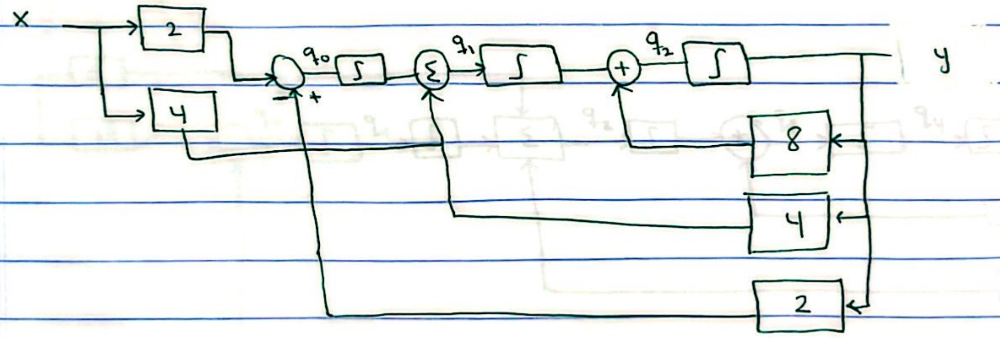
$q_1 = \int q_0 - 4y + 4x$

$y' - 8y = \int q_1$

$\int y' = \int \underline{8y + \int q_1}$

$q_2 = 8y + \int q_1$

$y = \int q_2$



$\boxed{6}$ $\frac{d^5}{dt^5} y(t) - 5\frac{d^3}{dt^3} y(t) - 6\frac{d^2}{dt^2} y(t) + 7y(t) = 6\frac{d^2}{dt^2} x(t) - 19x(t)$

$\int y^{(5)} - \int 5y^{(3)} - \int 6y^{(2)} - \int 6x^{(2)} = \int \underline{-7y - 19x}$ $\qquad q_0 = -7y - 19x$

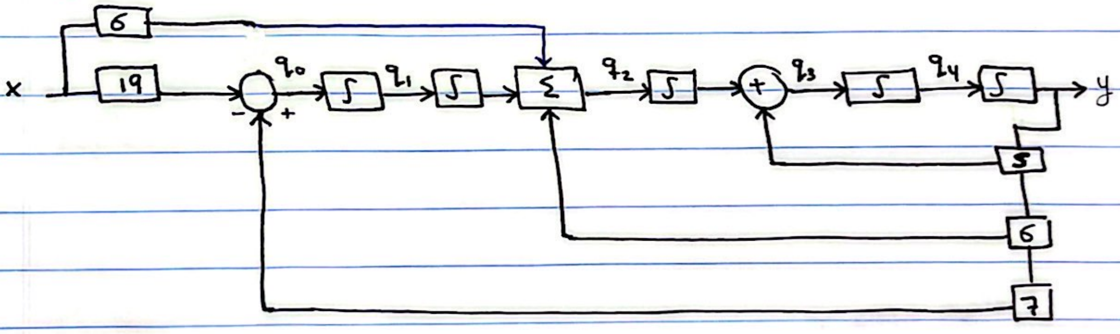
$\int y^{(4)} - \int 5y^{(2)} - \int 6y' - \int 6x' = \int \underline{\int q_0}$ $\qquad q_1 = \int q_0$

$\int y^{(3)} - \int 5y' = \int \underline{6y + 6x + \int q_1}$ $\qquad q_2 = 6x + 6y + \int q_1$

$\int y^{(2)} = \int \underline{5y + \int q_2}$ $\qquad q_3 = 5y + \int q_2$

$\int y' = \int \underline{\int q_3}$ $\qquad q_4 = \int q_3$

$y = \int q_4$



$\boxed{7}$ $\frac{d}{dt} y(t) + 3y(t) = 18x''(t)$

$\rightarrow \frac{d}{dt} h(t) + 3h(t) = 18\delta''(t)$

$\lambda + 3 = 0 \Rightarrow \lambda = -3$

$h(t) = g(t)\,u(t) + B\delta(t) + C\delta'(t)$

$g(t) = Ae^{-3t}$

$g(0) = A$

$g'(t) = -3Ae^{-3t} \Rightarrow g'(0) = -3A$

$h'(t) = g(0)\delta(t) + g'(t)u(t) + B\delta'(t) + C\delta''(t)$

$\delta''(t)$: $\boxed{C = 18}$

$\delta'(t)$: $B + 3C = 0 \Rightarrow \boxed{B = -54}$

$\delta(t)$: $3B + g(0) = 0 \Rightarrow \boxed{A = 162}$

$h(t) = 162e^{-3t}u(t) - 54\delta(t) + 18\delta'(t) = y(t)$

8 & 9 : // same as prev. questions.

10 $\frac{d}{dt}y(t) + 4y(t) = 5x(t)$

. Frequency:

$h(t)$: $\frac{d}{dt}h(t) + 4h(t) = 5\delta(t)$

$\lambda + 4 = 0 \rightarrow \lambda = -4$

$h(t) = Ae^{-4t}.u(t)$; $g(t) = Ae^{-4t}$

$h'(t) = Ae^{-4(0)}.\delta(t) + A(-4)e^{-4t}u(t)$ $g(0) = A$

$\delta(t)$: $\boxed{5 = A}$

$h(t) = y(t) = 5e^{-4t}u(t)$

$\Rightarrow H(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\lambda}.h(\lambda)\,d\lambda = 5\int_0^{\infty} e^{-j\omega\lambda}.e^{-4\lambda}\,d\lambda$

$H(\omega) = 5\int_0^{\infty} e^{-\lambda(j\omega+4)}\,d\lambda = \frac{-5e^{-\lambda(j\omega+4)}}{j\omega+4}\Big|_0^{\infty} = -5(0) + \frac{5}{j\omega+4} = \boxed{\frac{5}{j\omega+4}}$

$H(s) = \frac{5}{s+4}$ ⟷ Laplace transform :

$\underline{H(s) = 5e^{-4t}}$

- steady state response:

$$x(t) = 10\cos(20t + \frac{\pi}{5}) + 10\sin(60t) = 10\cos(20t + \frac{\pi}{5}) + 10\cos(60 - \frac{\pi}{2})$$

$$|H(\omega)| = \frac{5.1 \measuredangle 0^\circ}{\sqrt{\omega^2+16} \measuredangle \tan^{-1}(\frac{4}{\omega})}$$

① at $w = 20$:

$$|H(20)| = \frac{5.1 \measuredangle 0^\circ}{20.3 \measuredangle 0.197} = 0.246 \measuredangle -0.197 \text{ rad}$$

② at $w = 60$

$$|H(60)| = \frac{5.1 \measuredangle 0^\circ}{60.13 \measuredangle 0.066} = 0.0831 \measuredangle -0.066 \text{ rad}$$

- using superposition → because it is an LTI system: // linear

$$\bar{x}(t) = 0.246 \cdot 10 \cdot \cos(20t + \frac{\pi}{5} - 0.197) + 10 \cdot 0.0831\cos(60t - \frac{\pi}{2} + -0.066)$$

$$\bar{x}(t) = 2.46\cos(20t + 0.4313) + 0.831\cos(60t - 1.63)$$

$\boxed{11}$ same as previous questions.

$\boxed{10}$ The ideal solution: $\frac{d}{dt}y(t) + 4y(t) = 5x(t)$

$x(t) \Leftrightarrow x(s)$, $\frac{d}{dt}x(t) = s\,x(s) - x(0)$ // $x(0)$ → zero state.

$$\Rightarrow s\,Y(S) + 4\,Y(S) = 5\,x(S)$$

$$Y(s) = \frac{5}{s+4} \cdot x(S)$$

$$H(S) = \frac{Y(s)}{x(S)} = \frac{5}{s+4} \qquad // \; L(h(t)) = \frac{5}{s+4}$$

$$H(\omega) = H(s)\Big|_{s=j\omega} = \frac{5}{j\omega+4}$$

Ch.3 : Fourier Series

① Trigonometric Fourier series : $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + \sum_{n=1}^{\infty} b_n \sin(n w_0 t)$

② Complex exponential Fourier series : $x(t) = \sum_{n=-\infty}^{n=\infty} x_n e^{jn w_0 t}$

a_0 : average value $\Rightarrow a_0 = \frac{1}{T} \int_T x(t)\, dt$

// we work on periodic signal , so we take one period & solve it

$$a_n = \frac{2}{T} \int_T x(t) \cos(n w_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_T x(t) \sin(m w_0 t)\, dt$$

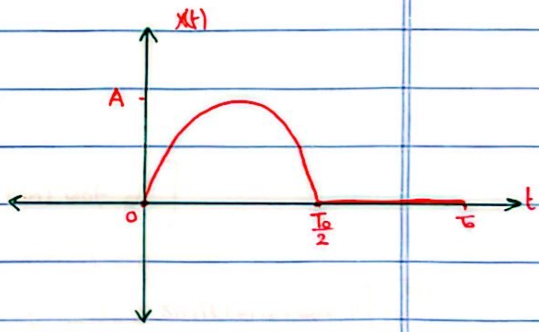
// if $x(t)$ even $\Leftrightarrow$ $b_n = 0$, a_n ✓ // if $x(t)$ has symmetry, & alternating $\rightarrow a_0 = 0$

// if $x(t)$ odd $\rightarrow$ $a_n = 0$, b_n ✓ ↳

or

Ex. Find the coefficients of the trignometric fourier series for half rectified sine wave , defined by:

$$x(t) = \begin{cases} A \sin(w_0 t) & , 0 \leq t \leq \frac{T_0}{2} \\ 0 & , o.w \end{cases}$$



$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt = \frac{1}{T_0} \int_{T_0} A \sin(w_0 t)\, dt$$

$$= \frac{-1}{T_0} \cdot \frac{A}{w_0} \cdot \cos(w_0 t) \Big|_0^{\frac{T_0}{2}} = \frac{-1}{\cancel{T_0}} \cdot \frac{A \cancel{T_0}}{2\pi} \left(\cos(\pi) - \cos(0) \right) = \boxed{\frac{A}{\pi}}$$

$$a_n = \frac{2}{T_0} \int_{T_0} A \sin(w_0 t) \cdot \cos(n w_0 t)\, dt \Rightarrow \sin(w_0 t) \cos(n w_0 t) = \frac{1}{2} \sin(w_0 t + n w_0 t) + \frac{1}{2} \sin(w_0 t - n w_0 t)$$

$$= \frac{2A}{T_0} \int_0^{\frac{T_0}{2}} \frac{1}{2} \sin((1+n) w_0 t)\, dt + \frac{2A}{T_0} \int_0^{T_0/2} \frac{1}{2} \sin((1-n) w_0 t)\, dt$$

$$= \frac{2A}{T_0 \cdot 2} \left[\frac{- \cos((1+n) w_0 t)}{(1+n) w_0} + \frac{- \cos((1-n) w_0 t)}{(1-n) w_0} \right] \Bigg|_0^{T_0/2}$$

$$a_n = \frac{-A}{2\pi (1+n)} \left(\cos((1+n)\pi) - 1 \right) - \frac{A}{2\pi(1-n)} \left(\cos((1-n)\pi) - 1 \right)$$

- For n even :

$$a_n = \frac{-A}{2\pi(1+n)}(-2) - \frac{A}{2\pi(1-n)}(-2) = \frac{2A}{(1-n^2)\pi}$$

- For n odd , $- \{1,-1\}$

$$a_n = 0$$

- For $n=1$ → we work on single sided → just we take (+)

$$a_1 = \frac{2 \cdot 1}{T_0}\int_{T_0} x(t)\cos(1 . \omega_0 t)\,dt = \frac{2 \cdot 1}{T_0}\int_0^{T_0/2} A\sin(2\omega_0 t)\cos(\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{T_0/2} \frac{A}{2}\sin(2\omega_0 t)\,dt = \frac{A}{T_0} \cdot \frac{-1}{2\omega_0}\cos(2\omega_0 t)\Big|_0^{T_0/2}$$

$$= \frac{A}{T_0} \cdot \frac{T_0}{4\pi}\Big(\cos(2\pi) - \cos(0)\Big) = 0$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt = \frac{2}{T_0}\int_{T_0} A\sin(\omega_0 t)\sin(n\omega_0 t)\,dt$$

