ENCS3340 - Artificial Intelligence

Constraint Satisfaction Problems

CSP Definitions

- Satisfies additional structural properties of the problem
  - may depend on the representation of the problem
- The problem is defined through a set of domain variables
  - variables can have possible values specified by the problem
  - constraints describe allowable combinations of values for a subset of the variables
- State in a CSP
  - defined by an assignment of values to some or all variables
- Consistent assignment
  - does not violate any constraints
  - also called legal assignment
- Complete assignment
  - every variable is mentioned
- Solution to a CSP
  - complete assignment that satisfies all constraints
  - solutions may be ranked according to an objective function

Example1: 3-SAT

Variables:

x_1, x_2, x_3, x_4, x_5

Domains:

{True, False}

Constraints: , = and

$(x_1 \vee x_2 \vee x_4)$,
$(x_2 \vee x_4 \vee \neg x_5)$,
$(x_3 \vee \neg x_4 \vee \neg x_5)$

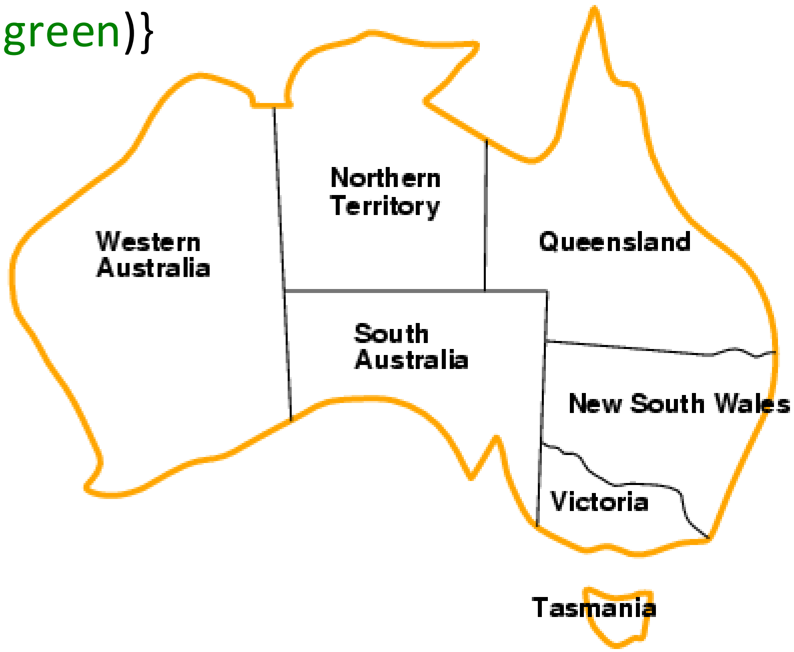
$$(x_1 \vee x_2 \vee x_4) \wedge$$
$$(x_2 \vee x_4 \vee \neg x_5) \wedge$$
$$(x_3 \vee \neg x_4 \vee \neg x_5)$$

Suggest a solution!

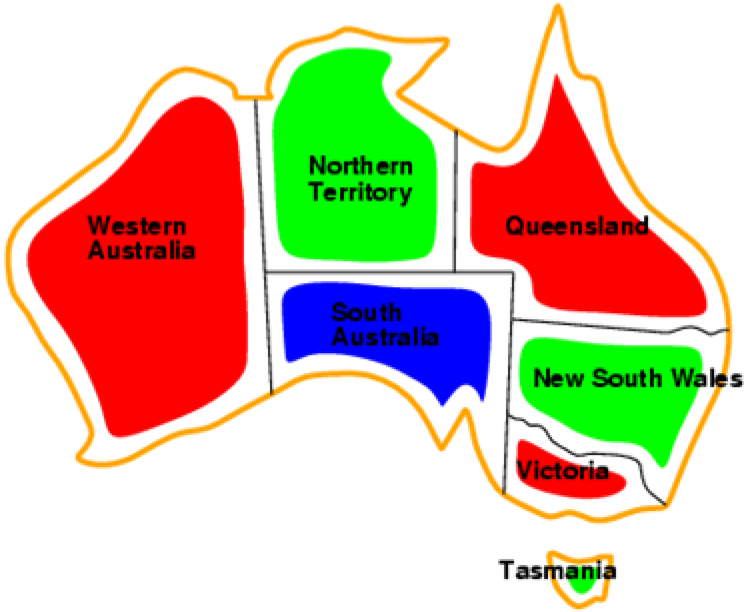
Example2: Map-Coloring Problem

- Variables *WA, NT, Q, NSW, V, SA, T*
- Domain D_i = {red, green, blue}
- Constraints: adjacent regions must have different colors
  - e.g., Color(WA) ≠ Color(NT) or in short WA ≠ NT
  - (WA, NT) ϵ {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)} OR
  - (WA, NT) ϵ/{(red, red), (blue, blue), (green, green)}

  - Graph Coloring Problem (more general)!



Example: Map-Coloring



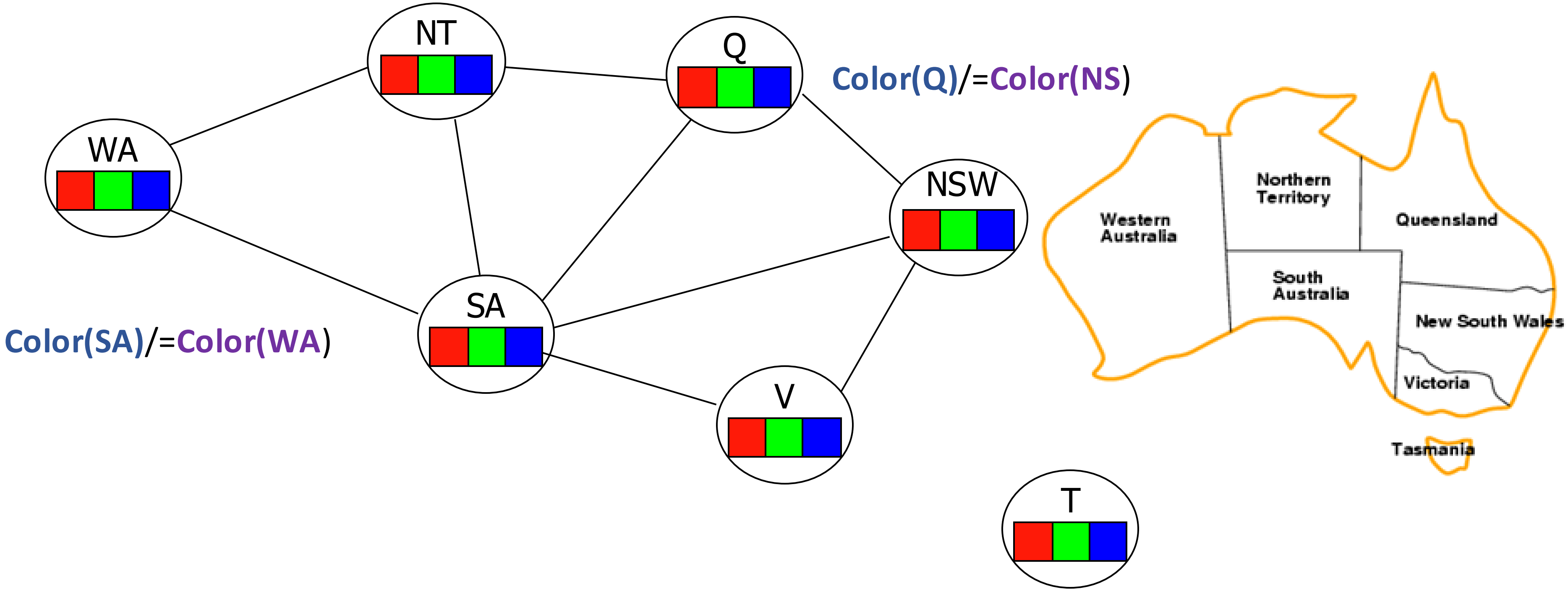
- Solutions are complete and consistent assignments, e.g.,

  WA = red, NT = **green**, Q = red, NSW = **green**, V = red, SA = **blue**, T = **green**

- Complete: all are assigned, consistent: obeys the constraints.
- A state may be incomplete e.g., just WA=red

Constraint graph

- It is helpful to visualize a CSP as a constraint graph
  - Binary CSP: each constraint relates two variables [here states]
  - Constraint graph: nodes are variables, arcs are constraints (e.g. color different)



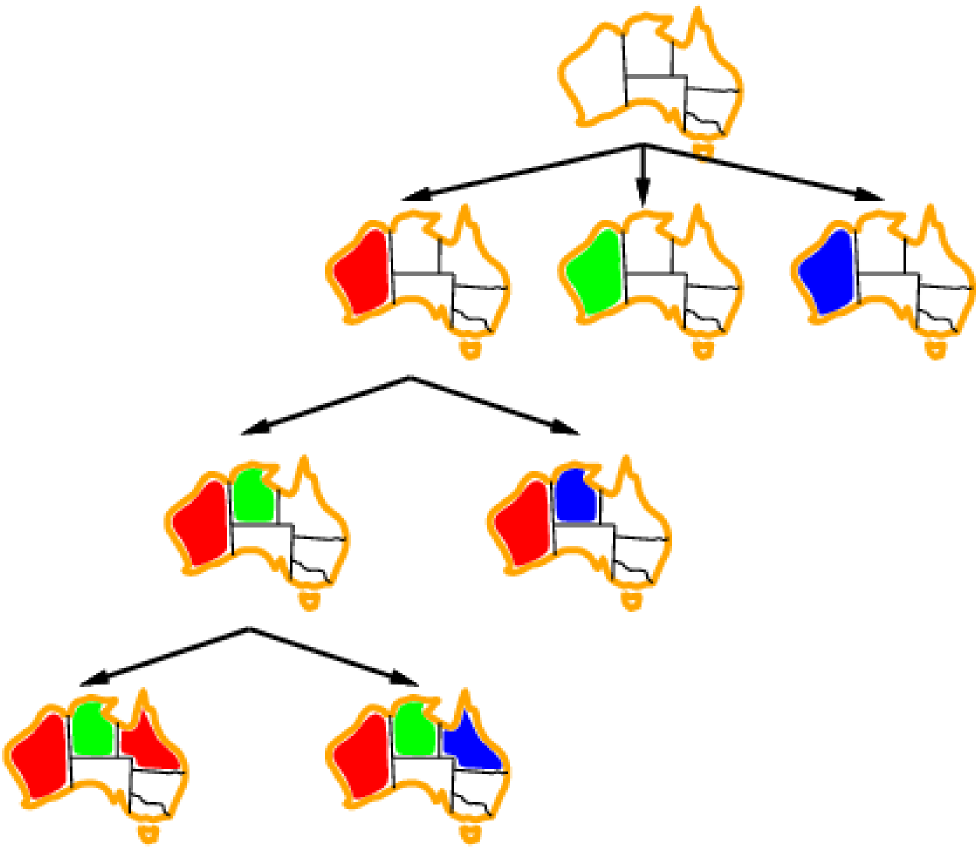
Varieties of CSPs

- Discrete variables
  - finite domains:
    - n variables, domain size d , $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., StartJob1 + 5 ≤ StartJob3
- Continuous variables
  - e.g., Time: start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

