0.4 Radicals and Rational Exponents

$\sqrt{4} = \sqrt[2]{2\times 2} = 2$ square root of 4 is 2

$\sqrt[3]{8} = \sqrt[3]{2\times 2\times 2} = 2$ cube root of 8 is 2

$\sqrt[3]{-8} = \sqrt[3]{(-2)(-2)(-2)} = -2$

$\sqrt[3]{-27} = \sqrt[3]{(-3)(-3)(-3)} = -3$

The expression $\sqrt[n]{a}$ is called radical (principal n^{th} root)

$\sqrt{\ }$ is called radical sign

n is called index

a is called radicand

- when no index is indicated $\Rightarrow n=2 \Rightarrow$ radical is square root

  Exp $\sqrt{9} = \sqrt[2]{3\times 3} = 3 \Leftrightarrow 3^2 = 9$

- If n is odd, then the radical $\sqrt[n]{a}$ is a real number

  Exp $\sqrt[3]{8} = \sqrt[3]{2\times 2\times 2} = 2 \Leftrightarrow 2^3 = 8$

  $\sqrt[3]{-27} = \sqrt[3]{(-3)(-3)(-3)} = -3 \Leftrightarrow (-3)^3 = -27$

- If n is even
  - and $a<0$ then $\sqrt[n]{a}$ is not real number

    Exp: $\sqrt{-4} = \sqrt[2]{-4}$ or $\sqrt[4]{-16}$
  - and $a>0$ then $\sqrt[n]{a}$ is positive

    Exp $\sqrt{81} = 9$, $\sqrt[4]{16} = 2$

- If $a=0$ then $\sqrt[n]{a} = 0$ for all n Exp $\sqrt[3]{0} = 0 \Leftrightarrow 0^3 = 0$

  $\sqrt[4]{0} = 0 \Leftrightarrow 0^4 = 0$

Exponent $\frac{1}{n}$: For any positive integer n $\Rightarrow$

$$a^{\frac{1}{n}} = \sqrt[n]{a^1} = \sqrt[n]{a} \qquad \text{" Assuming } \sqrt[n]{a} \text{ exists "}$$

Hence, $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{n}{n}} = a^1 = a$

Rational Exponent $\frac{m}{n}$: For any positive integer n
For any integer m

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = \left(a^m\right)^{\frac{1}{n}}$$

where $a \neq 0$ when $m \leq 0$
$a \geq 0$ when n is even

Exp write the following in radical form and simplify

1. $16^{\frac{3}{4}} = \sqrt[4]{16^3} \Rightarrow 16^{\frac{3}{4}} = (2 \cdot 2 \cdot 2 \cdot 2)^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^{\frac{3}{4} \cdot 4} = 2^3 = 8$

   or $(16)^{\frac{3}{4}} = \sqrt[4]{16^3} = \left(\sqrt[4]{16}\right)^3 = \left(\sqrt[4]{2.2.2.2}\right)^3 = 2^3 = 8$

2. $z^{-\frac{3}{4}} = \dfrac{1}{z^{\frac{3}{4}}} = \dfrac{1}{\sqrt[4]{z^3}}$

3. $(5m)^{-\frac{3}{2}} = \dfrac{1}{(5m)^{\frac{3}{2}}} = \dfrac{1}{\sqrt[2]{(5m)^3}} = \dfrac{1}{\sqrt{5^3 m^3}} = \dfrac{1}{\sqrt{125 m^3}}$

Exp write the following without radical signs

1. $\sqrt{x^5} = x^{\frac{5}{2}}$

2. $\dfrac{1}{\sqrt[3]{y^2}} = \dfrac{1}{y^{\frac{2}{3}}} = y^{-\frac{2}{3}}$

3. $\sqrt[5]{(ab)^5} = (ab)^{5/5} = ab$

Exp (Operation with Fractional Exponents)

Simplify the following expressions

1) $2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} = 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{3}{6}+\frac{2}{6}} = 2^{\frac{5}{6}}$

2) $\dfrac{3^{\frac{3}{4}}}{3^{\frac{1}{3}}} = 3^{\frac{3}{4}-\frac{1}{3}} = 3^{\frac{9}{12}-\frac{4}{12}} = 3^{\frac{5}{12}}$

3) $\left(2^{3} \cdot 4\right)^{\frac{2}{3}} = \left(2^{3}\right)^{\frac{2}{3}} 4^{\frac{2}{3}} = 2^{3\cdot\frac{2}{3}} 4^{\frac{2}{3}} = 2^{2} 4^{\frac{2}{3}} = 4\sqrt[3]{4^{2}} = 4\sqrt[3]{16}$

4) $\left(16^{\frac{3}{2}}\right)^{\frac{1}{2}} = 16^{\frac{3}{2}\cdot\frac{1}{2}} = 16^{\frac{3}{4}} = (2.2.2.2)^{\frac{3}{4}} = \left(2^{4}\right)^{\frac{3}{4}} = 2^{4\cdot\frac{3}{4}} = 2^{3} = 8$