// $\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

$$b_n = \frac{\cancel{2}A}{T_0}\left[\int_0^{T_0/2} \frac{1}{\cancel{2}}\cos((1-n)\omega_0 t)\,dt - \int_0^{T_0/2} \frac{1}{\cancel{2}}\cos((1+n)\omega_0 t\,dt\right]$$

$$= \frac{A}{T_0}\left[\frac{1}{(1-n)\omega_0}\sin((1-n)\omega_0 t\right]\Big|_0^{T_0/2} - \frac{A}{T_0}\left[\frac{1}{(1+n)\omega_0}\sin((1+n)\omega_0 t)\right]\Big|_0^{T_0/2}$$

$$= A\left[\frac{1}{(1-n)}\sin((1-n)\pi) - 0) - \frac{1}{2\pi(1+n)}\sin((1+n)\pi) - 0)\right]$$

- For n even : $b_n = 0$
- For n odd $- \{1,-1\} = 0$
- For $n = 1$:

$$b_1 = \frac{2}{T_0}\int_{T_0} A\sin(\omega_0 t) . \sin(\omega_0 t)\,dt = \frac{2}{T_0}\left[\int_0^{T_0/2} \frac{A}{2} - \frac{A}{2}\cos(2\omega_0 t)\,dt\right]$$

$$b_1 = \boxed{\frac{A}{2}}$$

- $a_0 = \frac{A}{\pi}$ • $a_n = \begin{cases} \frac{2A}{(1-n^2)\pi} & , n \text{ even} \\ 0 & , o.w \end{cases}$ $b_n = \begin{cases} \frac{A}{2} & , n=1 \\ 0 & , o.w \end{cases}$

- Ex. Consider the following signal

$$x(t) = \frac{A}{\pi} + \sum_{n=1,\ \text{even}}^{\infty} \frac{2A}{(1-n^2)\pi} \cos(\underbrace{30\pi}_{w_0} n t) + \frac{A}{2} \sin(\underbrace{30\pi}_{n=1} t)$$

→ same wo

① Specify the type of Fourier series:

Trigonometric Fourier series → it has cos, sin parts

② evaluate the dc-value / avg-value:

// لايكون مضروب ب sin / cos

$= \frac{A}{\pi}$

③ evaluate the fund. freq. :

// لولم يكن احدهما مضروبة ب n ← ch1 ← ratio

$w_0 = 30\pi$

$F_0 = \frac{w_0}{2\pi} = \frac{30\pi}{2\pi} = 15 \text{ Hz}.$

④ evaluate the trigonometric for the fourie series :

$a_n = \begin{cases} \frac{A}{(1-n^2)\pi} & , n \text{ even} \\ 0 & , o.w \end{cases}$

// المضروبة ب cos.

$b_n = \begin{cases} \frac{A}{2} & , n=1 \\ 0 & , o.w \end{cases}$

// مضروبة ب sin.

← يجب ان ننتبه ب n و wo

. Complex exponential fourier :

⇒ In trig. series :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + \sum_{n=1}^{\infty} b_n \sin(n w_0 t)$$

// using Euler's eq. :

$$\cos(n w_0 t) = \frac{e^{j n w_0 t} + e^{-j n w_0 t}}{2} \quad , \quad \sin(n w_0 t) = \frac{e^{j n w_0 t} - e^{-j n w_0 t}}{2j}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{j n w_0 t} + e^{-j n w_0 t}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{j n w_0 t} - e^{-j n w_0 t}}{2j} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n j}{2} \right) e^{n j w_0 t} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n j}{2} \right) e^{-j n w_0 t}$$

↗ +ve freq. ↗ -ve freq.

$$x(t) = \underbrace{a_0}_{x_0} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2} - \frac{b_n j}{2} \right)}_{x_n} e^{j n w_0 t} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a_n}{2} + \frac{b_n j}{2} \right)}_{x_{-n}} e^{-j n w_0 t}$$

$$\cdot \; x(t) = x_0 + \sum_{n=1}^{\infty} x_n e^{j n w_0 t} + \sum_{n=1}^{\infty} x_{-n} e^{-j n w_0 t}$$

; let $u = -n$

when $n = 1 \rightarrow u = -1$

when $n = \infty \rightarrow u = -\infty$

$$= x_0 + \sum_{n=1}^{\infty} x_n e^{j n w_0 t} + \sum_{u=-\infty}^{-1} x_u e^{j u w_0 t}$$

$$= x_0 + \sum_{n=1}^{\infty} x_n e^{j n w_0 t} + \sum_{n=-\infty}^{-1} x_n e^{j n w_0 t} = \sum_{n=-\infty}^{\infty} x_n e^{j n w_0 t}$$

. To evaluate x_n ;

$$x(t) \cdot e^{-j m w_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{j n w_0 t} \cdot e^{-j m w_0 t}$$

$$\int_T x(t) \cdot e^{-j m w_0 t} \, dt = \sum_{n=-\infty}^{\infty} x_n \int_T e^{j(n-m) w_0 t} \, dt$$ ← periodic

if $m \neq n \rightarrow \sin, \cos \Rightarrow e^{j(n-m) w_0 t} = \cos((n-m) w_0 t) + j \sin((n-m) w_0 t)$

$$\int_T x(t) \cdot e^{-jm\omega_0 t} \, dt = \sum_{n=-\infty}^{\infty} x_n \int_T [\cos((n-m)\omega_0 t) + j\sin((n-m)\omega_0 t) \, dt$$

when; $n \neq m \rightarrow x_n$ undefined

when; $n = m \rightarrow x_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} \, dt$ // double sided.

$$x_n = \begin{cases} \frac{1}{2}(a_n - jb_n) & , n > 0 \\ \frac{1}{2}(a_n + jb_n) & , n < 0 \end{cases}$$

and; $a_n = 2\mathcal{R}\{x_n\}$, $b_n = -2I_m\{x_n\}$

// $|x_n| = |x_{-n}| \rightarrow$ even.

// $\measuredangle \phi x_n = -\measuredangle \phi x_{-n} \rightarrow$ odd.

.prev. example : $x(t) = \begin{cases} A\sin(\omega_0 t) & , 0 \leq t \leq \frac{T_0}{2} \\ 0 & , o.w \end{cases}$

① compute T.S coef.

② compute C.E.F.S

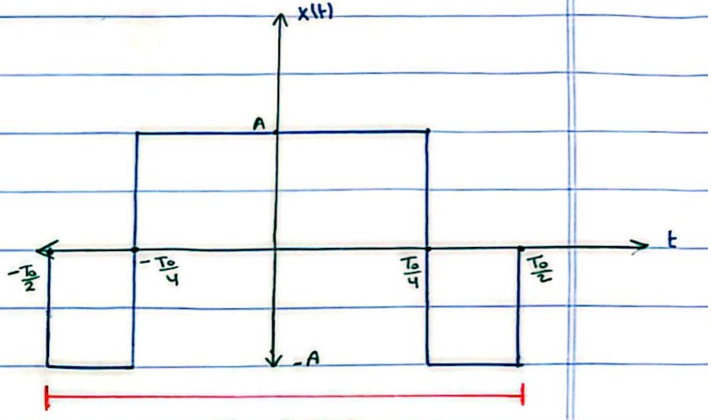
$$\Rightarrow x_n = \begin{cases} \frac{1}{2}(a_n - jb_n) & , n \geq 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & , n < 0 \end{cases}$$

$$x_n = \begin{cases} \frac{1}{2}\left(\left(\frac{A}{(1-n^2)\pi}\right) - j(0)\right) & , n \text{ even.} \\ \frac{1}{2}\left(0 - \frac{jA}{2}\right) & , n = 1 \\ 0 & , o.w \\ \frac{1}{2}\left(\frac{jA}{2}\right) & , n = -1 \end{cases}$$

Ex. Consider the following signal:

$$x(t) = \begin{cases} A & , -\frac{T_o}{4} \leq t \leq \frac{T_o}{4} \\ -A & , -\frac{T_o}{2} \leq t \leq -\frac{T_o}{4} \text{ and } \frac{T_o}{4} \leq t \leq \frac{T_o}{2} \end{cases}$$

① evaluate C.E. F.S:



T = To → full period

$$\Rightarrow x_n = \frac{1}{T}\int_T x(t)\, e^{-jnw_ot}\, dt$$

$$= \frac{1}{T_o}\int_{T_o} x(t)\, e^{-jnw_ot}\, dt$$

$$= \frac{1}{T_o}\left[\int_{-\frac{T_o}{2}}^{-\frac{T_o}{4}} -A e^{-jnw_ot}\, dt + \int_{-\frac{T_o}{4}}^{\frac{T_o}{4}} A e^{-jnw_ot}\, dt + \int_{\frac{T_o}{4}}^{\frac{T_o}{2}} -A e^{-jnw_ot}\, dt\right]$$

$$= \frac{1}{T_o}\left[\frac{A}{jnw_o} e^{-jnw_ot}\Big|_{-\frac{T_o}{2}}^{-\frac{T_o}{4}} + \frac{A}{-jnw_o} e^{-jnw_ot}\Big|_{-\frac{T_o}{4}}^{\frac{T_o}{4}} + \frac{A}{jnw_o} e^{-jnw_ot}\Big|_{\frac{T_o}{4}}^{\frac{T_o}{2}}\right]$$

$$= \frac{A}{jn2\pi}\left(e^{jn\frac{\pi}{2}} - e^{jn\pi}\right) - \frac{A}{jn2\pi}\left(e^{-jn\frac{\pi}{2}} - e^{jn\frac{\pi}{2}}\right)$$

$$+ \frac{A}{jn2\pi}\left(e^{-jn\pi} - e^{-jn\frac{\pi}{2}}\right)$$

$$= \frac{2A}{jn2\pi} e^{jn\frac{\pi}{2}} - \frac{2A}{nj2\pi} e^{-jn\frac{\pi}{2}} - \frac{A}{2jn\pi} e^{jn\pi} + \frac{A}{jn2\pi} e^{-jn\pi}$$

$$= \frac{2A}{n\pi}\left(\frac{e^{jn\frac{\pi}{2}} - e^{-jn\frac{\pi}{2}}}{2j}\right) - \frac{A}{n\pi}\left(\frac{e^{jn\pi} - e^{-jn\pi}}{2j}\right)$$

$$= \frac{2A}{n\pi}\sin\left(n\frac{\pi}{2}\right) - \underset{\uparrow}{\frac{A}{n\pi}}\sin(n\pi)$$

n = 0 → undefined.

. for n even : $x_n = 0$ — $\{0\}$

. for n odd : $x_n = \begin{cases} \frac{2A}{n\pi} , & n = 1, 9, 5, \dots \\ -\frac{2A}{n\pi} , & n = 3, 7, 11, \dots \end{cases}$

$$x_n = \begin{cases} (-1)^{(n-1)/2} \cdot \frac{2A}{n\pi} , & n \text{ odd.} \\ 0 , & n \text{ even.} \end{cases}$$

. when $n = 0$;

$$x_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt = a_0$$

⇒ after we solve the integration ⇒ $x_0 = a_0 = 0$

② evaluate T.C for F.S :

$$a_n = 2\,\mathrm{Re}\{x_n\} = (-1)^{(n-1)/2} \cdot \frac{4A}{n\pi}$$

$$a_0 = x_0 = 0$$

$b_n = -2\,\mathrm{Im}\{x_n\} = 0$ (there is no imaginary part)

// x(t) is even func. & periodic → a_n exist, $b_n = 0$ ✓

// x_n → is real. (no imag.) ✓

// if x(t) is odd → x_n → imaginary. ✓

$$\Rightarrow x(t) = \sum_{n=-\infty \,,\, n \text{ odd}}^{\infty} (-1)^{(n-1)/2} \cdot \frac{2A}{n\pi} \cdot e^{+jn\omega_0 t}$$

ex. $x(t) = \sum_{-\infty \to n \text{ odd}}^{\infty} (-1)^{(n-1)/2} \cdot \frac{2A}{n\pi} \cdot e^{j30\pi n t}$

① specify the type for the F.S: complex exp. F.S

② evaluate the complex exp. coeff. : $x_0 = 0$

$$x_n = \begin{cases} (-1)^{(n-1)/2} \cdot \frac{2A}{n\pi} , & n \text{ odd} \\ 0 , & \text{o.w} \end{cases}$$

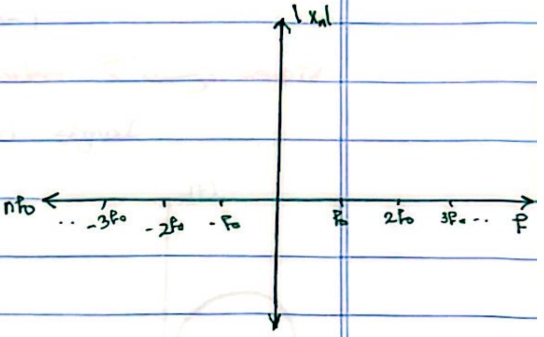
③ evaluate the first harmonic exp. coeff. : //تختلف التسمية عند دكتور جمال

$$x_1 = \frac{2A}{\pi}$$

④ evaluate the T.C for F.S :

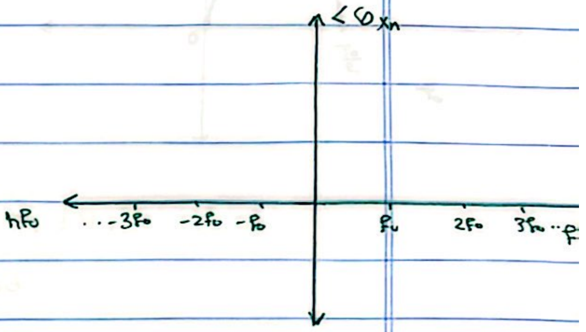
$$a_n = 2\,R\{x_n\} = \begin{cases} (-1)^{(n-1)/2} \cdot \frac{2 \cdot 2A}{n\pi} & , n \text{ odd} \\ 0 & , \text{o.w} \end{cases}$$

$$b_n = -2\,I_m\{x_1\} = 0$$



⑤ specify the type of signal, even or odd:

$x(t)$ = even ; $x_n \rightarrow$ real.