CSP as Incremental Search Problem

- initial state
  - all (or at least some) variables UNassigned
- successor function
  - assign a value to an UNassigned variable
  - must not conflict with previously assigned variables
- goal test
  - all variables have values assigned
  - no conflicts exist (in the assignments)
- path cost
  - e.g. constant for each step [some colors may be expensive]
  - may be problem-specific

Example



CSPs and Search

- In principle, any search algorithm can be used to solve a CSP, but:
  - awful branching factor
    - n*d for n variables with d values at the top level, (n-1)*d at the next level, etc.
  - not very efficient, since they neglect some CSP properties
    - commutativity: the order in which values are assigned to variables is irrelevant, since the outcome is the same

Backtracking Search for CSPs

A variation of depth-first search that is often used for CSPs

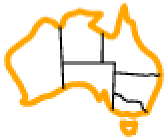
- values are chosen for one variable at a time
- if no legal values are left, the algorithm backs up and changes a previous assignment
- very easy to implement
  - initial state, successor function, goal test are standardized
- not very efficient
  - can be improved by trying to select more suitable unassigned variables first

Backtracking search Algorithm

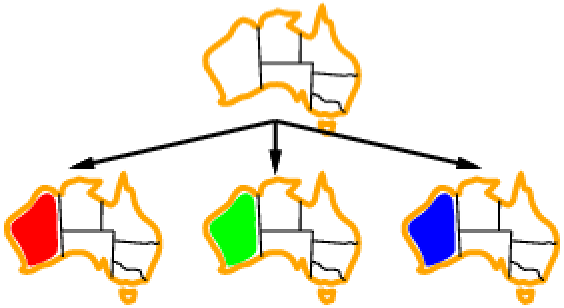
```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
    return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to Constraints[csp] then
            add { var = value } to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failue then return result
            remove { var = value } from assignment
    return failure
```

Backtracking example



Backtracking example



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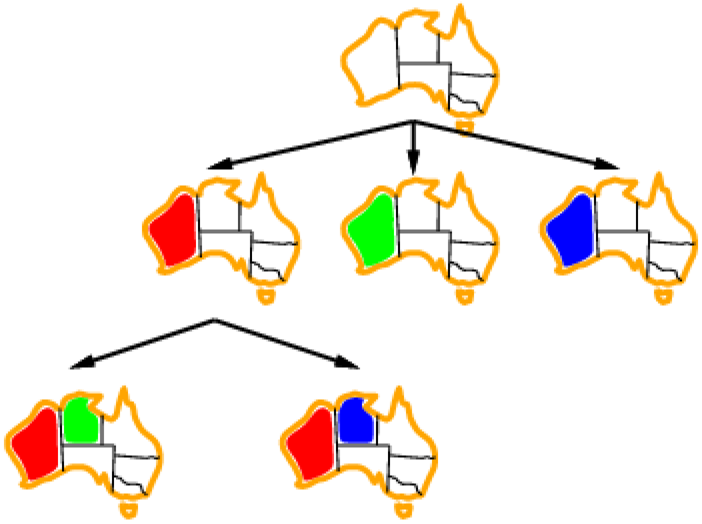
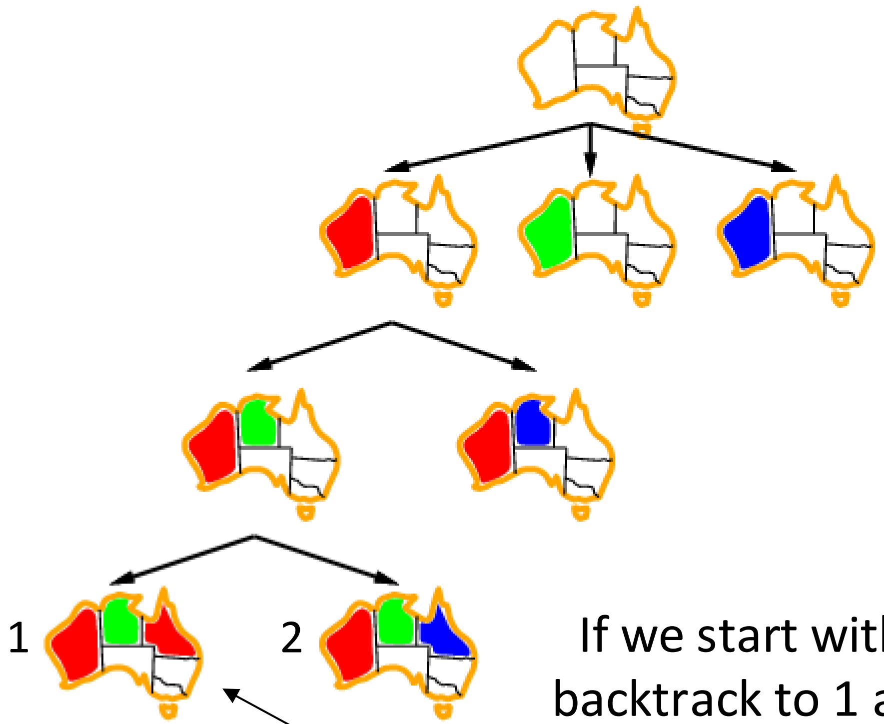
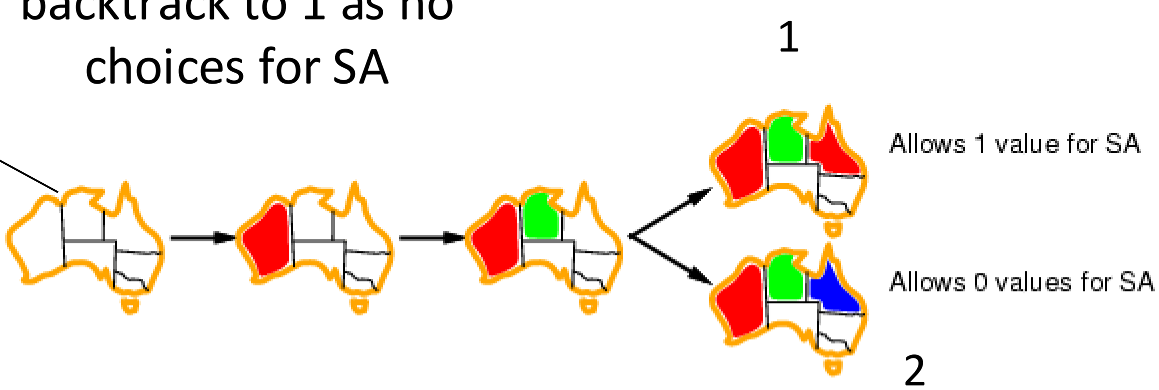


Backtracking example



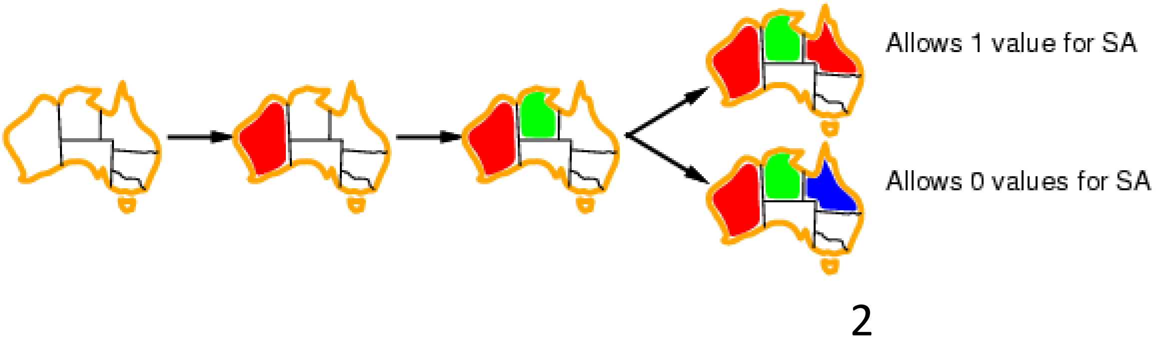
If we start with 2, backtrack to 1 as no choices for SA

Which assignment for Q should we choose?



