4.1 Experiments and Counting Rules

* Experiment is an process that generates well-define outcomes

Sample Space (S) for an experiment is the set of all experimental outcomes

Example: What is the sample space for the following experiments

[a] Toss a coin $S = \{\text{Head}, \text{Tail}\} = \{H, T\}$

[b] Roll a die $S = \{1, 2, 3, 4, 5, 6\}$

[c] Play a game $S = \{\text{win}, \text{lose}, \text{tie}\}$

* Experimental outcomes are also called sample points

Counting Rules

[1] Multiple-step experiment [2] Combinations [3] Permutations

[1] Multiple step experiment:

If an experiment has K steps with

n_1 possible outcomes for step 1,

n_2 = = = step 2,

⋮

n_K = = = step K

Then the total number of experimental outcomes is $n_1 n_2 \cdots n_K$

Example: Consider the experiment of tossing two coins. How many experimental outcomes are possible for this experiment

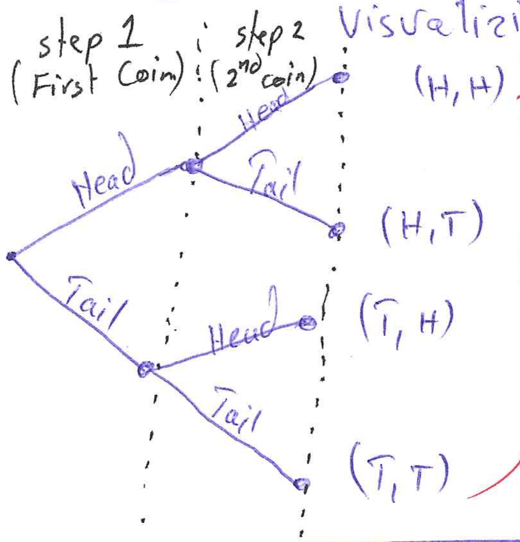
Tossing one coin ($n_1 = 2$) and tossing the other coin ($n_2 = 2$)

The number of experimental outcomes $= (n_1)(n_2) = (2)(2) = 4$.

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

If we toss 5 coins, then the number of experimental outcomes is $(2)(2)(2)(2)(2) = 32$.

* Tree diagram: A graphical representation that helps in visualizing a multiple-step experiment.



Experimental outcom = Sample point

$\boxed{2}$ Combinations (when the experiment involves slecting n objectis from a set of N objectis, $N \geq n$) (The order is not important)

$$C_n^N = \binom{N}{n} = \frac{N!}{n!\,(N-n)!}$$

where

we use combination to find the number of different samples of size n that can be slected.

$N! = N(N-1)(N-2)\cdots(2)(1).$

$n! = n(n-1)(n-2)\cdots(2)(1)$

$0! = 1$

$4! = 4\times3\times2\times1 = 24$ * see an example on the back ?!

$5! = 5\times4\times3\times2\times1 = 120$

Example (Q2 page 150): How many ways can three items be slected from a group of six items? Use the letters A, B, C, D, E, F to identify them, and list each of the different combinations of the three item.

$C_3^6 =$

$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = \frac{6!}{3!\,3!}$$

$$= \frac{6\times5\times4\times\cancel{3\times2\times1}}{3\times2\times1\times\cancel{3\times2\times1}} = 20$$

ABC	ACD	ADF	BCF	CDE
ABD	ACE	AEF	BDE	CDF
ABE	ACF	BCD	BDF	CEF
ABF	ADE	BCE	BEF	DEF

46,1

Example: How many ways 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7, 9 which are even without repeating the digits

$$(2)(5)(4) = 40$$

Example: A hotel surveyed 100 guests with the following data:

	satisfied	unsatisfied
Female	42	2
Male	40	16

If two guests are randomly selected, what is the prob. that both are unsatisfied?