. Line spectra → plot the mag. & phase for x_n

. $|x_n| = |x_{-n}|$ even

. $<\phi_{x_n} = -<\phi_{x-n}$ odd

$$\cdot\; x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jnw_0t} = \sum_{n=-\infty}^{-1} x_n e^{jnw_0t} + x_0 + \sum_{n=1}^{\infty} x_n e^{jnw_0t}$$

$$= \sum_{n=1}^{\infty} x_{-n} e^{-jnw_0t} + x_0 + \sum_{n=1}^{\infty} x_n e^{jnw_0t}$$

since; $x_n = |x_n| e^{jnw_0t}$

$= |x_n| e^{j\phi_{x_n}}$

$$x(t) = \sum_{n=1}^{\infty} |x_{-n}| e^{j\phi_{x-n}} \cdot e^{-jnw_0t} + x_0 + \sum_{n=1}^{\infty} |x_n| e^{j\phi_{x_n}} \cdot e^{jnw_0t}$$

since, $|x_{-n}| = |x_n|$, $e^{j\phi_{x-n}} = e^{-j\phi_{x_n}}$

$$x(t) = \sum_{n=1}^{\infty} |x_n| e^{-j\phi_{x_n}} \cdot e^{-jnw_0t} + x_0 + \sum_{n=1}^{\infty} |x_n| e^{j\phi_{x_n}} \cdot e^{jnw_0t}$$

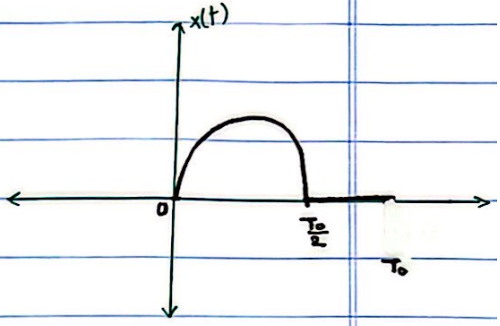
$$x(t) = x_0 + \sum_{n=1}^{\infty} |x_n| \left(e^{-j\phi_{x_n}} \cdot e^{-jnw_0t} + e^{j\phi_{x_n}} e^{jnw_0t} \right) \cdot \frac{2}{2}$$

$$x(t) = x_0 + 2\sum_{n=1}^{\infty} |x_n| \cos(n\omega_0 t + \phi_{x_n})$$

اذا حبينا نرسمه كـ single sided. بدل double sided.

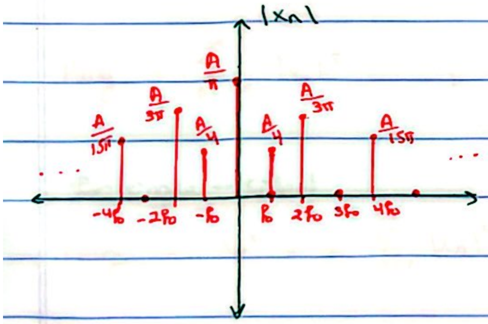
Ex. plot line spectra for the half-wave rectified signal

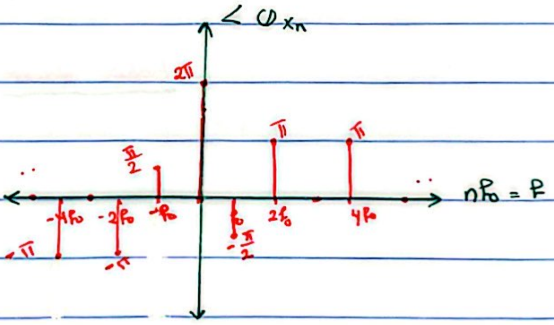
$$x_n = \begin{cases} \frac{A}{\pi(1-n^2)} & , n = 0, \pm 2, \pm 4, \dots \\ 0 & , n \text{ odd} \\ \mp \frac{1}{4} j n A & , n = \pm 1 \end{cases}$$



- نقوم بتحويل x_n بدلالة mag. و phase.

$$x_n = \begin{cases} |x_n| \angle \phi_{x_n} \\ 0 \\ |x_n| \angle \phi_{x_n} \end{cases} = \begin{cases} \frac{A}{\pi(1-n^2)} \angle 0 & , n = 0, \pm 2, \pm 4, \dots \\ 0 & , n \text{ odd} \\ \frac{1}{4} A \angle -\frac{\pi}{2} & , n = 1 \\ \frac{1}{4} A \angle \frac{\pi}{2} & , n = -1 \end{cases}$$





- if we want to draw single sided → we mul. each value by 2.

$$|x_2| = \frac{A}{\pi(1-4)} = \frac{-A}{3\pi} \angle 0 = \frac{A}{3\pi} \angle \pi$$

- Parseval's Theorem :

$$P_{avg} = \frac{1}{T}\int_T |x(t)|^2 \, dt = \frac{1}{T}\int_T x(t)\, x^*(t)\, dt$$

- In general; $Z = X + jY$, $Z^* = X - jY$

$$ZZ^* = X^2 + Y^2 = |Z|^2$$

- by using F.S :

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

$$P_{avg} = \frac{1}{T}\int_T \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} \, x^*(t)\, dt$$

// المجموع لا يعتمد على t ← يخرج من التكامل

$$= \sum_{n=-\infty}^{\infty} x_n \underbrace{\frac{1}{T}\int_T e^{jn\omega_0 t} \, x^*(t)\, dt}_{x_n^*}$$

- In general ; $x_n^* = \frac{1}{T}\int_T x^*(t)\, e^{jn\omega_0 t}$

$$P_{avg} = \sum_{n=-\infty}^{\infty} |x_n|^2 \rightarrow \underline{\text{double sided}}$$

- for single - sided

$$P_{avg} = x_0^2 + 2\sum_{n=1}^{\infty} |x_n|^2$$

Ex. For the following signal: $x(t) = 4\sin(50\pi t)$

evaluate the average power : Method ① $P_{avg} = \frac{1}{T}\int_T |x(t)|^2 \, dt$

$$P_{avg} = \frac{1}{0.04}\int_{0.04} 16\sin^2(50\pi t)\, dt = 8 \text{ Watt}.$$

Method ② : Parseval's theorem : $P_{avg} = \sum_{n=-\infty}^{\infty} |x_n|^2$

$$x_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= 25\int_0^{0.04} 4\underline{\sin(50\pi t)}\, e^{-jn50\pi t}\, dt$$

↳ convert to exp.

when , $n =$ even or odd $- \{-1, 1\}$ $\Rightarrow x_n = 0$

$x_1 = -j2$, $x_{-1} = 2j$

$P_{avg} = x_0^2 + 2\sum_{n=1}^{\infty} |x_n|^2 = 0 + 2(2)^2 = 8$ watt

ex. $x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{j\pi n+1} e^{j(\pi n 3t/2)}$

① specify the type of fourier series

$\Rightarrow$ complex exp. F.S

② evaluate the third harmonic of complex exp. F.S :

$\Rightarrow$ we need to find x_n first

$x_n = \frac{1}{1 + jn\pi}$

$x_3 = \frac{1}{1 + 3j\pi}$, $x_{-3} = \frac{1}{1 - 3j\pi}$

③ evaluate the mag. & phase of third harmonic .

$|x_3| = \frac{1}{\sqrt{(1)^2 + (3\pi)^2}} = \frac{1}{\sqrt{1 + (3\pi)^2}} = |x_{-3}|$

$\angle \theta x_3 = -\tan^{-1}(3\pi)$

$\angle \theta_{-3} = \tan^{-1}(3\pi)$

④ evaluate T.C of F.S :

$x_n = \frac{1}{1 + nj\pi} \cdot \frac{1 - jn\pi}{1 - jn\pi} = \frac{1}{1^2 + (n\pi)^2} - \frac{jn\pi}{1^2 + (n\pi)^2}$

$a_n = \frac{2}{1^2 + (n\pi)^2}$, $b_n = \frac{2n\pi}{1^2 + (n\pi)^2}$ $\rightarrow$ we can't determine whether it is even or odd

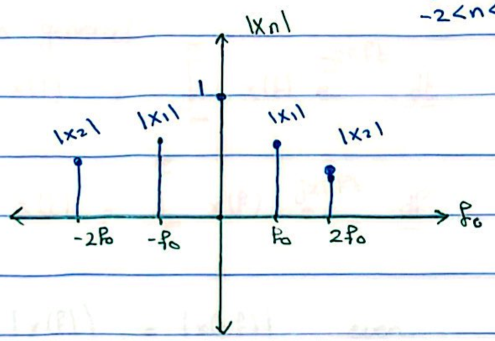
⑤ evaluate the third harmonic of T.F.S:

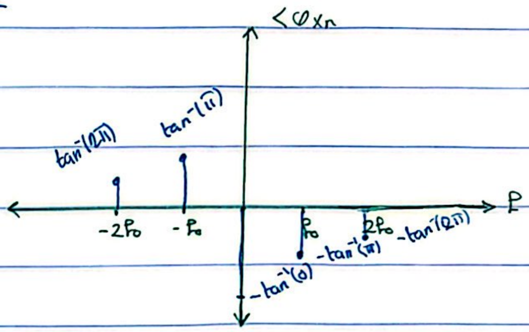
$a_3 = \frac{2}{1+(3\pi)^2}$, $b_3 = \frac{2\pi(3)}{1+(3\pi)^2}$

⑥ plot line spectra $[-2, 2]$:

$\downarrow$

$-2<n<2$





// we calculate :

$x_2 = \frac{1}{\sqrt{(2\pi)^2+1}}$, $-\tan^{-1}(2\pi)$

$x_1 = \frac{1}{\sqrt{\pi^2+1}}$, $-\tan^{-1}(\pi)$

$x_0 = \; = 1$

⑦ evaluate avg. power. using parseval's theorem $(-2<n<2)$

$P_{avg} = |x_0|^2 + 2|x_1|^2 + 2|x_2|^2$

$= 1^2 + 2\cdot\left(\frac{1}{\sqrt{\pi^2+1}}\right)^2 + 2\cdot\left(\frac{1}{\sqrt{(2\pi)^2+1}}\right)^2$ watt

Ch.4: Fourier transform

Time domain → frequency domain

$$\begin{tikzcd} x(t) \arrow[r, shift left, "F(x(t))"] & x(f) \arrow[l, shift left, "F^{-1}(x(f))"] \end{tikzcd}$$