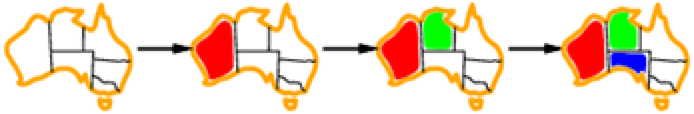
Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next? {WA, NT, Q, NSW, V, SA, T }
  - In what order should its values be tried? [R, B, G], [R, G, B],…
  - Can we detect inevitable failure early? Case 2 below

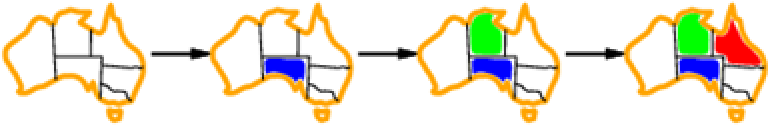


Heuristics for CSP

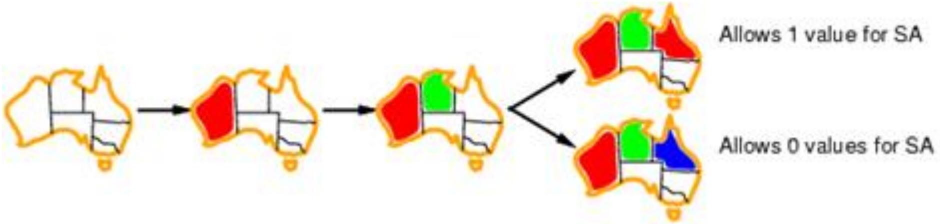
- most-constrained variable (Minimum Remaining Values: MRV, “fail-first”)
  - variable with the **fewest** possible values is selected
  - tends to minimize the branching factor



- most-constraining variable MCV
  - variable with the **largest** number of constraints on other unassigned variables

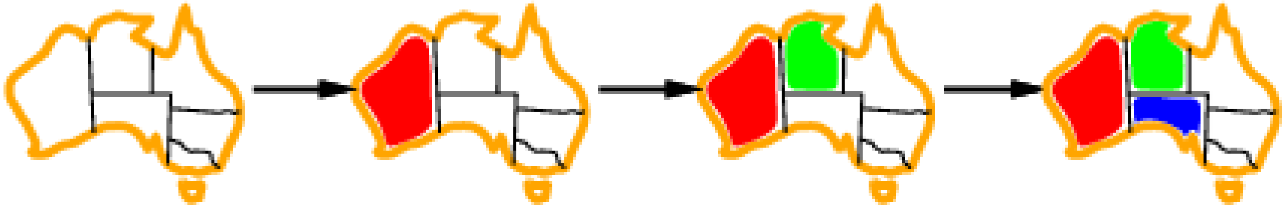


- least-constraining value LCV
  - for a selected variable, choose the **value** that leaves more freedom for future choices



Most constrained variable
Minimum Remaining Values (MRV)

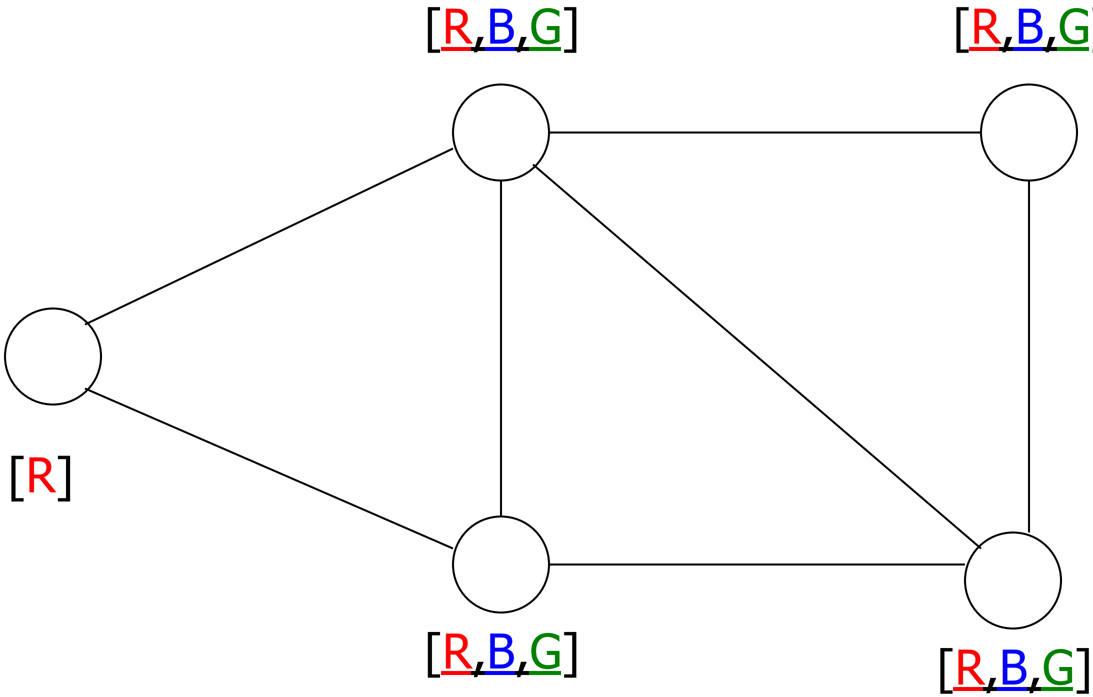
- Most constrained variable
  - choose the variable with the fewest legal values



- Called minimum remaining values (MRV) heuristic

- “fail-first” heuristic: Picks a variable which will cause failure as soon as possible, allowing the tree to be pruned.

Example: backtracking with MRV heuristic



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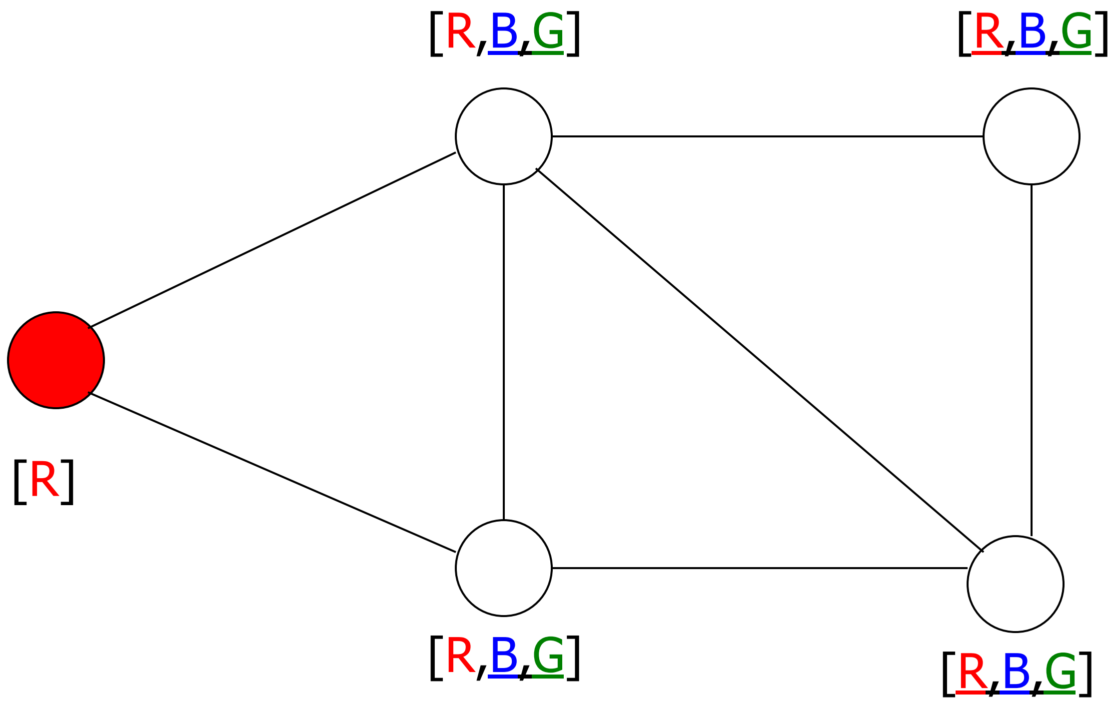

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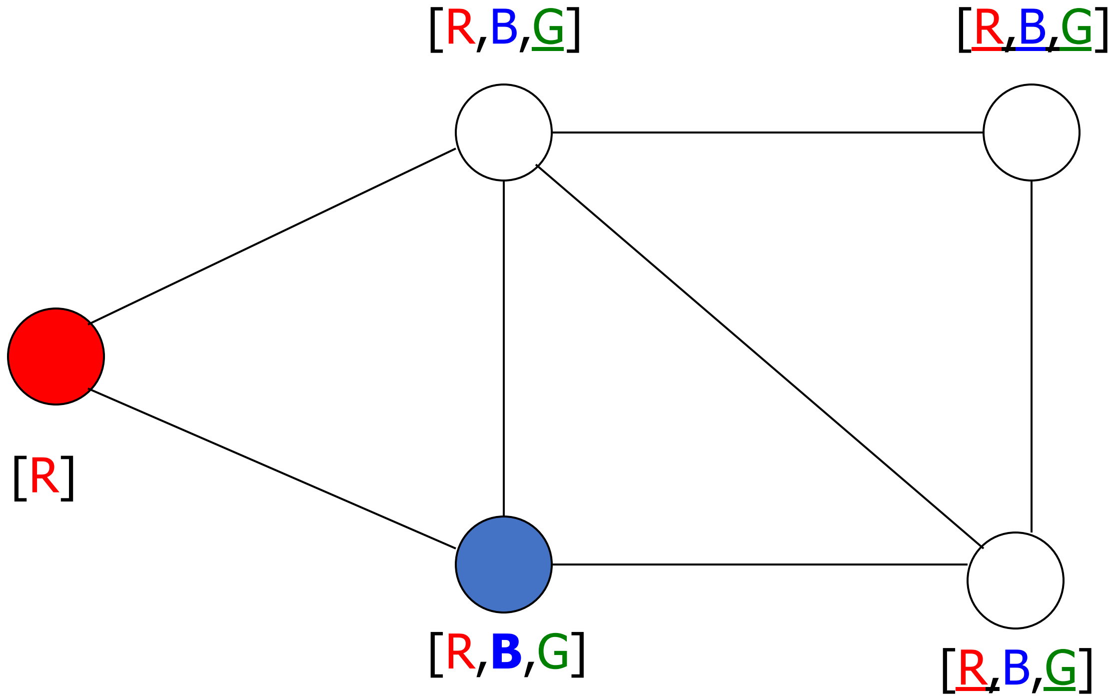

