Exercises:

$0! = 1$
$1! = 1$
$2! = 2$
$3! = 6$

Q89: Find the mean and variance if they exist:

a. $F(x) = \begin{cases} \frac{3! = 6}{x!\,(3-x)!}\left(\frac{1}{2}\right)^3 & , x = 0,1,2,3 \\ 0 & , \text{elsewhere} \end{cases}$ discrete.

$\rightarrow E(x) = \sum_x x f(x) = \sum_0^3 x\left(\frac{3!}{x!(3-x)!}\left(\frac{1}{2}\right)^3\right) = 0(f(x)) + 1\left(\frac{6}{2}\right)\left(\frac{1}{8}\right) + 2\left(\frac{6}{2}\right)\left(\frac{1}{8}\right) + 3\left(\frac{6}{6}\right)\left(\frac{1}{8}\right)$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 1.5$

$\rightarrow E(x) = 1.5 = \mu$.

$\rightarrow \delta^2 = E(x^2) - \mu^2$

• $E(x^2) = \sum_0^3 x^2 f(x) = 0 f(x) + 1\left(\frac{3}{8}\right) + 4\left(\frac{3}{8}\right) + 9\left(\frac{1}{8}\right) = 3$

• $Var(x) = E(x^2) - \mu^2 = 3 - (1.5)^2 = 3 - 2.25 = 0.75$.

b. $F(x) = \begin{cases} 6x(1-x) & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$ continuous.

$\rightarrow E(x) = 6\int_0^1 x(x - x^2)\,dx = 6\int_0^1 x^2 - x^3\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1\right) = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{6}{12} = 0.5$ μ

$\rightarrow E(x^2) = 6\int_0^1 x^2(x - x^2)\,dx = 6\int_0^1 x^3 - x^4\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{6}{20} = 0.3$

$\rightarrow Var(x) = E(x^2) - \mu^2 = 0.3 - 0.25 = 0.05$ Var(x).

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c. $f(x) = \begin{cases} \frac{2}{x^3} , & 1 < x < \infty \\ 0 , & \text{elsewhere} . \end{cases}$ continuous.

$\rightarrow E(x) = \int_1^\infty x\left(\frac{2}{x^3}\right)dx = \int_1^\infty \frac{2}{x^2}\,dx = \lim_{b\to\infty}\left(-\frac{2}{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{2}{b}+2\right) = 0+2 = 2.$

$\rightarrow E(x^2) = \int_1^\infty x^2\left(\frac{2}{x^3}\right)dx = \int_1^\infty \frac{2}{x}\,dx = 2\ln x\Big|_1^\infty = \lim_{b\to\infty}\left(2\ln x\Big|_1^b\right) = \lim_{b\to\infty}(2\ln b - 2\ln 1) = \infty$

So $Var(x)$ does not exists since $E(x^2)$ diverge.

Q9: $f(x) = \begin{cases} \left(\frac{1}{2}\right)^x , & x = 1,2,3,\dots \\ 0 , & \text{elsewhere} . \end{cases}$ be the P.d.f of the r.v X. Find the m.g.f and mean and variance of X.

discrete

$\rightarrow E(e^{tx}) = \sum_1^\infty e^{tx}\left(\frac{1}{2}\right)^x = \sum_1^\infty \left(\frac{e^t}{2}\right)^x = \underbrace{\sum_{x=0}^\infty \left(\frac{e^t}{2}\right)^x - 1}_{}$ geometric

$\sum_0^\infty \left(\frac{e^t}{2}\right)^x - \sum_0^0 1 = \frac{1}{1-\frac{e^t}{2}} - 1 = \frac{2}{2-e^t} - 1 = \frac{e^t}{2-e^t}$

$M(t) = \frac{e^t}{2-e^t}$.

$\rightarrow \mu = \bar{M}(0) = \frac{2e^t}{(2-e^t)^2}\Big|_{t=0} = \frac{2(1)}{(2-1)^2} = 2.$

$\rightarrow var(x) = E(x^2) - \mu^2$

$\bullet\ E(x^2) = \bar{\bar{M}}(0) = \frac{2e^t(2-e^t)(2+e^t)}{(2-e^t)^4}\Big|_{t=0} = \frac{(2)(1)(3)}{1} = \underline{\underline{6}}$

$\rightarrow var(x) = 6-(2)^2 = \underline{\underline{2}}$

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Q92: If the variance of the Random variable X exists show that $E(X^2) \geq (E(X))^2$.