$$\frac{18}{100} \times \frac{17}{99} = 0.03 \quad \text{or} \quad \frac{\binom{18}{2}}{\binom{100}{2}} = 0.03$$

Example In how many ways can the letters of the word (Formula) be rearranged.

$$P^7_7 = \frac{7!}{(\cancel{7}-7)!} = \frac{7!}{0!} = 7! = 5040$$

$\boxed{46,2}$

3 **Permutations:** (when the experiment involves selecting n objects from a set of N objects $N \geq n$, where the order of selection is important.)

$$P_n^N = \frac{N!}{(N-n)!}$$

Example (Q3 page 150) How many permutations of three items can be selected from a group of six?

$$P_3^6 = \frac{6!}{3!} = \frac{6\times5\times4\times3\times2\times1}{3\times2\times1} = 120$$

ABC, ACB, BAC, BCA, CAB, CBA are different outcomes here

~~see the back for one more example:-~~

Assigning Probabilities

- Basic requirements for assigning probabilities:
  1. Each experimental outcome E_i must have $0 \leq P(E_i) \leq 1$
  2. Considering all experimental outcomes, we must have $P(E_1) + P(E_2) + \cdots + P(E_n) = 1$

- Three methods for assigining probabilites:

  1 Classical method: when all the experimental outcomes are equally likely. $\left(\frac{1}{n}\right)$

Example ① Toss a fair coin $\Rightarrow S = \{H, T\}$ $n=2$

$P(H) = P(T) = \frac{1}{2}$ "equally likely" $\frac{1}{2}$

② Roll a die $\Rightarrow S = \{1, 2, 3, 4, 5, 6\}$ $n=6$

$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$ $\frac{1}{6}$

2 Relative frequency method (when data are available to estimate the proportion of the time the experimental outcome will occure if the experiment is repeated a large number of items.

Example:

Number of patients wating	outcome Number of days
0	2
1	5
2	6
3	4
4	3
	$\boxed{20 \text{ days}}$

number of sample points

2 days were 0 patients watinig
5 days wrer one " "
6 days were 2 " "
4 days were 3
3 days were 4 " "

Using the relative frequency method:

the probability of 0 patients wree wating $= \frac{2}{20}$

" " " 1 " " " $= \frac{5}{20}$

" " " 2 " " " $= \frac{6}{20}$

" " " 3 " " " $= \frac{4}{20}$

" " " 4 " " " $= \frac{3}{20}$

3 Subjective Method (when we can not assume that the experimental outcomes are equally likely

- This method express the person's degree of belief (scal 0-1)

Example: Suppose that student A and student B gave an excuse to there teach about their absance.

E_1: the excuse is accepted student A $\Rightarrow P(E_1) = 0.8$
$\Rightarrow P(E_2) = 0.2$

E_2: the excuse is rejected

student B $\Rightarrow P(E_1) = 0.6$ $P(E_2) = 0.4$

~~4.3~~ + $\boxed{4.2}$ Events and their Probability

Event: is a collection of sample points.

Probability of an event: is the sum of the probabilities of the sample points in the event.

Example (Q14 page 154) An experiement has four equally likely outcomes E_1, E_2, E_3, E_4.

[a] what is the prob. that E_2 occures? $\frac{1}{4}$

[b] " " " " " any two of the outcomes occure? يعني i.e. (E_1 or E_3)

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

[c] " " " " " " three " " " " ? يعني (E_1 or E_2 or E_4)

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Example (Q21 page 156)

Assume a person is randomly choosen.

Age	Number (million of people)
≤ 19	80.5
20 – 24	19.0
25 – 34	39.9
35 – 44	45.2
45 – 54	37.7
55 – 64	24.3
≥ 65	35.0
	281.6 million

(a) what is the prob. the person is 20-24 years old?

$$\frac{19}{281.6} = 0.067$$

(b) What is the prob. the person is 20 – 34 years old?

$$\frac{19 + 39.9}{281.6} = \frac{58.9}{281.6} = 0.21$$

(c) what is the prob. the person is 45 years or older?