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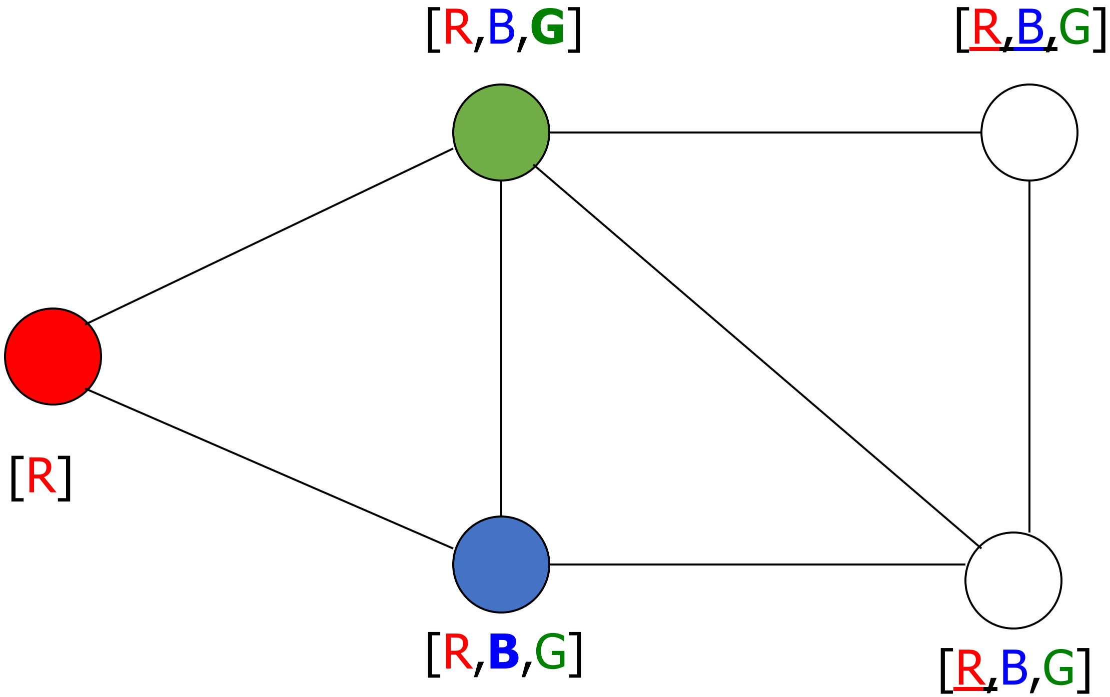

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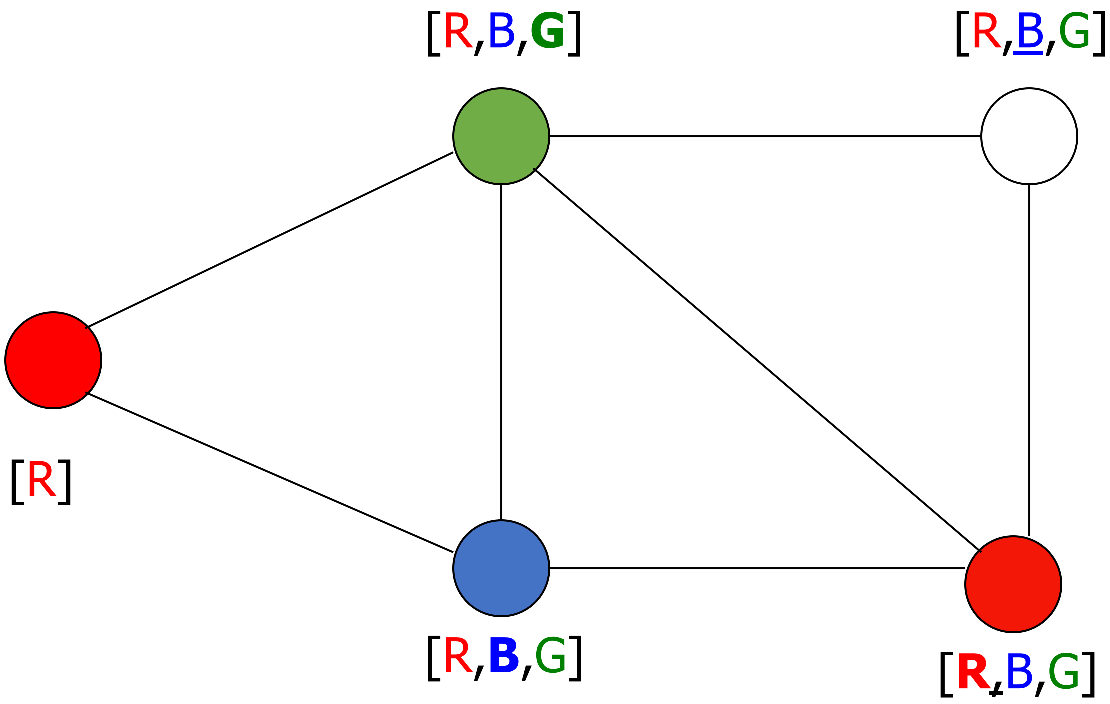

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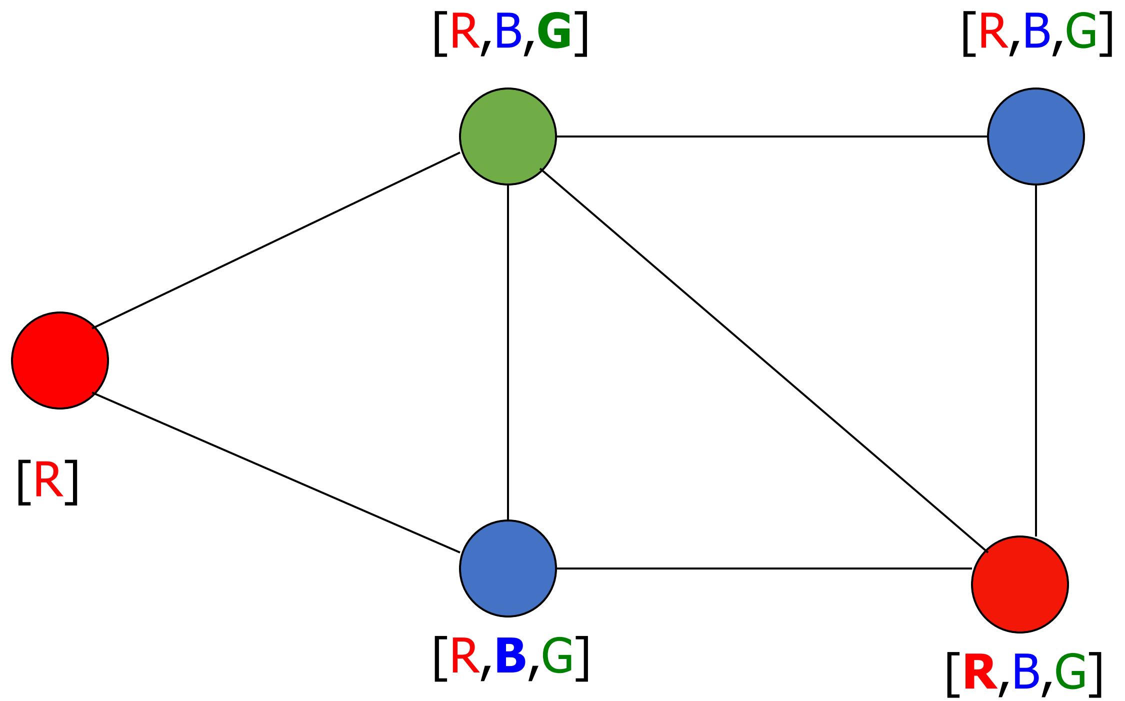
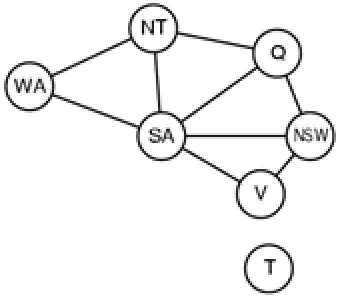


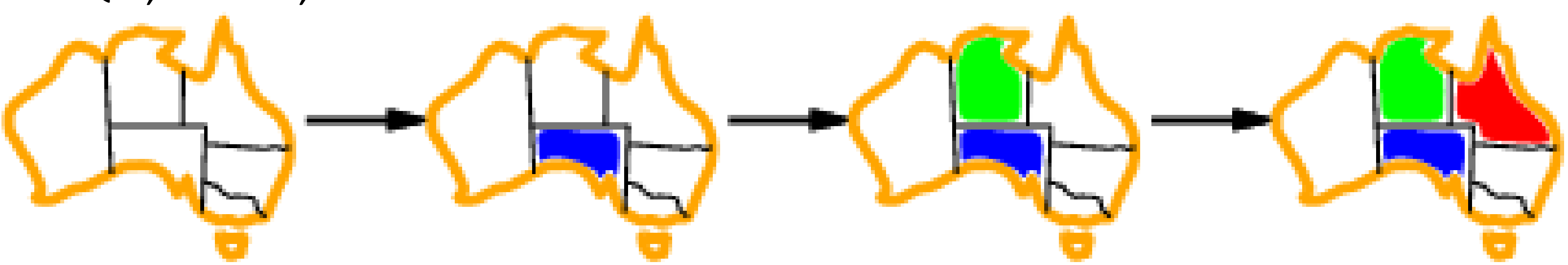
Solution !!!