In general:

$$x(f) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, \underline{dt}$$

$$x(t) = \int_{-\infty}^{\infty} x(f)\, e^{j2\pi ft}\, \underline{df}$$

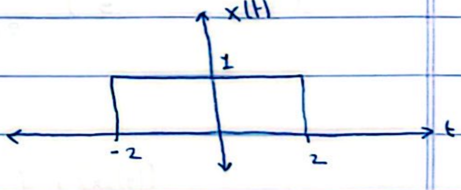
$|x(f)| = |x(-f)|$ even

$\emptyset_{xf} = -\emptyset_{x-f}$ odd

$\Rightarrow x(f) = x^*(-f)$

$x(f) = F[x(t)]$, $x(t) = F^{-1}[x(f)]$

ex. For the following signal, find x(f)

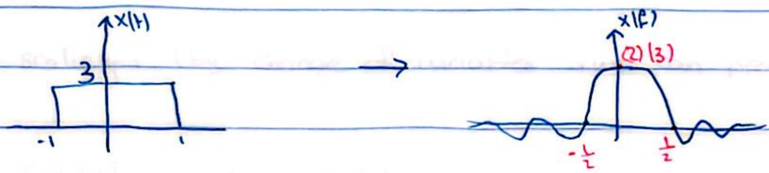


$$x(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi j ft}\, dt$$

$$= \int_{-2}^{2} (1) \cdot e^{-2\pi j ft}\, dt = \left.\frac{e^{-2\pi j ft}}{-2\pi j f}\right|_{-2}^{2} = \frac{-1}{j2\pi f}\left(e^{-j2\pi(2)f} - e^{-j(2\pi)f(-2)}\right.$$

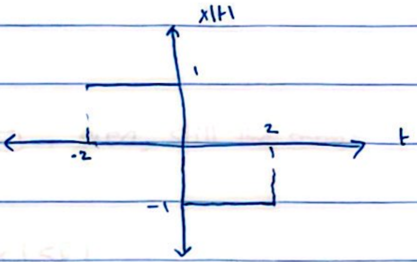
$$= \frac{1}{\pi f}\, \underline{\sin(4\pi f)} = 4\operatorname{sinc}(4f)$$

even → even



ex.

$X(f) = \int_{-2}^{0} e^{-j2\pi f t}\, dt - \int_{0}^{2} e^{-j2\pi f t}\, dt$



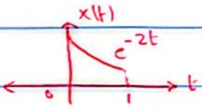
$X(f) = (j4\, \underset{\text{even}}{\underset{\downarrow}{\text{sinc}^2(2f)}})\, \underset{\text{odd}}{\underset{\downarrow}{\pi f}}$

even . odd = odd

x(t) → odd → X(f) → imaginary , odd

x(t) → even → X(f) → real , even

ex. $x(t) = e^{-2t}[u(t) - u(t-1)]$



$X(f) = \int_{-\infty}^{\infty} e^{-2t} \cdot e^{-j2\pi f t}\, dt = \int^{1} e^{-(2+2\pi j f t)}\, dt$

$= \frac{-1}{2+2\pi j f} e^{-(2+2j\pi f t)}\Big|_{0}^{1} = \frac{-1}{2+2\pi j f}\left(e^{-(2+2\pi j f)} - 1\right)$

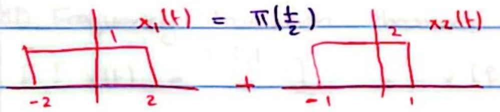
Theorems of Fourier transform:

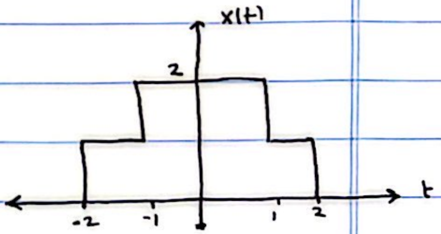
① $F[a x_1(t) \mp b x_2(t)] = a F[x_1(t)] \mp b F[x_2(t)]$

$= a X_1(f) \mp b X_2(f)$

ex. consider the following signal:

$x_1(t) = \Pi(\frac{t}{2})$ $x_2(t) = \Pi(\frac{t}{4})$





$x(t) = x_1(t) + x_2(t) = 2\Pi(\frac{t}{2}) + \Pi(\frac{t}{4})$

$X(f) = 2\,\text{sinc}(2f) + 4\,\text{sinc}(4f)$ (by linearity & scaling)

② scaling: (by change of variables → we can prove it)

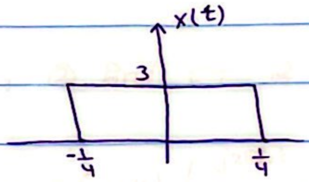
$F[x(at)] = \frac{1}{|a|} \cdot x\left(\frac{f}{a}\right)$

ex. find $x(f)$ for $x(t) = 3\Pi(2t)$

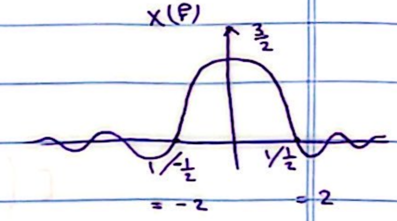
$x(f) = \frac{3}{2}\,\text{sinc}\left(\frac{f}{2}\right)$ // if it was ⊟ → even; still the same.

$x(t) = 15\ \Pi\left(\frac{t}{5}\right) \Rightarrow x(f) = 15.5\ \text{sinc}(5f)$

$x(t) = 3\Pi(2t)$



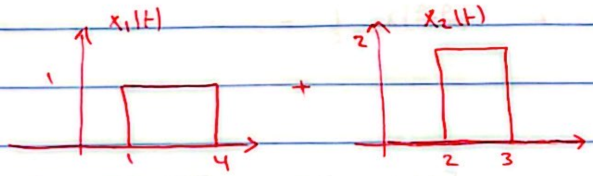
→

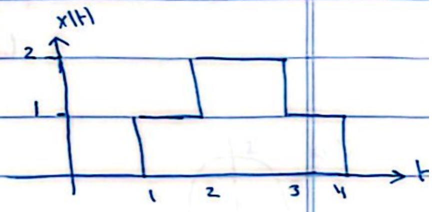


③ Time-delay Theorem:

$F[x(t-t_0)] = e^{-j2\pi f t_0}\ x(f)$

ex.





$x(t) = x_1(t) + x_2(t)$

$x(t) = 2\,\Pi\left(\frac{t-2.5}{1}\right) + \Pi\left(\frac{t-2.5}{3}\right)$

1) linearity. 2) scaling.
3) time-delay.

$x(f) = 2e^{-j2\pi f(2.5)}\ \text{sinc}(f) + 3e^{-j2\pi f(2.5)}\ \text{sinc}(3f)$

④ Frequency transition theorem:

$F[x(t)\ e^{j2\pi f_0 t}] = x(f-f_0)$

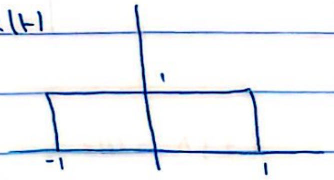
⑤ Modulation theorem: طريقة لنقل الإشارات حتى لا تبعث متداخلة مع بعضها البعض → c(t). messa

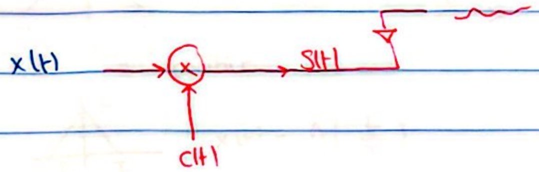
$F[x(t)\cos(2\pi f_0 t)]$

every signal will be at base band signal → at origin ex. consider the following modulation system:

$s(t) = x(t) \cdot c(t)$

where x(t) ; $c(t) = 4\cos(2000\pi t)$



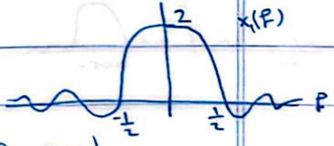


① find the F.T of s(t) , ② find FT of x(t), c(t)

$$\Rightarrow F[x(t)\cos(2\pi f_0 t)] = F\left[x(t)\left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right)\right]$$

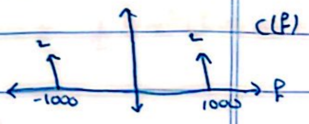
$$= F\left[\frac{1}{2}x(t)\,e^{j2\pi f_0 t}\right] + F\left[\frac{1}{2}x(t)\,e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$



Answer: ① $F[x(t)] = F[\Pi(\frac{t}{2})] = 2\sin(2f)$

$F[c(t)] = F[4\cos(2000\pi t)] = \frac{4}{2}\delta(f-1000) + \frac{4}{2}\delta(f+1000)$

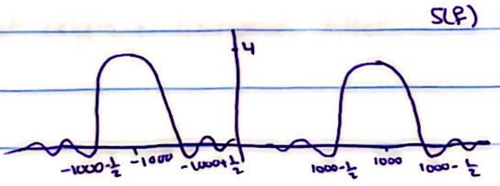


transmitted signal → message → carrier

② $F[s(t)] = F[x(t) \cdot c(t)]$

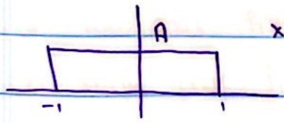
$= F[x(t) \cdot 4\cos(2000\pi t)] = F[\Pi(\frac{t}{2}) \cdot 4\cos(2000\pi t)]$

$= 4\left[\frac{1}{2} \cdot 2 \cdot \text{sinc}(2(f-1000)) + \frac{1}{2} \cdot 2 \cdot \sin(2(f+1000))\right]$

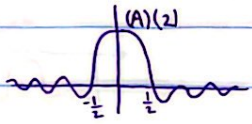


// when it is not at origin → band pass signal.

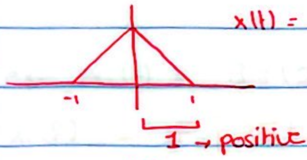
*** Functions** *** Fourier transform.**



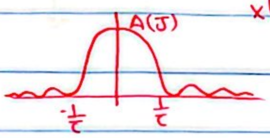
$x(t) = A\Pi\left(\frac{t}{2}\right)$ $\xrightarrow{F}$



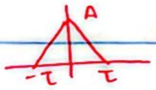
$x(f) = 2A\ \sin(2f)$



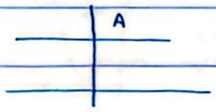
$x(t) = \Lambda\left(\frac{t}{1}\right)$ $\xrightarrow{F}$



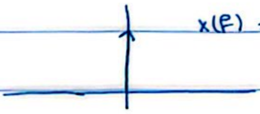
$x(f)$



$x(t) = \Lambda\left(\frac{t}{\tau}\right)$

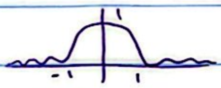


$x(t) = A$ $\xrightarrow{F}$

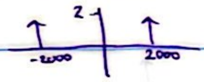


$x(f) = \delta(t)$

ex. $F[\Lambda(t)] = \text{sinc}^2(f)$

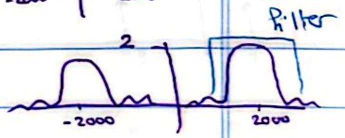


$F[4\cos(2000\pi t)] = \frac{4}{2}\delta(f-2000) + \frac{4}{2}\delta(f+2000)$



$F[S_2(t)] = F[\Lambda(t) \cdot 4\cos(2000t\pi)]$

$= 4\left[\frac{1}{2} \cdot \text{sinc}^2(f-2000) + \frac{1}{2}\ \text{sinc}^2(f+2000)\right]$



⇒ Band pass theorem.

. $F[\Lambda(t-5) \cdot 4\cos(2000\pi t)] = 4\left[\frac{1}{2}\text{sinc}^2(f-2000)\,e^{-j2\pi f(5)} + \frac{1}{2}\text{sinc}^2(f+2000)\,e^{-2\pi j f(5)}\right]$

. $F[3\Lambda(2t-5) \cdot \cos(2000\pi t)] = F[3\Lambda(2(t-\frac{5}{2}) \cdot \cos(2000\pi t)]$

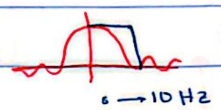
$= \left(\frac{1}{2} \cdot \frac{3}{2}\ \text{sinc}^2\left(\frac{f-1000}{2}\right) + \frac{1}{2} \cdot \frac{3}{2}\ \text{sinc}^2\left(\frac{f+1000}{2}\right)\right) e^{-2\pi j f(5)}$

// if we want to take signal → at origin → low pass filter

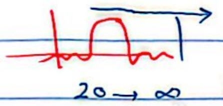
// Band pass filter → reference.