$Var(X) = E\left((X-\mu)^2 \right)$, $\mu = E(X)$

$\rightarrow (X-\mu)^2 \geq 0$

$\rightarrow X^2 - 2\mu X + \mu^2 \geq 0$

$\rightarrow E(X^2 - 2\mu X + \mu^2) \geq 0$

$\rightarrow E(X^2) - 2\mu E(X) + (E(X))^2 \geq 0$

$\rightarrow E(X^2) - 2\mu\mu + \mu^2 \geq 0$

$\rightarrow E(X^2) \underbrace{- 2\mu^2 + \mu^2}_{} \geq 0$

$\rightarrow E(X^2) \geq (E(X))^2$ $\blacksquare$

Q95: Show that the m.g.f of the Random variable X having the p.d.f

$$f(x) = \begin{cases} \frac{1}{3} & , -1 < x < 2 \\ 0 & , \text{elsewhere} \end{cases} \quad \text{is} \quad M(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$$

$\rightarrow E(e^{tX}) = \int_{-1}^{2} e^{tx} \frac{1}{3} dx = \frac{1}{3} \int_{-1}^{2} e^{tx} dx$: 2-cases.

case 1 : $t = 0$: $\frac{1}{3} \int_{-1}^{2} \cancel{e^0}^{1} dx = \frac{1}{3} \left[x \Big|_{-1}^{2} \right] = \frac{1}{3} \left[2 - -1 \right] = 1$

case 2 : $t \neq 0$: $\frac{1}{3} \int_{-1}^{2} e^{tx} dx = \frac{1}{3} \left[\frac{e^{tx}}{t} \Big|_{-1}^{2} \right] = \frac{1}{3} \left[\frac{e^{2t}}{t} - \frac{e^{-t}}{t} \right] = \frac{e^{2t} - e^{-t}}{3t}$

$$\text{So } M(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & , t \neq 0 \\ 1 & , t = 0 \end{cases}$$ $\blacksquare$

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Q96: let X be a Random variable s.t $E\left((X-b)^2\right)$ exists for all real b. Show that $E\left((X-b)^2\right)$ is a minimum when $b = E(X)$.

Q105: Find the mean and the variance of the distribution that has the c.d.f

$$F(x) = \begin{cases} 0 & , \quad x<0 \\ \frac{x}{8} & , \quad 0 \le x < 2 \\ \frac{x^2}{16} & , \quad 2 \le x < 4 \\ 1 & , \quad 4 \le x . \end{cases}$$

First: Find p.d.f .

$$f(x) = \bar{F}(x) = \begin{cases} 0 & , \quad x<0 \\ \frac{1}{8} & , \quad 0 \le x < 2 \\ \frac{1}{8}x & , \quad 2 \le x < 4 \\ 0 & , \quad 4 \le x \end{cases} \Rightarrow f(x) = \begin{cases} \frac{1}{8} & , \quad 0 \le x < 2 \\ \frac{1}{8}x & , \quad 2 \le x < 4 \\ 0 & , \quad \text{elsewhere} . \end{cases}$$

Now Find $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{2} x \frac{1}{8}\,dx + \int_{2}^{4} x \frac{1}{8}x\,dx + \cancel{\int_{4}^{\infty} x\,(0)\,dx}$

$$= \frac{1}{4} + \frac{7}{3}$$

$$= \frac{31}{12} \rightarrow$$

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continue : $Var(x) = E(x^2) - \mu^2$

$$\Rightarrow E(x^2) = \int_0^2 x^2 \frac{1}{8}\, dx + \int_2^4 x^2\left(\frac{1}{8}x\right) dx + \int_4^\infty x^2(0)\, dx$$

$$= \frac{1}{3}{\scriptstyle\frac{\times 8}{\times 8}} + \frac{60}{8}{\scriptstyle\frac{\times 3}{\times 3}} = \frac{188}{24}$$

$$\Rightarrow Var(x) = \frac{188}{24} - \left(\frac{31}{12}\right)^2 = \frac{6\times 188}{6\times 24} - \frac{961}{144} = \frac{167}{144}.$$

Q III :

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