Most Constraining Variable - MCV

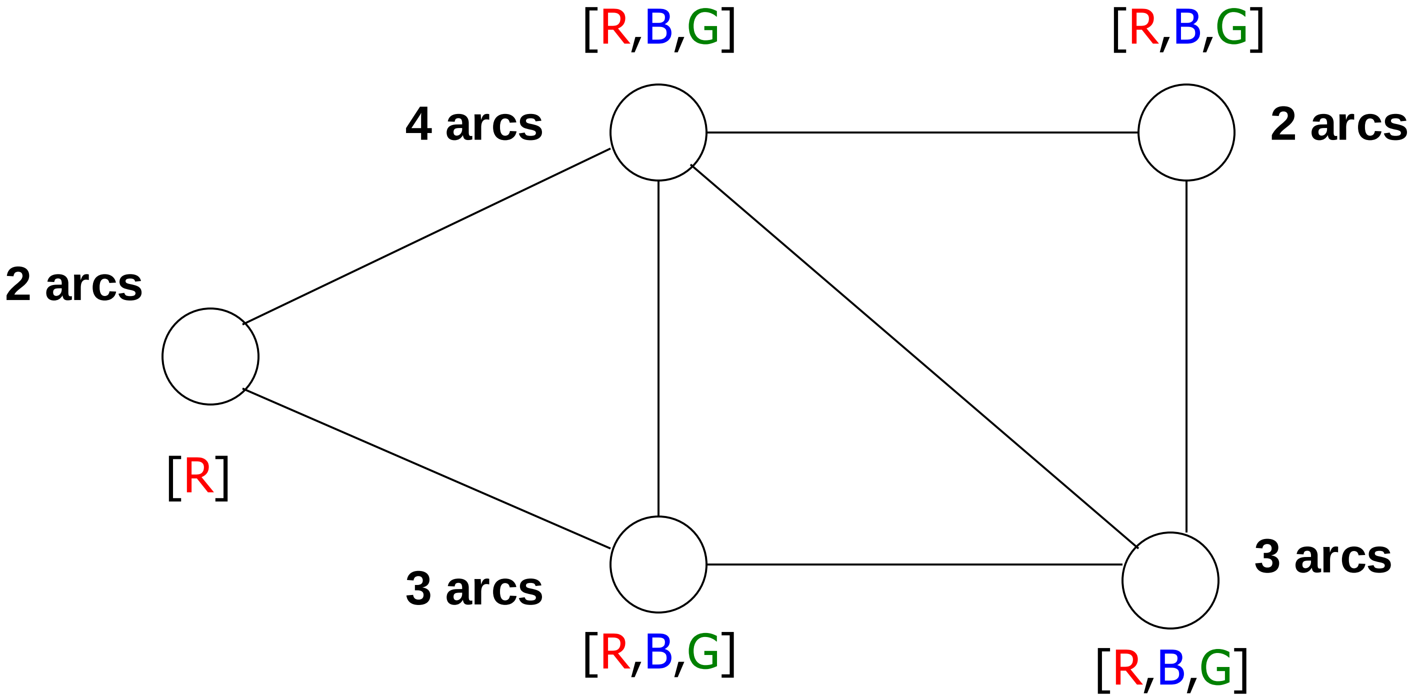
- **Tie-breaker** among most constrained variables (MRV)



- Most constraining variable:
  - choose the variable **with the most constraints on remaining variables** (select variable that is involved in the largest number of constraints - edges in graph on other unassigned variables: SA:5, WA:2, NT:3, Q:3, NSW:3, V:2 then:
  - WA:1, NT:2, Q:2, NSW:2, V:1 then
  - Q:1, NSW:2, V:1 then …



Example: backtracking with MCV heuristic



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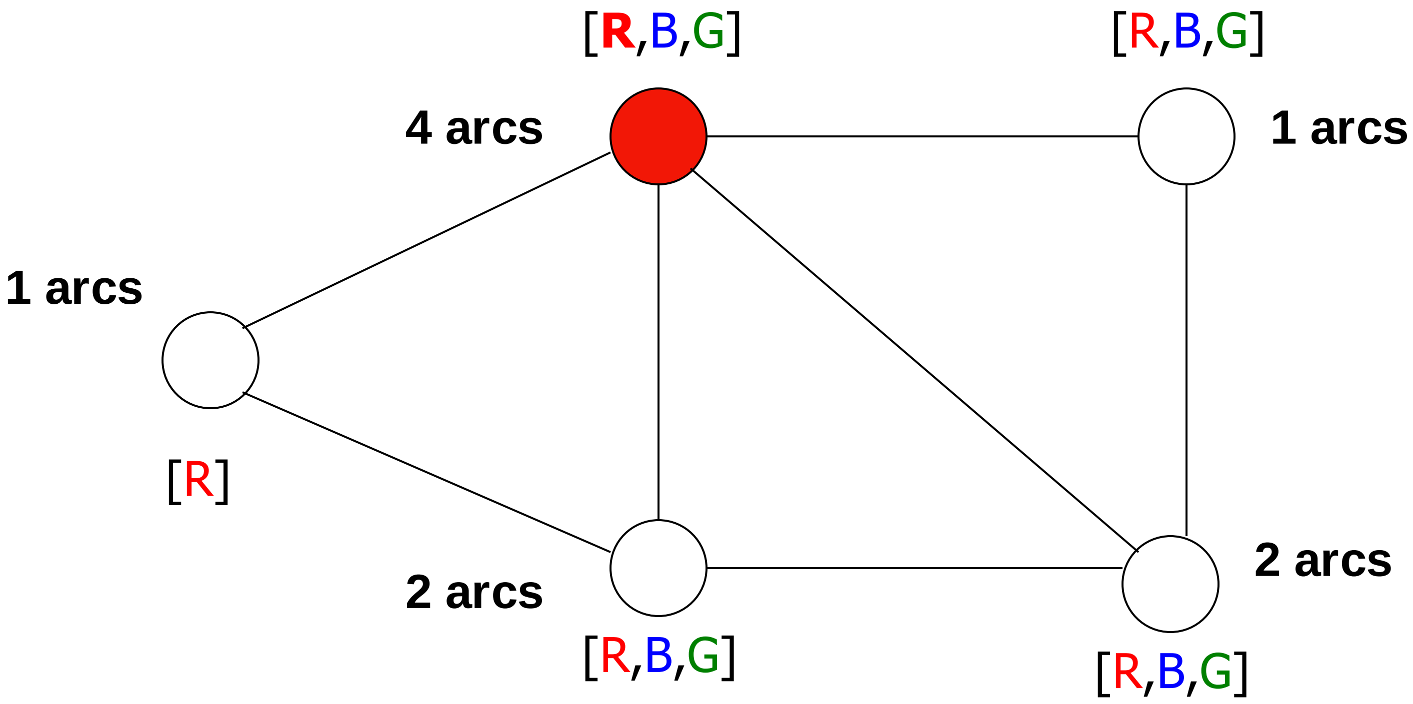

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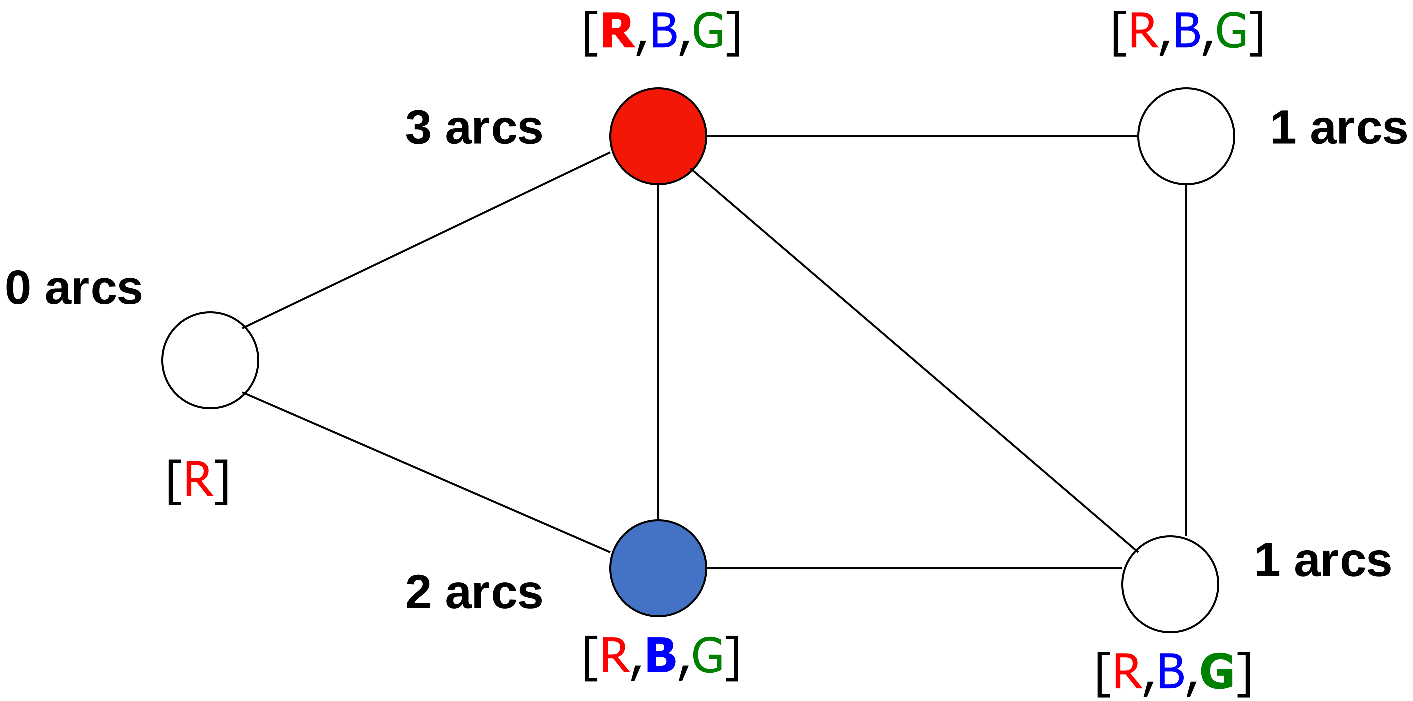