· Types of filters :

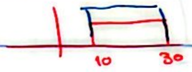
① low pass filter → تمرر f قليلة



② High pass filter →



③ Band pass filter →



ex. $x_1(t) = 3\pi(2t-10)\cos(14\pi t)$

$x_1(f) = F[3\pi(2t-10)\cos(14\pi t)]$

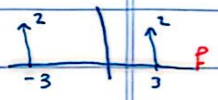
$$= 3\left(\frac{1}{2}\cdot\frac{1}{2}\,\text{sinc}\left(\frac{f-7}{2}\right) + \frac{1}{2}\cdot\frac{1}{2}\,\text{sinc}\left(\frac{f+7}{2}\right)\right)\cdot e^{-j2\pi f(5)}$$

ex. Find and plot F.T of x(t), c(t) :

$x(t) = 2\cos(6\pi t)$

$c(t) = 4\cos(12\pi t)$

$\Rightarrow F[2\cos(6\pi t)] = \frac{1}{2}\cdot 2[\delta(f-3) + \delta(f+3)]$



$\Rightarrow F[4\cos(12\pi t)] = \frac{1}{2}\cdot 4[\delta(f-6) + \delta(f+6)]$

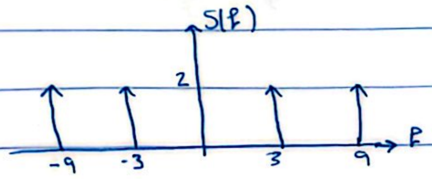
Find & plot F.T of S(t)

$F[S(t)] = F[2\cos(6\pi t)\cdot 4\cos(12\pi t)]$

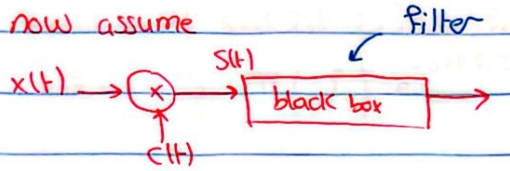
$= 8F[\frac{1}{2}\cos(6\pi t) + \frac{1}{2}\cos(18\pi t)]$

$= 4\ F[\cos(6\pi t)] + 4F[\cos(18\pi t)]$

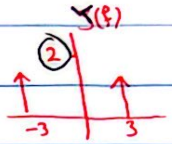
$= 4[\frac{1}{2}\delta(f-3) + \frac{1}{2}\delta(f+3)] + 4[\frac{1}{2}\delta(f-9) + \frac{1}{2}\delta(f+9)]$

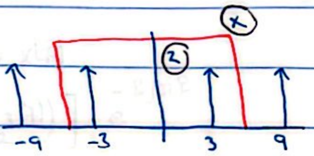


· now assume

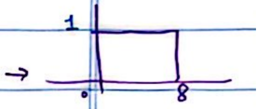


$y(t) = 4\cos(6\pi t)$





$s(t) = 4\cos(18\pi t) + 4\cos(6\pi t)$



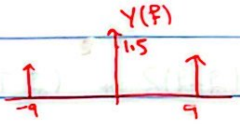
⇒ low pass filter → (لايكون عند ال 9)

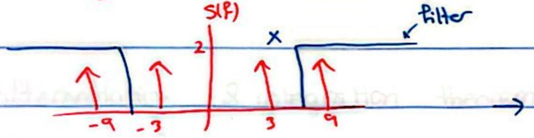
· The amp. of filter can be calculated as:

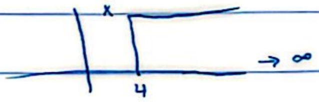
$(x)(2) = 2$

$\boxed{x = 1}$

· new: $y(t) = 3\cos(18\pi t)$

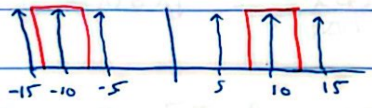






$(x)(2) = 1.5$

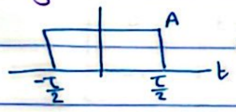
$\boxed{x = \frac{3}{4}}$ high pass filter.



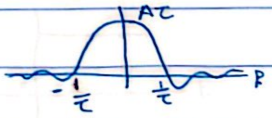
→ Band pass filter

· Duality theorem:

①



$x(t) = A\,\Pi\left(\frac{t}{\tau}\right) \xleftrightarrow{F}$



$x(f) = A\tau\,\text{sinc}(\tau f$

· if $x(t) = 2\,\text{sinc}(2t)$, find $x(f)$

⇒ by using duality theorem:

$x(f) = \frac{2}{2}\,\Pi\left(\frac{-f}{2}\right) = \Pi\left(\frac{f}{2}\right)$

- $x(t) = 4\ \mathrm{sinc}(2(t-2))$, find $x(f)$

$$x(f) = \frac{4}{2}\ \Pi\left(\frac{-f}{2}\right) e^{-j2\pi f(2)} = 2\ \Pi\left(\frac{f}{2}\right) e^{-4j\pi f}$$

- $x(t) = 3\ \mathrm{sinc}\ (2t-8) \cos(14\pi t)$, find $x(f)$

$$x(f) = \frac{3}{2}\left[\frac{1}{2} \Pi\left(\frac{-(f-7)}{2}\right) + \frac{1}{2} \Pi\left(\frac{-(f+7)}{2}\right)\right] e^{-8j\pi f}$$

$$x(f) = \left(\frac{3}{4}\ \Pi\left(\frac{f-7}{2}\right) + \frac{3}{4}\ \Pi\left(\frac{f+7}{2}\right)\right) e^{-8j\pi f}$$

- $A \longleftrightarrow A\delta(f)$ · $A\delta(t-t_0) \longleftrightarrow A e^{-j2\pi t_0 f}$

$A\delta(f) \longleftrightarrow A$

- $A\ \delta(t) \cos(2\pi f_0 t) \longleftrightarrow \frac{A}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$

· Differentiation & integration theorem:

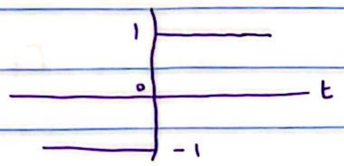
$$F\left[\frac{d}{dt} x(t)\right] = 2j\pi f\ x(f)$$

. In general : $F\left[\frac{d^n}{dt^n} x(t)\right] = (j2\pi f)^n\ x(f)$

$$F\left[\int_{-\infty}^{t} x(\lambda)\, d\lambda\right] = \frac{x(f)}{j2\pi f} + c\ \delta(f)$$

ex. Find the F.T for a signal function. (signum function)

$$\mathrm{sgn}(t) = \begin{cases} 1 & , t>0 \\ 0 & , t=0 \\ -1 & , t<0 \end{cases}$$



$\mathrm{sgn}(t) = 2u(t) - 1$

$\Rightarrow F[\mathrm{sgn}(t)] = F[2u(t) - 1]$

$F\left[\frac{d}{dt} \mathrm{sgn}(t)\right] = F[2\delta(t)] = 2 \Rightarrow F[\mathrm{sgn}(t)] = \frac{1}{j\pi f}$

$2j\pi f\ F\{\mathrm{sgn}(t)] = 2$

. To evaluate F.T for u(t)

$sgn(t) = 2u(t) - 1$

$\Rightarrow u(t) = \frac{1}{2} + \frac{1}{2} sgn(t)$

$F[u(t)] = \frac{1}{2}\delta(f) + \frac{1}{2j\pi f}$

. if $x(t) = \frac{1}{\pi t} \Longrightarrow$ find $x(f)$

$\Rightarrow F[sgn(t)] = \frac{1}{j\pi f}$

$\frac{1}{\pi t} \cdot \frac{j}{j} = \frac{j}{j\pi t}$

$\Rightarrow x(f) = j\, sgn(-f) = -j\, sgn(f)$; using duality. function.

. Convolution theorem:

$F[x_1(t) * x_2(t)] = x_1(f) . x_2(f)$

<u>ex.</u> if $x(t) = \Pi(\frac{t}{3}) * \Pi(\frac{t}{3})$; find $x(f)$

$F[x(t)] = x(f) = F[\Pi(t.\frac{1}{3}) * \Pi(\frac{t}{3})]$

$= 3\,sinc(3f) . 3\,sinc(3f)$

$= 9\,sinc^2(3f) = 3 . 3\,sinc^2(3f)$

$= 3\,(F[\Lambda(\frac{t}{3})])$

. In general:

$F[\Pi(\frac{t}{\tau}) * \Pi(\frac{t}{\tau})] = \tau\, F[\Lambda(\frac{t}{\tau})]$

<u>ex.</u> find F.T for the hilbert transform function (بعمل شيفت لل sig. بمقدار 90°)

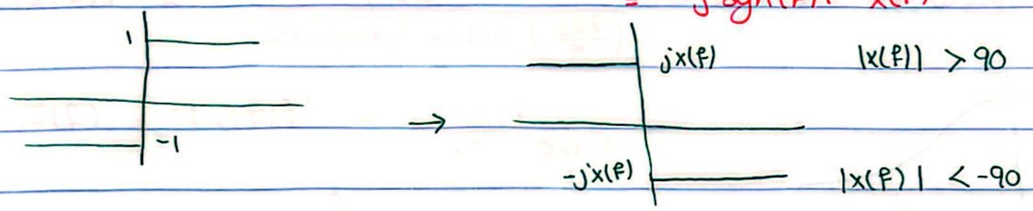
$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda}\, d\lambda = \frac{1}{\pi t} * x(t)$

// بكون بداخله ال sgn ← لعمل شيفت

$$\cdot\ F[\hat{x}(t)] = F\left[\frac{1}{\pi t} * x(t)\right] = \frac{1}{\pi} F\left[\frac{1}{t}\right] \cdot F[x(t)]$$

$$= j\,\mathrm{sgn}(-f) \cdot x(f)$$

$$= -j\,\mathrm{sgn}(f) \cdot x(f)$$



→ تبقى ال energy كما هي

.energy spectral density. function

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad ; \quad |x(t)|^2 = x(t)^* \cdot x(t)$$

$$= \int_{-\infty}^{\infty} x(t)^* \cdot x(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)^* \left(\int_{-\infty}^{\infty} x(f)\, e^{j2\pi f t}\, df \right) dt$$

$$= \int_{-\infty}^{\infty} x(f) \left(\int_{-\infty}^{\infty} x(t)^*\, e^{j2\pi f t}\, df \right) dt$$

$$= \int_{-\infty}^{\infty} x(f)\, x^*(f)\, df = \boxed{\int_{-\infty}^{\infty} |x(f)|^2\, df}$$

where, $G(f) = |x(f)|^2$ → energy spectral density.

ex. $x(t) = e^{-\alpha t} u(t)$, $\alpha > 0$ (energy)

① find F.T for $x(t)$:

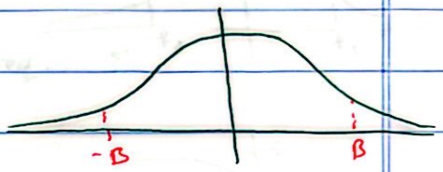
$F[e^{-\alpha t} u(t)] = \dfrac{1}{\alpha + j2\pi f}$ (using integration)

② find the energy S.D function.

$$X(f) = \frac{1}{\alpha + j2\pi f}$$

$$|X(f)| = \frac{1\angle 0}{\sqrt{\alpha^2+(2\pi f)^2} \angle \tan^{-1}\left(\frac{2\pi f}{\alpha}\right)} = |X(f)| \angle \varphi_{X(f)}$$

$$G(f) = |X(f)|^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$



$$\Rightarrow E = \lim_{B\to\infty} \int_{-\infty}^{B} \frac{1}{\alpha^2+(2\pi f)^2}\, df$$

$-\infty \swarrow -B$

. System analysis F.T : $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

For LTI system:

$$y(t) = h(t) * x(t)$$

$$F[y(t)] = H(f) \cdot X(f) \Rightarrow |X(f)|\, e^{j\varphi_{X(f)}} \cdot |H(f)|\, e^{j\varphi_{H(f)}} = Y(f)$$

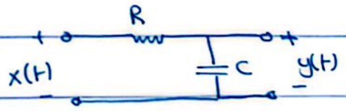
$$Y(f) = |X(f)|\,|H(f)|\, e^{j(\varphi_{X(f)} + \varphi_{H(f)})}$$

. In general:

$$Y(f) = X(f) \cdot H(f) \Rightarrow \text{transfer function } H(f) = \frac{Y(f)}{X(f)}$$

ex. For the RC circuit; ① find the amp. & phase of the system.

$$F\left[RC \frac{d}{dt} y(t) + y(t) = x(t)\right] \qquad -\infty < t < \infty$$



$$RC\, F\left[\frac{d}{dt} y(t)\right] + F[y(t)] = F[x(t)]$$

$$RC \cdot j2\pi f \cdot Y(f) + Y(f) = X(f)$$

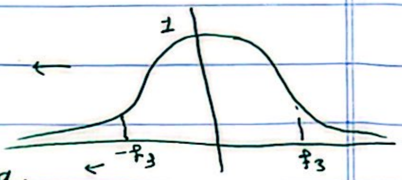
$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + RC \cdot j2\pi f} = \frac{1}{RC\left(\frac{1}{RC} + j2\pi f\right)} \Rightarrow \frac{1}{RC}\, e^{-\frac{t}{RC}}\, u(t) = h(t)$$

inverse.

