Exercises:

Q26: if the p.d.f of X is $f(x)=\begin{cases} 2xe^{-x^2}, & 0<x<\infty \\ 0, & \text{elsewhere} \end{cases}$

determine the p.d.f of $y=x^2$. 1-1

$y=x^2 \rightarrow x=\sqrt{y}$ $\qquad w(y)=\sqrt{y}=y^{\frac{1}{2}}$

$\bar{U}: A \rightarrow B$ $\qquad \bar{w}(y)=\frac{1}{2}y^{-\frac{1}{2}}=\frac{1}{2\sqrt{y}}$

$A: \{x: 0<x<\infty\}$

$B: \{y: 0<y<\infty\}$

$$\Rightarrow g(y)=\begin{cases} f(\sqrt{y})\,|\bar{w}(y)|, & 0<y<\infty \\ 0, & \text{else} \end{cases}$$

$$=\begin{cases} 2\sqrt{y}\,e^{-(\sqrt{y})^2}\left|\frac{1}{2\sqrt{y}}\right|, & 0<y<\infty \\ 0, & \text{else} \end{cases}$$

$$=\begin{cases} e^{-y}, & 0<y<\infty \\ 0, & \text{else} \end{cases}$$

Q30: let X_1 and X_2 denote a Random sample of size 2 from a distribution that is $N \sim (\mu, \delta^2)$, let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint p.d.f of Y_1 and Y_2 and show that these random variables are indp.

$$X_1 \sim N(\mu, \delta^2) \rightarrow f(x_1) = \frac{1}{\sqrt{2\pi}\,\delta} e^{\frac{-(x_1-\mu)^2}{2\delta^2}}, \quad x_1 \in \mathbb{R}$$

$$X_2 \sim N(\mu, \delta^2) \rightarrow f(x_2) = \frac{1}{\sqrt{2\pi}\,\delta} e^{\frac{-(x_2-\mu)^2}{2\delta^2}}, \quad x_2 \in \mathbb{R}$$

$$f(x_1, x_2) = \frac{1}{2\pi\delta^2} e^{-\frac{(x_1-\mu)^2-(x_2-\mu)^2}{2\delta^2}}, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R}$$

(2-variables, 1-1, condi.)

$$Y_1 = X_1 + X_2 \rightarrow X_1 = \frac{y_1 + y_2}{2}$$

$$Y_2 = X_1 - X_2 \rightarrow X_2 = \frac{y_1 - y_2}{2}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\Rightarrow g(y_1, y_2) = f\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) |J|$$

$\rightarrow$

$$g(y) = \frac{1}{2\pi \sigma^2} e^{\frac{-\left(\frac{y_1+y_2}{2}-\mu\right)^2 - \left(\frac{y_1-y_2}{2}-\mu\right)^2}{2\sigma^2}}, \quad y_1, y_2 \in \mathbb{R}$$

$$= \frac{1}{2\pi\sigma^2} \exp\left[-\left(\frac{(y_1+y_2)^2}{4} - \mu(y_1+y_2) + \mu^2 + \frac{(y_1-y_2)^2}{4} - \mu(y_1-y_2) + \mu^2\right)\right.$$

$$= \frac{1}{2\pi\sigma^2} \exp\left[-\left(\frac{y_1^2}{4} + \cancel{\frac{y_1 y_2}{2}} + \frac{y_2^2}{4} - \cancel{\mu y_1} + \cancel{\mu y_2} + \mu^2 + \frac{y_1^2}{4} - \cancel{\frac{y_1 y_2}{2}} + \frac{y_2^2}{4} - \cancel{\mu y_1} + \cancel{\mu y_2} + \mu^2\right)\right.$$

$$= \frac{1}{2\pi\sigma^2} \exp\left(\frac{-y_1^2}{4} - \frac{y_2^2}{4} - \mu^2 - \frac{y_1^2}{4} - \frac{y_2^2}{4} - \mu^2\right)$$

$$= \frac{1}{2\pi\sigma^2} \exp\left(\frac{-y_1^2}{2} - \frac{y_2^2}{2} - 2\mu^2\right)$$

to show its indep:

$\exists\ h_1(y_1) = \frac{1}{2\pi\sigma^2} e^{-\frac{y_1^2}{2} + 2\mu y_1}, \quad y_1 \in \mathbb{R}$

and $h_2(y_2) = e^{-\frac{y_2^2}{2} - 2\mu^2}, \quad y_2 \in \mathbb{R}$

s.t $h_1(y_1) \cdot h_2(y_2) = g(y_1, y_2)$

so Y_1 and Y_2 indep.

Q33: let X_1 and X_2 have the joint p.d.f ~~$h(x_1, x_2)$~~.

$$h(x_1, x_2) = \begin{cases} 2e^{-x_1 - x_2} & , 0 < x_1 < x_2 < \infty \\ 0 & , \text{else} \end{cases}$$

Find the joint p.d.f of $y_1 = 2X_1$ and $y_2 = X_2 - X_1$ and argue that Y_1 and Y_2 independent.

$$y_1 = 2X_1 \rightarrow X_1 = \frac{y_1}{2}$$

$$y_2 = X_2 - X_1 \rightarrow X_2 = y_2 + \frac{1}{2} y_1$$

$$A = \{ 0 < x_1 < x_2 < \infty \}$$

$$B = \{ y_1 > 0 , y_2 > 0 \}.$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} - 0 = \frac{1}{2}$$

$$\Rightarrow g(y_1, y_2) = \begin{cases} h\left(\frac{1}{2}y_1 , y_2 + \frac{1}{2}y_1\right) \left|+\frac{1}{2}\right| & , y_1, y_2 \in B \\ 0 & , \text{else} \end{cases}$$

$$= \begin{cases} \cancel{2} e^{\left(-\frac{1}{2}y_1 - y_2 - \frac{1}{2}y_1\right)} \cdot \frac{1}{\cancel{2}} & , y_1, y_2 \in B \\ 0 & , \text{else} \end{cases}$$

$$g(y_1, y_2) = \begin{cases} e^{-y_1 - y_2} & , \quad y_1, y_2 \in B \\ 0 & , \quad \text{else} . \end{cases}$$

to show its indep.

$\exists \; h_1(y_1) = e^{-y_1}$ and $h_2(y_2) = e^{-y_2}$

s.t $h_1(y_1) \cdot h_2(y_2) = e^{-y_1} e^{-y_2}$

$$= e^{-y_1 - y_2}$$

$$= g(y_1, y_2)$$

so its indep.