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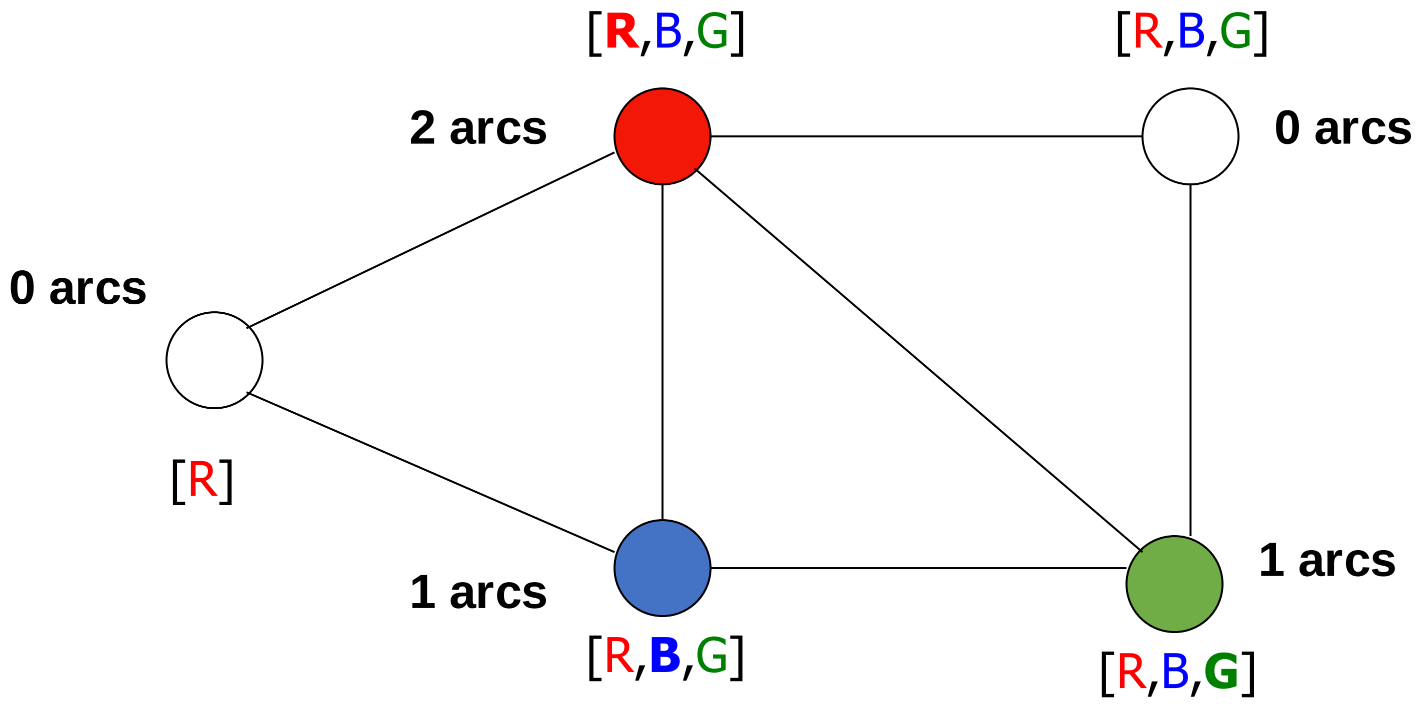

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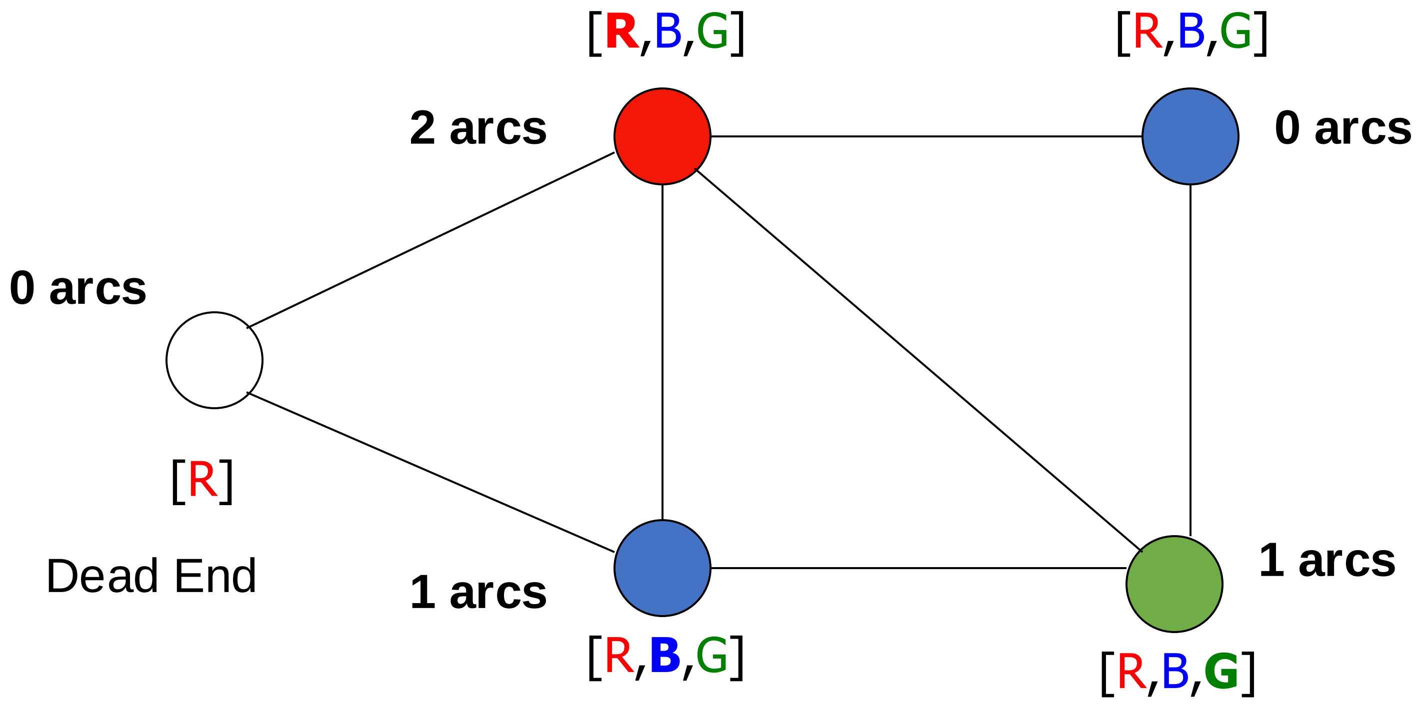

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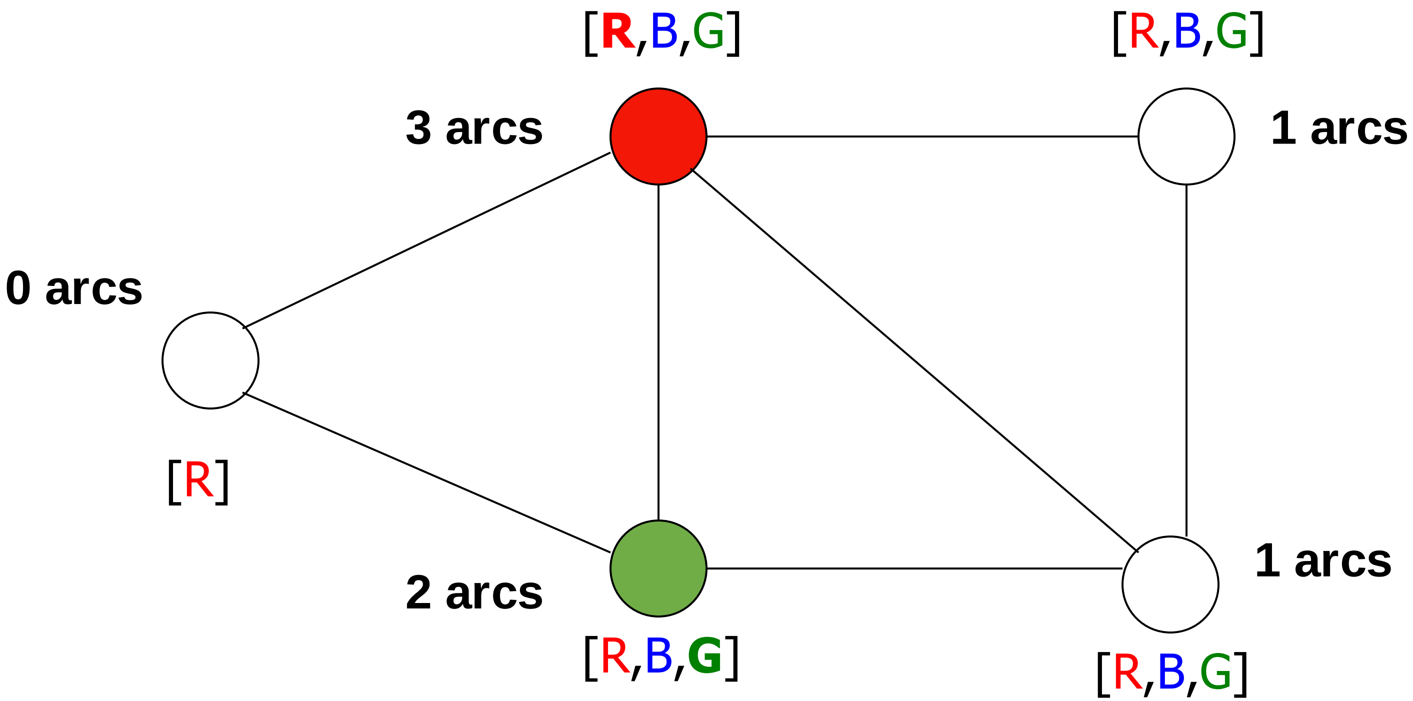