② plot amp. & phase response.

$$H(f) = \frac{1}{1 + j2\pi f RC} = \frac{1 \angle 0}{\sqrt{(1)^2 + (2\pi f RC)^2} \angle \tan^{-1}\left(\frac{2\pi f RC}{1}\right)} = |H(f)| \angle \varphi_{H(f)}$$

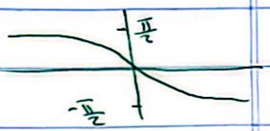
low pass filter ←



$f_3 = \frac{1}{2\pi RC}$ (يجب ان يكون واحد) cut off freq.

ونأخذ المقلوب

$$\varphi_{H(f)} = -\tan^{-1}(2\pi f RC) = -\tan^{-1}\left(\frac{f}{f_3}\right)$$



$f_3 \Rightarrow \underline{\underline{1}} + \underline{\underline{2\pi f RC}}$

- Steady state resp. to sinusoidal input by means of the F.T

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$, for LTI

$$y(t) = h(t) * x(t)$$

// if $x(t)$ is periodic signal → it can be represented in F.S

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jnw_0t} = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nf_0t}$$

$$x(f) = F[x(t)] = \sum_{n=-\infty}^{n=\infty} x_n \, \delta(f - f_0 n)$$

$$Y(f) = X(f) \cdot H(f) = \sum_{n=-\infty}^{\infty} x_n \, \delta(f - f_0 n) H(f) = \sum_{n=-\infty}^{\infty} x_n \, H(nf_0) \, \delta(f - nf_0)$$

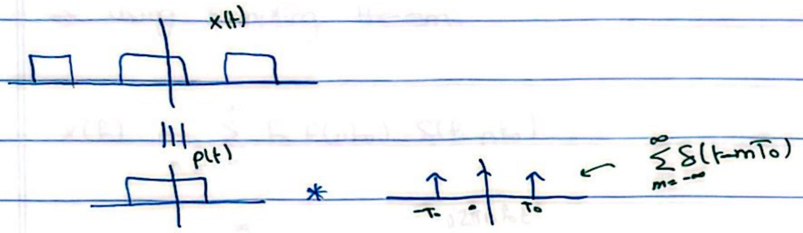
// $|x_n| e^{j\varphi_{x_n}} = x_n$

// $|H(nf_0)| e^{j\varphi_{H(nf_0)}} = H(nf_0)$

$$Y(f) = \sum_{n=-\infty}^{n=\infty} |x_n| \, |H(nf_0)| \, e^{j(\varphi_{x_n} + \varphi_{H(f_0)})} \quad \underline{\delta(f - nf_0)} \quad \hookrightarrow e^{j2\pi nf_0t}$$

$$Y(f) = \sum_{n=-\infty}^{\infty} |x_n| \, |H(nf_0)| \, e^{j(\varphi_{x_n} + \varphi_{H(nf_0)} + 2\pi nf_0t)} = y(t)$$

- Fourier transform for a periodic signal.



In general;

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$\Sigma\ x(t) * \delta(t-t_0) = \sum_{m=-\infty}^{\infty} x(t) * \delta(t-mT_0) = \sum_{m=-\infty}^{\infty} x(t-mT_0)$$

$$x(t) = p(t) * \sum_{m=-\infty}^{\infty}\delta(t-mT_0)$$

$$F[x(t)] = F[p(t)] \cdot F[\Sigma\delta(t-mT_0)]$$

$$= P(f) \cdot \Sigma\ F[\delta(t-mT_0)]$$

$$= P(f) \cdot \sum_{n=-\infty}^{\infty} F[C_n e^{j2\pi n f_0 t}]$$

$$= P(f) \cdot \Sigma\ C_n\ \delta(f-nf_0)$$

In general; $F[\sum_{-\infty}^{\infty}\delta(t-mT_0)] = \sum_{-\infty}^{\infty}\underset{\downarrow}{C_n} e^{j2\pi n f_0 t} = \sum_{-\infty}^{\infty} C_n\ \delta(f-nf_0)$

complex exp. F.S → fund. Freq. → f_0

$$C_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\ e^{-j2\pi n f_0 t}\ dt$$

by using sifting theorem.

$$C_n = \frac{1}{T_0} = f_0$$

$$\Rightarrow \sum_{-\infty}^{\infty}\delta(t-mT_0) = \sum_{-\infty}^{\infty} f_0\ e^{j2\pi n f_0 t}$$

$$\Rightarrow F[\underset{\substack{\uparrow \\ \text{train of delta}}}{\Sigma\ \delta(t-mT_0)}] = \sum_{-\infty}^{\infty} f_0\ \delta(f-nf_0)$$

$$x(f) = P(f) \sum_{-\infty}^{\infty} f_0\, \delta(f - nf_0) = \sum_{-\infty}^{\infty} P(f)\, f_0\, \delta(f - nf_0)$$

⇒ using sampling theorem.

$$x(f) = \sum_{n=-\infty}^{\infty} f_0\, P(nf_0)\, \delta(f - nf_0)$$

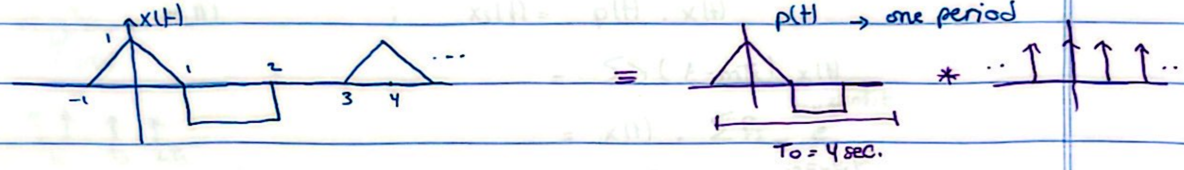
$$x(t) = \sum_{n=-\infty}^{\infty} f_0\, P(nf_0)\, e^{j2\pi n f_0 t}$$

since; x(t) periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} x_n\, e^{j2\pi n f_0 t}$$

$X_n = P(nf_0) \cdot f_0$ //another way to calculate F.S

ex.



$f_0 = \frac{1}{4}$ Hz

1. evaluate x(f)

since x(t) is periodic signal. ⇒ $x(t) = p(t) * \sum_{-\infty}^{\infty} \delta(T - mT_0)$

① $x(f) = \sum_{-\infty}^{\infty} f_0\, p(nf_0)\, \delta(f - nf_0)$

② $p(f) = ?$

$p(t) = \Lambda(t) + (-1)\, \Pi\left(\frac{t-1.5}{1}\right)$

$p(f) = \text{sinc}^2(f) - \text{sinc}(f)\, e^{-j(2\pi) f (1.5)}$

③ $p(nf_0) = p(\frac{n}{4}) = \text{sinc}^2(\frac{n}{4}) - \text{sinc}(\frac{n}{4})\, e^{-j2\pi(\frac{n}{4})(1.5)}$

④ $f_0\, P(nf_0) = \frac{1}{4} p(\frac{n}{4})$ s $\frac{1}{4}\text{sinc}^2(\frac{n}{4}) - \frac{1}{4}\text{sinc}(\frac{n}{4})\, e^{-j2\pi \frac{n}{4}(1.5)}$

⑤ $x(f) = \sum_{n=-\infty}^{\infty} \frac{1}{4}\text{sinc}^2(\frac{n}{4}) - \frac{1}{4}\text{sinc}(\frac{n}{4})\, e^{-j2\pi \frac{n}{4}(1.5)}\, \delta(f - \frac{n}{4})$

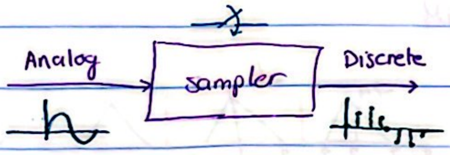
2. evaluate x(t)

$$x(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\text{sinc}^2(\frac{n}{4}) - \frac{1}{4}\text{sinc}(\frac{n}{4})\, e^{-j2\pi(\frac{n}{4})(1.5)}\right) e^{j2\pi(\frac{n}{4})t}$$

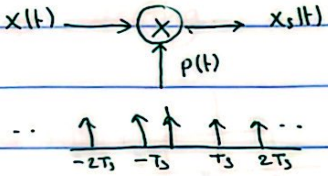
3. evaluate x_n

$$x_n = f_0\, p(n f_0) = \frac{1}{4}\,\text{sinc}^2\left(\frac{n}{4}\right) - \frac{1}{4}\,\text{sinc}\left(\frac{n}{4}\right) e^{-j(2\pi)\left(\frac{n}{4}\right)(1.5)}$$

. Analog to digital converter (ADC) ⇒ Ch.8



. Sampler :



; $x_s(t) = p(t)\,.\,x(t)$

$$= \sum \delta(t - mT_s)\; x(t)$$

$$= x(t)\,.\,\sum f_s\,.\, e^{j2\pi n f_s t}$$

$$= \sum f_s\, x(t)\, e^{j2\pi n f_s t}$$

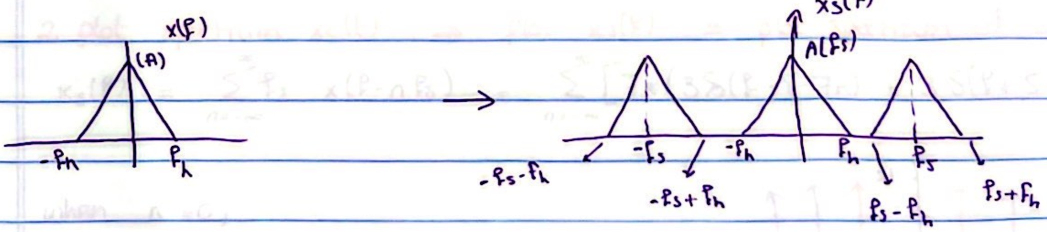
$$\sum_{n=-\infty}^{\infty} \delta(mT_s + t) = \sum C_n\, \delta(t - mT_s)$$

$$= \sum C_n\, e^{j2\pi n f_s t} = f_s \sum e^{j2\pi n f_s t}$$

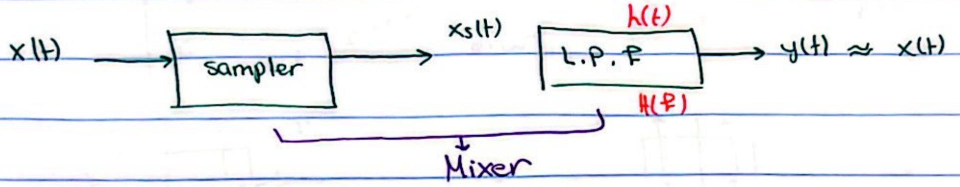
. using freq. translation:

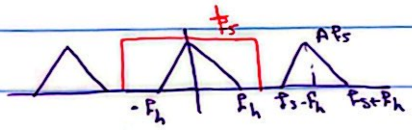
$$x_s(f) = \sum_{n=-\infty}^{\infty} f_s\, x(f - nf_s)$$, Fourier transform for sample signal

. In general



. Data Reconstruction signal. إعادة بناء ال



①


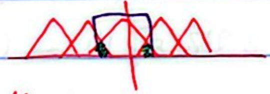
$f_s - f_h \geqslant f_h$

$f_s \geqslant 2f_h \Rightarrow$ Nyquistrate

$f_s = 2f_h$

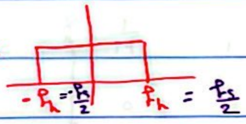
(at least)

↳ sampling frequency

②


if $f_s < 2f_h$

$\Rightarrow$ aliasing.