5) $3^{-\frac{1}{2}} \cdot 3^{-\frac{3}{2}} = 3^{-\frac{1}{2}-\frac{3}{2}} = 3^{-\left(\frac{1}{2}+\frac{3}{2}\right)} = 3^{-2} = \dfrac{1}{3^{2}} = \dfrac{1}{9}$

6) $\sqrt[5]{-1} = (-1)^{\frac{1}{5}} = -1$ since $(-1)(-1)(-1)(-1)(-1) = -1$

7) $\sqrt[6]{-1} = (-1)^{\frac{1}{6}}$ is undefined

8) $-8^{-\frac{2}{3}} = -\dfrac{1}{8^{\frac{2}{3}}} = -\dfrac{1}{\left(2^{3}\right)^{\frac{2}{3}}} = -\dfrac{1}{2^{3\cdot\frac{2}{3}}} = -\dfrac{1}{2^{2}} = -\dfrac{1}{4}$

Operations with Radicals

Given $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are reals ($a \geq 0$ and $b \geq 0$ when n is even)

- $\sqrt[n]{a^{n}} = \left(\sqrt[n]{a}\right)^{n} = a^{\frac{n}{n}} = a$ Exp $\sqrt[7]{5^{7}} = 5$

- $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ Exp $\sqrt[3]{2}\,\sqrt[3]{4} = \sqrt[3]{(2)(4)} = \sqrt[3]{8} = \sqrt[3]{2^{3}} = 2$

- $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$, $b \neq 0$ Exp $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt[2]{\dfrac{18}{2}} = \sqrt[2]{9} = 3$

Note that $\sqrt{(-2)^{2}} = \sqrt{4} = 2$ so if $a < 0$ then $\sqrt{a^{2}} = -a > 0$

This means $\sqrt{a^{2}} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Exp (Simplifying Radicals) - (Multiplying Radicals) - (Division Radicals) $\boxed{20}$

Simplify the following radicals

1) $\sqrt{48x^5y^6} = \sqrt[2]{16\cdot 3\cdot x^2\cdot x^2\cdot x\cdot y^3\cdot y^3} = 4\sqrt{3}\,x^2y^3\sqrt{x}$
$= 4\sqrt{3x}\,x^2y^3$

2) $\sqrt[3]{72a^3b^4} = \sqrt[3]{8\cdot 9\cdot a^3\cdot b^4} = \sqrt[3]{8}\,\sqrt[3]{a^3}\,\sqrt[3]{b^3}\,\sqrt[3]{9b} = 2ab\sqrt[3]{9b}$

3) $\sqrt{64x^4} = \sqrt{64}\sqrt{x^4} = 8x^2$

4) $\sqrt[3]{16x^2y}\cdot\sqrt[3]{3x^2y} = \sqrt[3]{16\cdot 3\cdot x^2\cdot x^2\cdot y\cdot y} = \sqrt[3]{8\cdot 2\cdot 3\cdot x^3\cdot x\cdot y^2}$
$= \sqrt[3]{8}\,\sqrt[3]{x^3}\,\sqrt[3]{6xy^2} = 2x\sqrt[3]{6xy^2}$

5) $\sqrt[3]{2xy}\cdot\sqrt[3]{4x^2y} = \sqrt[3]{2xy\cdot 4x^2y} = \sqrt[3]{x^3}\,\sqrt[3]{8}\,\sqrt[3]{y^2} = 2x\sqrt[3]{y^2}$

6) $\sqrt{8xy^3z}\,\sqrt{4x^2y^3z^2} = \sqrt[2]{32x^3y^6z^3} = \sqrt[2]{16x^2y^6z^2}\,\sqrt[2]{2xy}$
$= 4|x|\,y^3\,|z|\,\sqrt{2xy}$

7) $\dfrac{\sqrt[3]{32}}{\sqrt[3]{4}} = \sqrt[3]{\dfrac{32}{4}} = \sqrt[3]{8} = 2$

8) $\dfrac{\sqrt{16a^3x}}{\sqrt{2ax}} = \sqrt{\dfrac{16a^3x}{2ax}} = \sqrt{8a^2} = \sqrt{4a^2}\sqrt{2} = 2|a|\sqrt{2}$

Exp Express the following with no radicals in the denominator

1) $\dfrac{2}{\sqrt{x}} = \dfrac{2}{\sqrt{x}}\,\dfrac{\sqrt{x}}{\sqrt{x}} = \dfrac{2\sqrt{x}}{x}$

2) $\dfrac{2x}{\sqrt{18xy}} = \dfrac{2x}{\sqrt{18xy}}\cdot\dfrac{\sqrt{2xy}}{\sqrt{2xy}} = \dfrac{2x\sqrt{2xy}}{\sqrt{36x^2y^2}} = \dfrac{2x\sqrt{2xy}}{6xy} = \dfrac{\sqrt{2xy}}{3y}$ $x>0$ $y>0$