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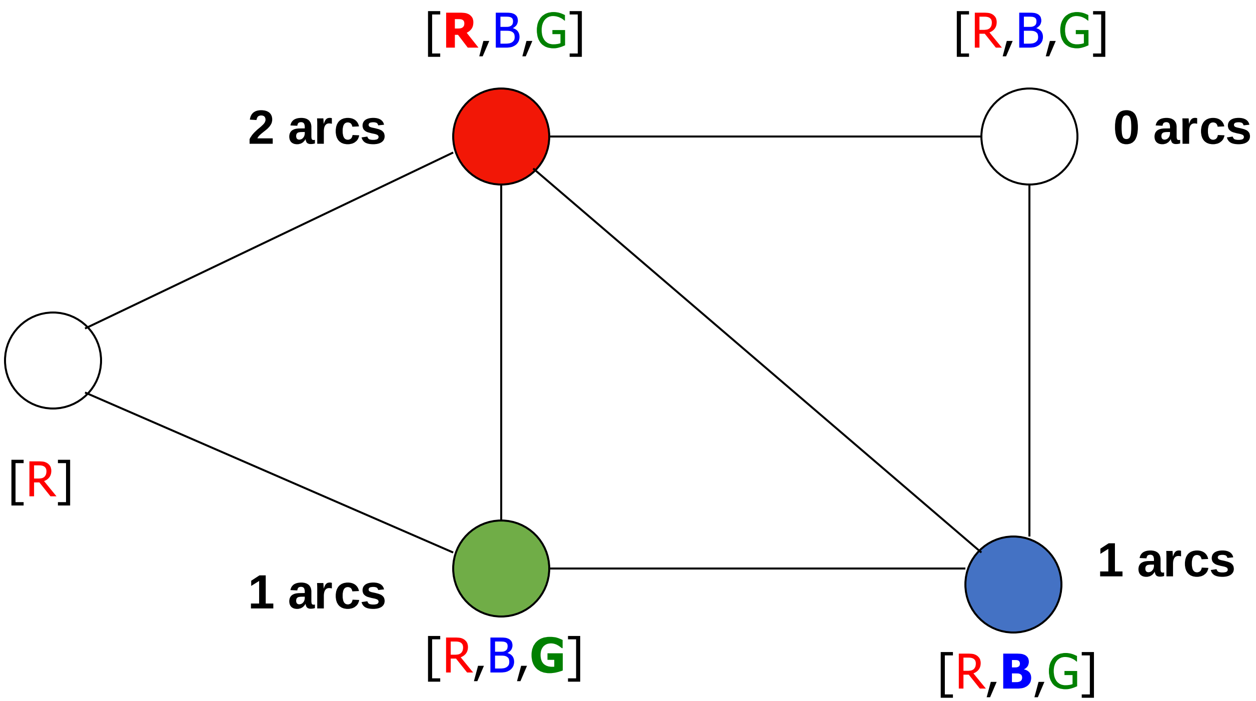

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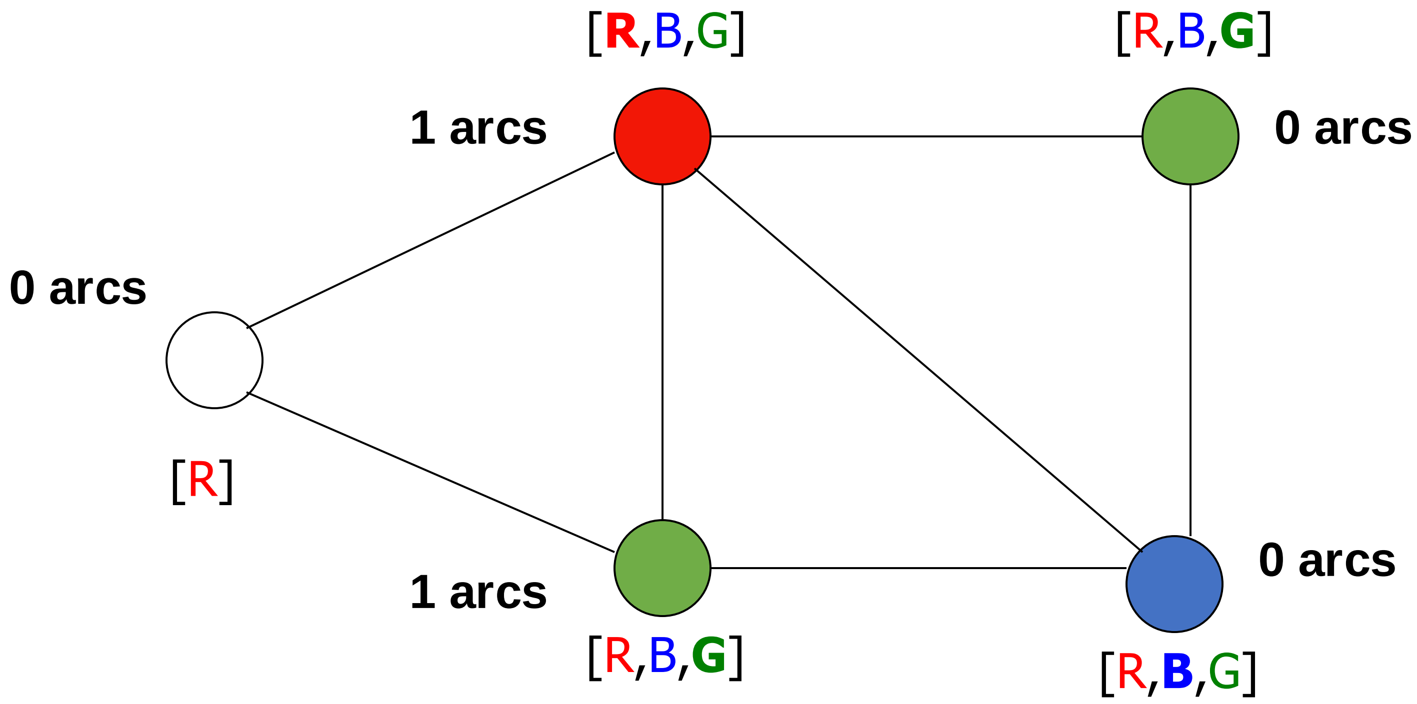

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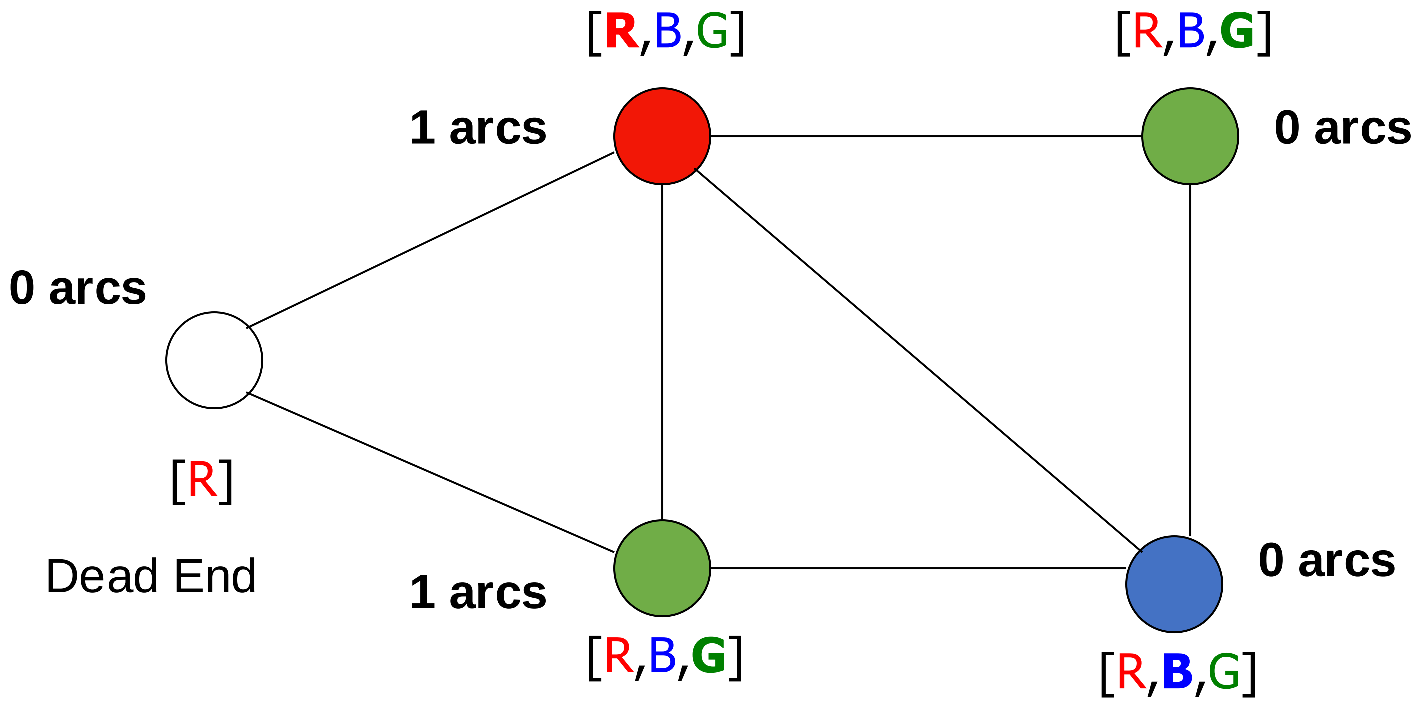