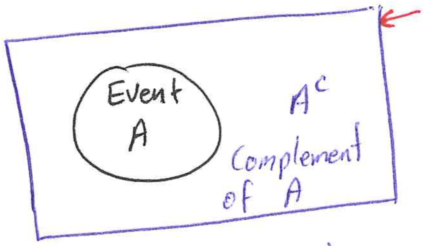
$$\frac{37.7 + 24.3 + 35}{281.6} = \frac{97}{281.6} = 0.34$$

4.3 Some Basic Relationships of Probability

Given an event A. The complement of A, denoted by A^c, consists of all sample points that are not in A.

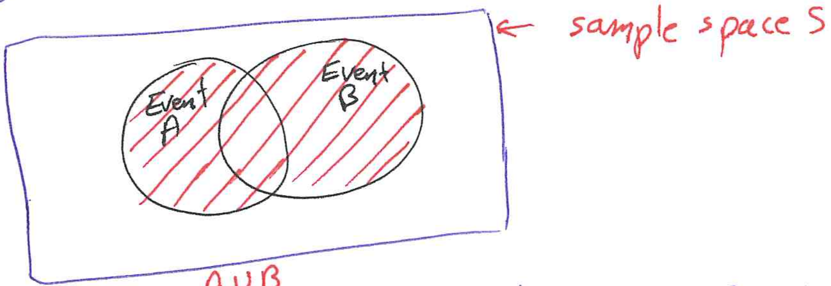
$P(S) = 1$

$P(A) + P(A^c) = 1$



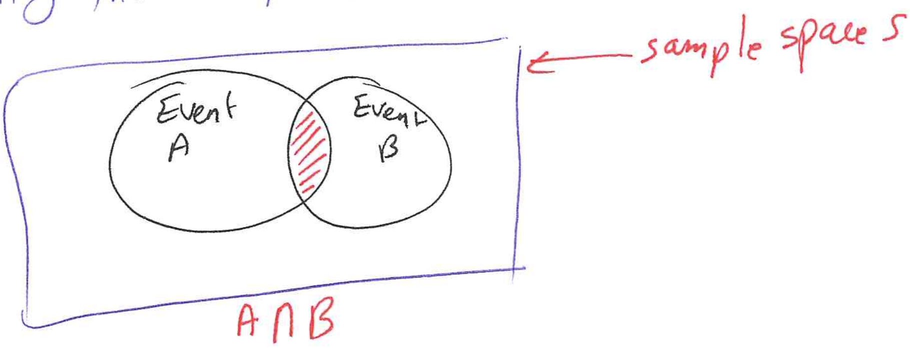
The figure above is called Venn diagram.

- Given the events A and B.
  - The union of A and B is the event containing all sample points belonging to A or B or Both. Denoted by $A \cup B$.



$A \cup B$

- The intersection of A and B, denoted by $A \cap B$, is the event containing the sample points belonging to both A and B.



$A \cap B$

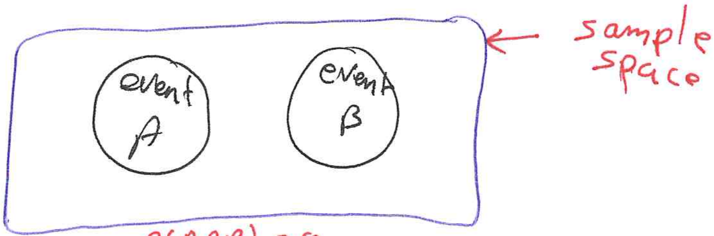
- Additiona law:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

* Two events A and B are said to be mutually exclusive متنافين if the events A and B have no sample points in common ($P(A\cap B)=0$).

* Adition law for mutually exclusive events:

$$P(A\cup B) = P(A) + P(B)$$



$P(A\cap B)=0$

Example: (Q23 page 161) Suppose we have a sample space

$S=\{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ where E_i are the sample points.