. for filter:


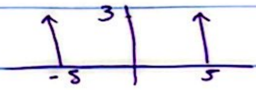
$$H(f) = \begin{cases} \frac{1}{f_s} = T_s & , |f| \leqslant 0.5 f_s \\ 0 & , \text{o.w} \end{cases}$$

. ex. consider the following signal
$x(t) = 6\cos(10\pi t)$ sampled at 7Hz.

$\swarrow f_s \overset{??}{\geqslant} 2f_h \Rightarrow 7 \overset{??}{\geqslant} 5*2 = 10$ X → aliasing

1. plot spectrum x(t) ⇒ plot F.T of x(t)

$x(f) = 3\,\delta(f-5) + 3\,\delta(f+5)$



2. plot spectrum xs(t) ⇒ plot xs(f) ≡ plot spectrum of sampled signal xs(t)

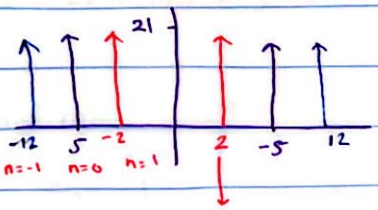
$$x_s(f) = \sum_{n=-\infty}^{\infty} f_s \; x(f - n f_s) = \sum_{n=-\infty}^{\infty} \left[7 * (3\delta(f-5-7n) + 3\delta(f+5-7n))\right]$$

when n = 0;

$x_s(f) = 21\,\delta(f-5) + 21\,\delta(f+5)$

when n = 1;

$x_s(f) = 21\,\delta(f-2) + 21\,\delta(f+12)$

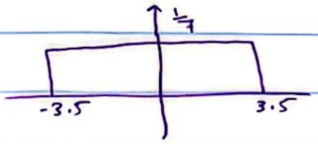


يوجد signal داخل الأصلية

③ plot the output of reconstruction filter

$$H(f) = \begin{cases} \frac{1}{7} & , \quad -3.5 \leq f \leq 3.5 \\ 0 & , \quad o.w \end{cases}$$

$\frac{1}{2}\cdot -7$ $\frac{1}{2}\cdot 7$

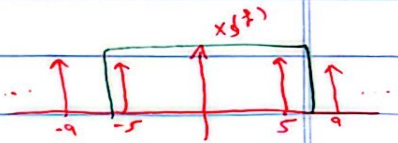


ليست ال sig. الأصلية

now; $f_s = 14$

1. spectrum of $x_s(t)$

2. $x_s(f) = \sum (14 (3 . \delta(f-5-14n) + 3 \delta(f-5+14n)))$



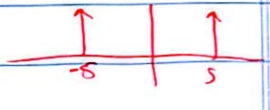
since; $f_s \geqslant 2f \rightarrow$ nyquist rate $\Rightarrow$ we take only $n=0$

$x_s(f) = 14 [3 \delta(f-5) + 3\delta(f+5)]$

3. plot D.R.F

$$H(f) = \begin{cases} \frac{1}{14} & , \quad |f| \leqslant 7 \\ 0 & , \quad o.w \end{cases}$$

$\rightarrow$ جميع الدلتا تكون بعد ال 7.



ل ال signal الأصلية.

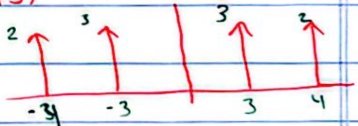
Ex. $x(t) = 4 \cos(8\pi t) + 6 \cos(6\pi t)$

1. minimum required sampling frequency to avoid aliasing

$f_s = 2 \max(f_1, f_2) = 2 \cdot 4 = 8$

2. plot spectrum $x(t)$

$x(f) = 2 \delta(f-4) + 2\delta(f+4) + 3\delta(f-3) + 3\delta(f+3)$

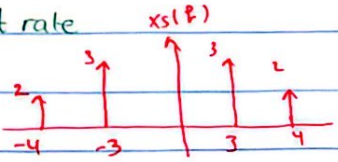


3. plot spectrum of sampled signal $x_s(t)$

$$x_s(f) = \sum_{n=-\infty}^{\infty} 8 [2.\delta(f-4-8n) + 3\delta(f-3-8n) + 2\delta(f+4-8n) + 3\delta(f+4-8n)]$$

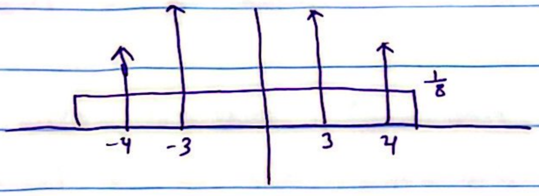
since we have $f_s = 2\max(f_1, f_2) \rightarrow$ nyquist rate

$\rightarrow$ we take $n = 0$



4. D. R

$$H(f) = \begin{cases} \frac{1}{8} & , |f| \leq 8 \cdot \frac{1}{2} \\ 0 & , \text{o.w} \end{cases} = \begin{cases} \frac{1}{8} & , |f| \leq 4 \\ 0 & , \text{o.w} \end{cases}$$



Ch.8 : Discrete - time signal & systems

. Z - transform

$$x_s(t) = \sum_{m=-\infty}^{\infty} x(t)\, \delta(t-mT_s)$$

. using sampling theorem.

$$x_s(t) = \sum_{m=-\infty}^{\infty} x(mT_s)\, \delta(t-mT_s)$$

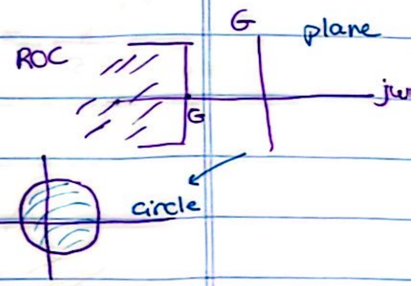
→ laplace transform : For sampling frequency.

$$x_s(s) = \sum_{m=-\infty}^{\infty} x(mT_s) \int_0^{\infty} \delta(s-mT_s)\, e^{-st}\, dt \quad ; \quad x(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

to be divergeness

$$x_s(s) = \sum_{m=-\infty}^{\infty} x(mT_s)\, e^{-mT_s s} \quad ; \quad z = e^{sT_s}$$

$$\boxed{x_s(z) = \sum_{m=-\infty}^{\infty} x(mT_s)\, z^{-m}}$$ → from laplace to z transform



. laplace : $s = G + jw$

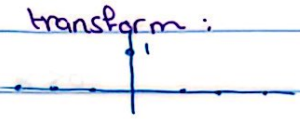
$Re\{s\} > 0$ → single sided (solution [illegible])

$$z = e^{sT_s} = e^{(G+jw)T_s} = e^{GT_s}\, e^{jwT_s} = |z|\ \angle z$$

// ROC : region of convergence.

Ex. For the following signals; find the z transform:

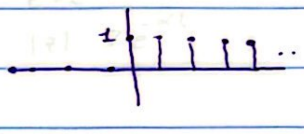
① $\delta[n] = \begin{cases} 1 & , n=0 \\ 0 & , \text{o.w} \end{cases} = x_1(z)$



$$x_1(z) = \sum_{n=-\infty}^{\infty} x(nT_s)\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n]\, \underset{\substack{\downarrow \\ n=0 \rightarrow z^{-n} = \boxed{1}}}{z^{-n}} = 1$$

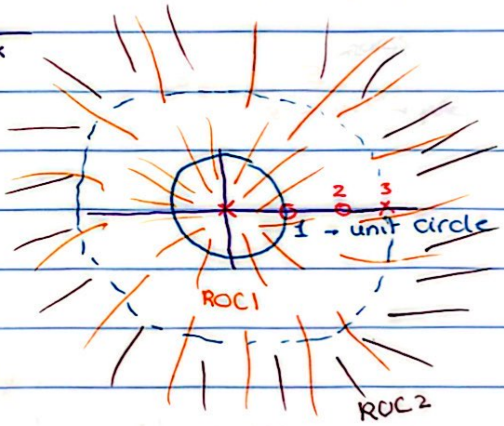
$x(nT_s) = x[n]$

② $u[n] = u(nT) = \begin{cases} 1 & , n \geq 0 \\ 0 & , \text{o.w} \end{cases}$



$$X_2(z) = \sum_{-\infty}^{\infty} x(nT)\, z^{-n} = \sum_{n=0}^{\infty} 1 \,.\, z^{-n} = \frac{1}{1-z^{-1}} = \frac{z \rightarrow \text{zeros}}{z-1 \rightarrow \text{poles}}$$

. In general; $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$



. In general;

$$x(z) = \frac{(z-1)(z-2)}{z\,(z-3)}$$

Find ROC : we look for poles

$(z)\ (z-3)$

$\downarrow \quad\quad \downarrow$

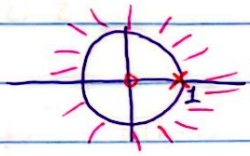
$z > 0 \quad\quad z-3 > 0$

$z > 3$

$\Rightarrow$ ROC $\rightarrow$ $z > 3$ $\Rightarrow$ we take the combined area

. For the ex. ;

ROC $\rightarrow$ $|z| > 1$



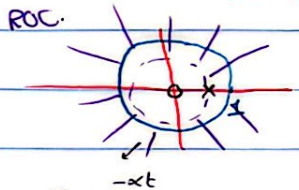
ex. $x(nT) = e^{-\alpha nT} u(nT)$

$= e^{-\alpha nT} \quad , \quad n \geq 0 \quad , \alpha > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-n\alpha T} z^{-n} = \sum_{0}^{\infty} (e^{-\alpha T} . z^{-1})^{n} = \frac{1}{1 - e^{-\alpha T} z^{-1}}$$

$$= \frac{z}{z - e^{-\alpha T}}$$



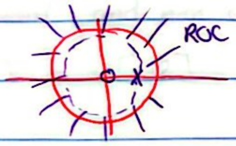
$|z| > e^{-\alpha t}$

. Consider the following signal:

$x[n] = 0.5^{n} u[n]$

① find $X(z)$ ② find ROC

$$X(z) = \sum_{0}^{\infty} 0.5^{n} z^{-n} = \sum_{0}^{\infty} (0.5 z^{-1})^{n} = \frac{1}{1 - 0.5 z^{-1}}$$



$$= \frac{\overset{\text{zeros}}{z}}{z - 0.5 \rightarrow \text{poles}}$$

$|z| > 0.5$

. properties of the z - transform:

① $x_1(nT) + x_2(nT)$ *linearity.*

$z[x_1(nT) + x_2(nT)] = z[x_1(nT)] + z[x_2(nT)] = X_1(z) + X_2(z)$

② time - delay:

$z[x[n-n_0]] = z[x(nT - n_0T)] = X(z)\, z^{-n_0}$

ex. $x[n] = 7 \left(\frac{1}{3}\right)^{n-2} u[n-2] - 6 \left(\frac{1}{2}\right)^{n-1} u[n-1]$

$X(z) = z\left[7\left(\frac{1}{3}\right)^{n-2} u[n-2]\right] - z\left[6\left(\frac{1}{2}\right)^{n-1} u[n-1]\right]$

$$= 7 . \frac{1}{1 - \frac{1}{3} z^{-1}} z^{-2} - 6 . \frac{1}{1 - \frac{1}{2} z^{-1}} . z^{-1}$$

. In general:

$$z[a^n u[n]] = \frac{1}{1-az^{-1}}$$

$$z[a^{n-n_0} u[n-n_0]] = \frac{1}{1-az^{-1}} z^{-n_0}$$

ex. For $x(nT) = a^n \cos(n\frac{\pi}{2})$

Find $x(z)$:

$$x(z) = \sum_{-\infty}^{\infty} x(nT)\, z^{-n} = \sum_{k=0}^{\infty} a^{2k} (-1)^k z^{-2k}$$

$$\cos(n\frac{\pi}{2}) = \begin{cases} 0 & , \text{odd } n \\ \pm 1 & , \text{even } n \end{cases}$$

. Properties of the system:

① causal and non causal:

- $y[n] = x[n-2] \rightarrow$ causal.