Given $P(E_1)=P(E_7)=0.05$, $P(E_2)=P(E_3)=0.20$

$P(E_4)=0.25$, $P(E_5)=0.15$, $P(E_6)=0.10$.

Let $A=\{E_1, E_4, E_6\}$, $B=\{E_2, E_4, E_7\}$, $C=\{E_2, E_3, E_5, E_7\}$.

[a] Find $P(A)$, $P(B)$, $P(C)$

$P(A)=P(E_1)+P(E_4)+P(E_6) = 0.05+0.25+0.1 = 0.4$

$P(B)=P(E_2)+P(E_4)+P(E_7) = 0.2+0.25+0.05 = 0.50$

$P(C)=P(E_2)+P(E_3)+P(E_5)+P(E_7) = 0.20+0.20+0.15+0.05 = 0.60$

[b] Find $A\cup B$ and $P(A\cup B)$

$A\cup B=\{E_1, E_2, E_4, E_6, E_7\}$

$P(A\cup B) = P(E_1)+P(E_2)+P(E_4)+P(E_6)+P(E_7) = 0.05+0.20+0.25+0.10+0.05 = 0.65$

[c] Find $A\cap B$ and $P(A\cap B)$

$A\cap B=\{E_4\}$, $P(A\cap B)=P(E_4)=0.25$

[d] Are the events A and C mutually exclusive?

Yes, they are mutually exclusive because $P(A\cap C)=0$

[e] Find B^c and $P(B^c)$

$B^c=\{E_1, E_3, E_5, E_6\} \Rightarrow P(B^c)=1-P(B)=1-0.5=0.5$

4.4 Conditional Probabilities

Let A and B be two events:

- The conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- The conditional probability of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- The events A and B are independent if

$$P(A \cap B) = P(A)\ P(B)$$

otherwise, they are dependent.

- If A and B are independent, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(B|A) = P(B)$$

- The events A and B are mutually exclusive if $P(A \cap B) = 0$

Example: suppose that we have two events A and B, with $P(A) = 0.2$, $P(B) = 0.4$, $P(A \cap B) = 0.08$

[a] Find $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.08}{0.4} = 0.2$

[b] Find $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.08}{0.2} = 0.4$

[c] Are A and B independent? why or why not?

Yes Because $P(A|B) = P(A)$
$P(B|A) = P(B)$

Example (Q32 page 168) Sample data representative of the national health insurance coverage are shown here:

		Health Insurance Yes	Health Insurance No	Total
Age	18-34	750	170	920
Age	35 and older	950	130	1080
Total		1700	300	$\boxed{2000}$

ⓐ Develop a joint probability table for these data

joint probabilities

		Health insurance Yes	Health insurance No	Total
Age	18-34	$\frac{750}{2000} = 0.375$	0.085	0.46
Age	35 and older	0.475	0.065	0.54
Total		0.850	0.150	1.00

Marginal probabilities

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(b) what do the marginal probabilities tell you about the age of the US population?

46 % of the population are within age 18-34
54 % of the population are 35 and older.

(c) what is the probability that a randomly selected individual does not have health insurance?

0.15

(d) If the individual is between the ages 18-34, what is the probability that the individual does not have health insurance?

$$P(No \mid 18\text{-}34) = \frac{0.085}{0.46} = 0.1848$$

(e) If the individual age is 35 or older, what is the probability that the individual does not have health insurance.

$$P(No \mid 35 \text{ or older}) = \frac{0.065}{0.54} = 0.1204$$

(f) If the individual does not have health insurance, what is the prob. that the individual is in 18-34 age?

$$P(18\text{-}34 \mid No) = \frac{0.085}{0.150} = 0.567$$

(g) what does the prob. information tell you about the health insurance in United state?

From d and e, we see higher probability of No for 18-34