② time variant and invariant:

$y[n] = x[n-2] + x[2-n]$

$$\left.\begin{array}{l} y_1[n-n_0] = x[n-n_0-2] + x[2-n+n_0] \\ y_2[n-n_0] = x[n-2-n_0] + x[2-n-n_0] \end{array}\right] \neq \text{invariant}$$

③ linearity

$y[n] = x[n] + 2$

$x_1[n] \rightarrow y_1[n] = x_1[n] + 2$

$x_2[n] \rightarrow y_2[n] \pm x_2[n] + 2$

$x[n] = x_1[n] + x_2[n] \rightarrow y[n] = x_1 + x_2 + 4 \rightarrow$ non linear

④ convolution theorem:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

in LTI system:

$$y[n] = x[n] * h[n] \qquad * \rightarrow \int \rightarrow \Sigma \text{ in Disctrete}$$

$$= \sum_{K=-\infty}^{\infty} x(KT)\, h(nT-KT)$$

$$\overset{OR}{=} \sum_{K=-\infty}^{\infty} x(nT-KT)\, h(KT)$$

ex. if $x(nT) = \left(\frac{1}{2}\right)^n u[n]$ and $h[nT] = \left(\frac{1}{3}\right)^n u[n]$

find $y[n] = x[n] * h[n]$

$$= \sum_{K=-\infty}^{\infty} x(KT)\, h(nT-KT)$$

$$= \sum_{K=-\infty}^{\infty} \left(\frac{1}{2}\right)^K u[K] \cdot \left(\frac{1}{3}\right)^{n-K} u[n-K]$$

$$= \left(\frac{1}{3}\right)^n \sum_{K=-\infty}^{\infty} \left(\frac{1}{2}\right)^K \cdot \left(\frac{1}{3}\right)^{-K} \underline{u[K]} \cdot \underline{u[n-K]}$$

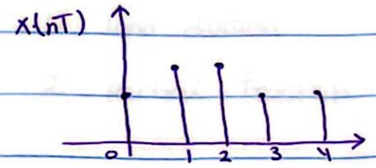
$$= \left(\frac{1}{3}\right)^n \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K \cdot \left(\frac{1}{3}\right)^{-k} = \left(\frac{1}{3}\right)^n \sum_{K=0}^{n} \left(\frac{3}{2}\right)^k = \frac{1-\left(\frac{3}{2}\right)^{n+1}}{1-\left(\frac{3}{2}\right)} \cdot \left(\frac{1}{3}\right)^n$$

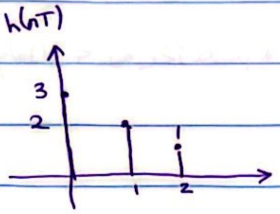
. In general

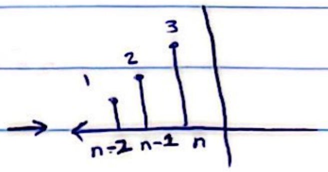
$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$$

$$//\left(\frac{1}{3}\right)^n \sum_{n=0}^{\overset{N}{n+1-1}} \left(\frac{3}{2}\right)^K$$

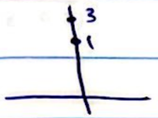
- Convolve the two functions







- For $n=0$



$y[n] = y[0] = 1.3 = 3$

For $n=1$

$y[1] = 1.2 + 2.3 = 8$

For $n=2$

$y[2] = 3.2 + 2.2 + 1.1 = 11$

⋮

$n=6$

$y[6] = 1.1 = 1$, for $n=7 \rightarrow y[7] = 0$

. For the following system, find the transfer function $H(z) = \frac{Y(z)}{X(z)}$

$$z\left[6y[n] - 5y[n-1] + y[n-2]\right] = z\left[x[n]\right]$$

$$6Y(z) - 5Y(z)z^{-1} + Y(z)z^{-2} = X(z)$$

$$Y(z)(6 - 5z^{-1} + z^{-2}) = X(z)$$

$$H(z) = \frac{1}{6 - 5z^{-1} + z^{-2}} \times \frac{z^{+2}}{z^{+2}}$$

$$H(z) = \frac{z^{+2}}{6z^2 - 5z + 1}$$

. find step response : $x(nT) = u(nT) \rightarrow x(z) = \frac{1}{1 - z^{-1}}$

$$Y(z) = H(z) . x(z) = \frac{1}{6 - 5z^{-1} + z^{-2}} . \frac{1}{1 - z^{-1}}$$

- inverse z – transform :

1. long division   قوة ال z في البسط أكبر من z بالمقام تساوي
2. partial fraction

. ex. $x(z) = \frac{z^2}{(z-1)(z-0.2)} = \frac{\cancel{z^2}}{\cancel{z}(1-z^{-1})\cancel{z}(1-0.2z^{-1})}$

$= \frac{1}{(1-z^{-1})(1-0.2z^{-1})}$

$= \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$

$1 = A(1-0.2z^{-1}) + B(1-z^{-1})$

let $z^{-1} = \frac{1}{0.2} \Rightarrow B = -0.25$,

let $z^{-1} = 1 \rightarrow A = 1.25$

$= \frac{1.25}{1-z^{-1}} + \frac{-0.25}{1-0.2z^{-1}}$; $z[a^{n-1} u[n-1]] = \frac{1}{1-az^{-1}} \cdot z^{-1}$

it must be ①

$x[n] = 1.25 (1)^n u[n] + -0.25 u[n] (0.2)^n$

ex. $x(n) = z^{-1}\left[\frac{2}{3(1-\frac{2}{3}z^{-1})}\right] = z^{-1}\left[\frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}z^{-1}}\right] = \frac{2}{3} \cdot \left(\frac{2}{3}\right)^n u[n]$

$x(z) = \frac{2}{3-2z^{-1}}$

ex. $Y(z) = \left[\frac{z^2}{z^2-1.2z+0.2}\right] z^{-2} = x(z)\, z^{-2}$

evaluate $Y[n] = x((n-2)T)$

$= 1.25 (1)^{n-2} u[n-2] - 0.25 (0.2)^{n-2} u[n-2]$

. $H(z) = Y(z)$ if $x[n] = \delta[n]$

$h[n] = y[n]$ $X(z) = 1$

. To check if the system is stable

$$\sum_{K=-\infty}^{\infty} |h(k)| < \infty$$

ex. $h[n] = \left[4\left(\frac{1}{3}\right)^n - 3\left(\frac{1}{4}\right)^n \right] u[n]$

① $\Sigma |h[n]| = \sum_{0}^{\infty} 4\left(\frac{1}{3}\right)^n \cancel{u[n]} - \sum_{0}^{\infty} 3\left(\frac{1}{3}\right)^n \cancel{u[n]}$

$= 4 \cdot \frac{1}{1-\frac{1}{3}} - 3 \cdot \frac{1}{1-\frac{1}{4}} =$ constant → BIBO (stable)

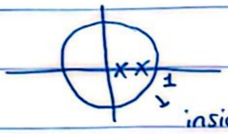
② Method 2: $H(z) = 4\,Z\left[\left(\frac{1}{3}\right)^n u[n]\right] - 3\,Z\left[\left(\frac{1}{4}\right)^n u[n]\right]$

$= 4 \cdot \frac{1}{1-\frac{1}{3}z^{-1}} - 3 \cdot \frac{1}{1-\frac{1}{4}z^{-1}}$

$= \frac{4z}{z-\frac{1}{3}} - \frac{3z}{z-\frac{1}{4}}$

// poles:

⇒ stable



inside the unit circle

Ch.9: Analysis & Design of digital Filters

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$\downarrow$ DFE → difference eq.

$$y[n] + 2y[n-1] - 3y[n-2] + \dots = x[n] + 2x[n-1] + 3x[n-2] + \cdots$$

$\downarrow$

z- transform

$$Y(z) + 2Y(z)z^{-1} + \cdots = X(z) - 2X(z)z^{-1} + 3X(z)z^{-2} + \cdots$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{[1 - 2z^{-1} + 3z^{-2} + \cdots]}{[1 + 2z^{-1} - 3z^{-1} + \cdots]} = \frac{\sum_{i=0}^{r} L_i z^{-i}}{\sum_{j=0}^{m} K_j z^{-j}}$$

$$= \frac{\sum_{i=0}^{r} L_i z^{-i}}{1 + \sum_{j=1}^{m} K_j z^{-j}} \Rightarrow Y(z)\left[1 + \sum_{j=1}^{m} K_j z^{-j}\right] = \sum_{i=0}^{r} L_i z^{-i} \cdot X(z)$$

$$Y(z) = \sum_{0}^{r} L_i z^{-i} \cdot X(z) - Y(z) \sum_{1}^{m} K_j z^{-j}$$

$$\Rightarrow Y(z) = L_0 X(z) \longrightarrow y[n] = L_0 x[n]$$

$$Y(z) = z^{-1} X(z) \longrightarrow y[n] = x[n-1] \leftarrow \text{delay}$$

in F domain /t /n domain

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \rightarrow \text{convolution}$$

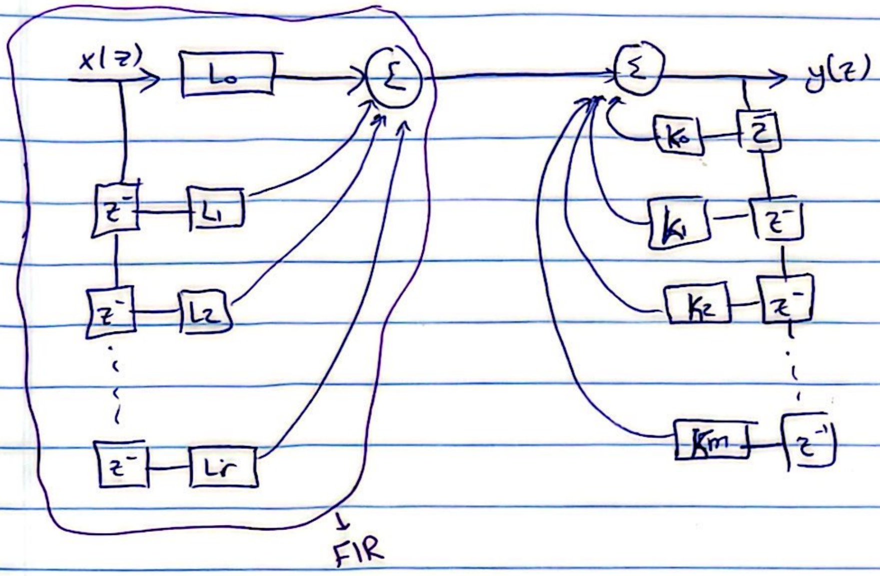
but in z domain

$$X(z) \rightarrow \boxed{L_0} \rightarrow Y(z) \rightarrow X(z) \cdot L_0 = Y(z)$$

// same as in F.T

$$X(z)\ \boxed{z^{-1}} - \boxed{z^{-1}} - Y(z) \longrightarrow Y(z) = z^{-2} X(z) \rightarrow y[n] = x[n-2]$$

Direct Form I



⇒ IIR

. Types of filters:

① FIR (finite impulse response) ⇒ we have no poles

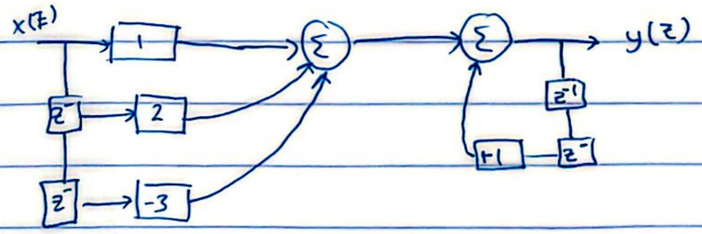
$$H(z) = \sum_{i=0}^{r} L_i z^{-i}$$

② IIR (infinite) ⇒ we have at least one pole

ex. $H(z) = \frac{1+2z^{-1}-3z^{-2}}{1-z^{-2}}$ 1) plot DFI

$$\frac{Y(z)}{X(z)} \Rightarrow Y(z)\cdot(1-z^{-2}) = X(z)\cdot(1+2z^{-1}-3z^{-2})$$

it must be 1



. IIR Filter