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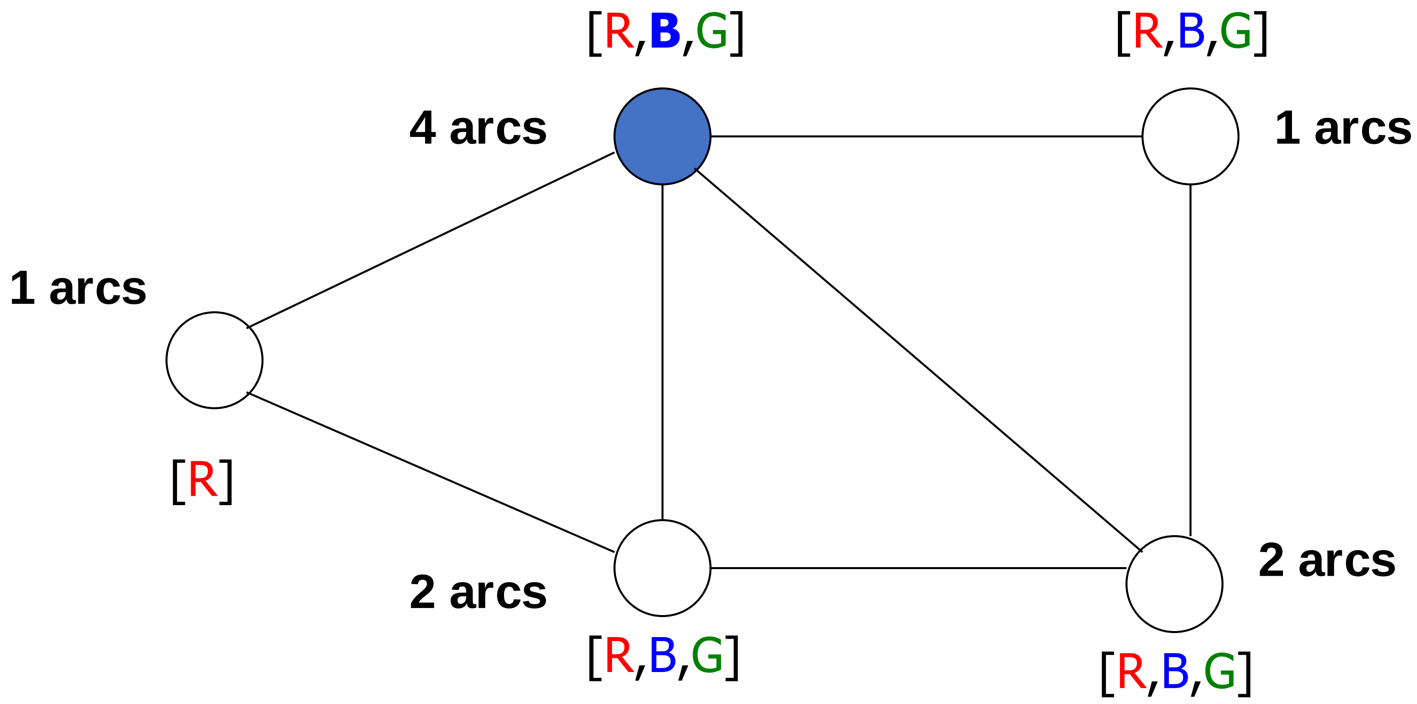

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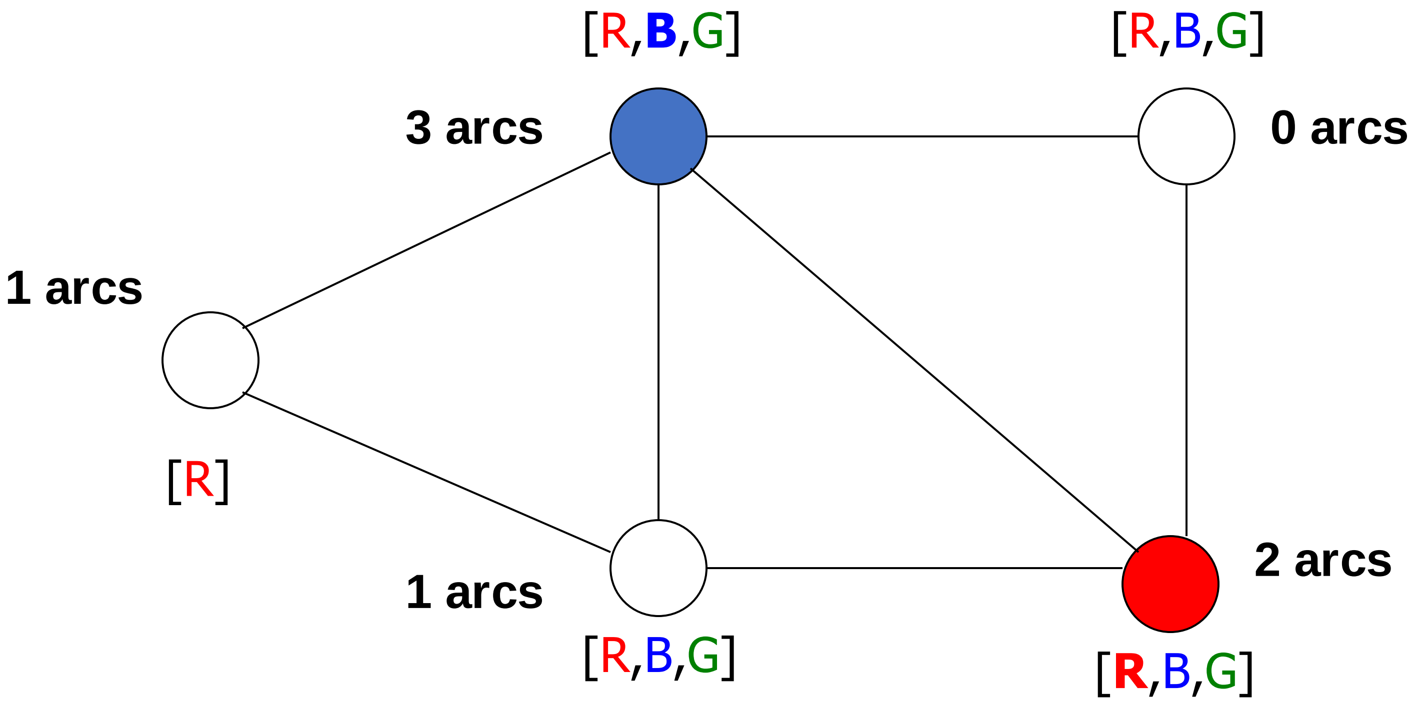

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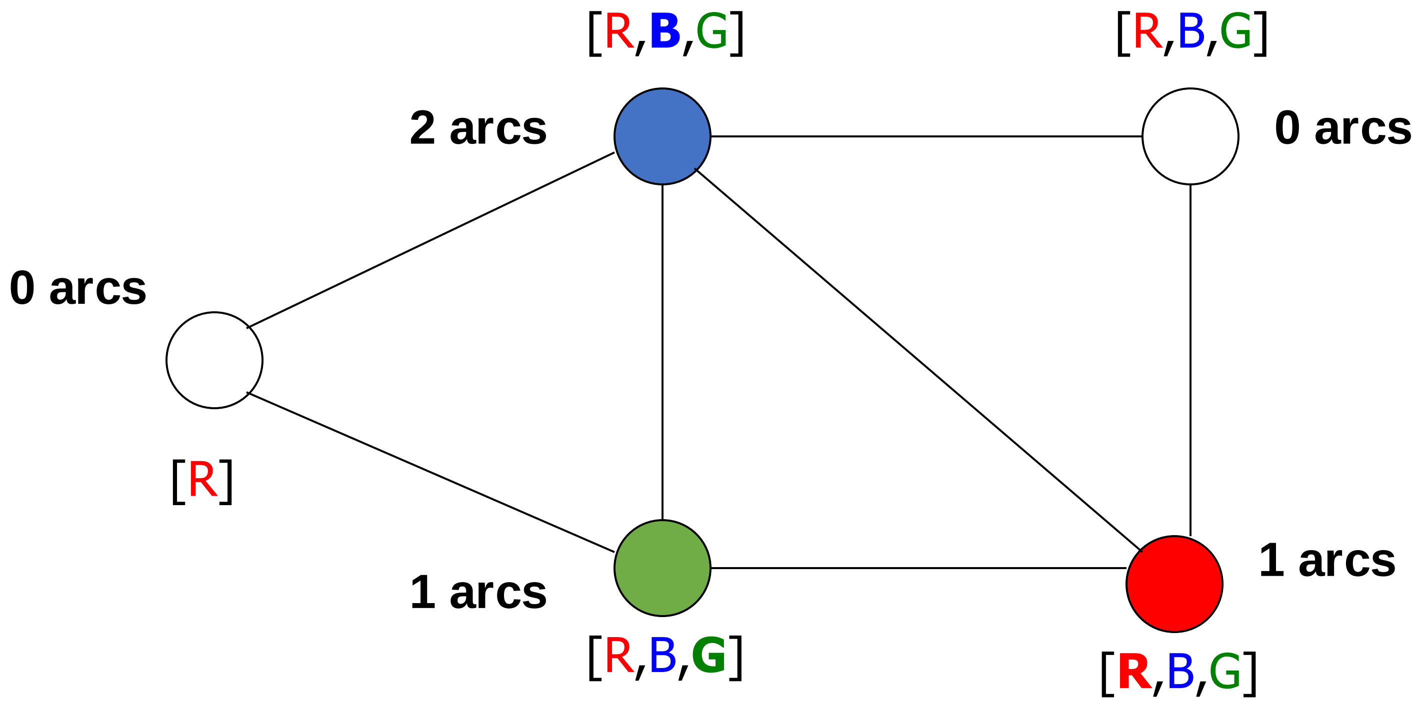

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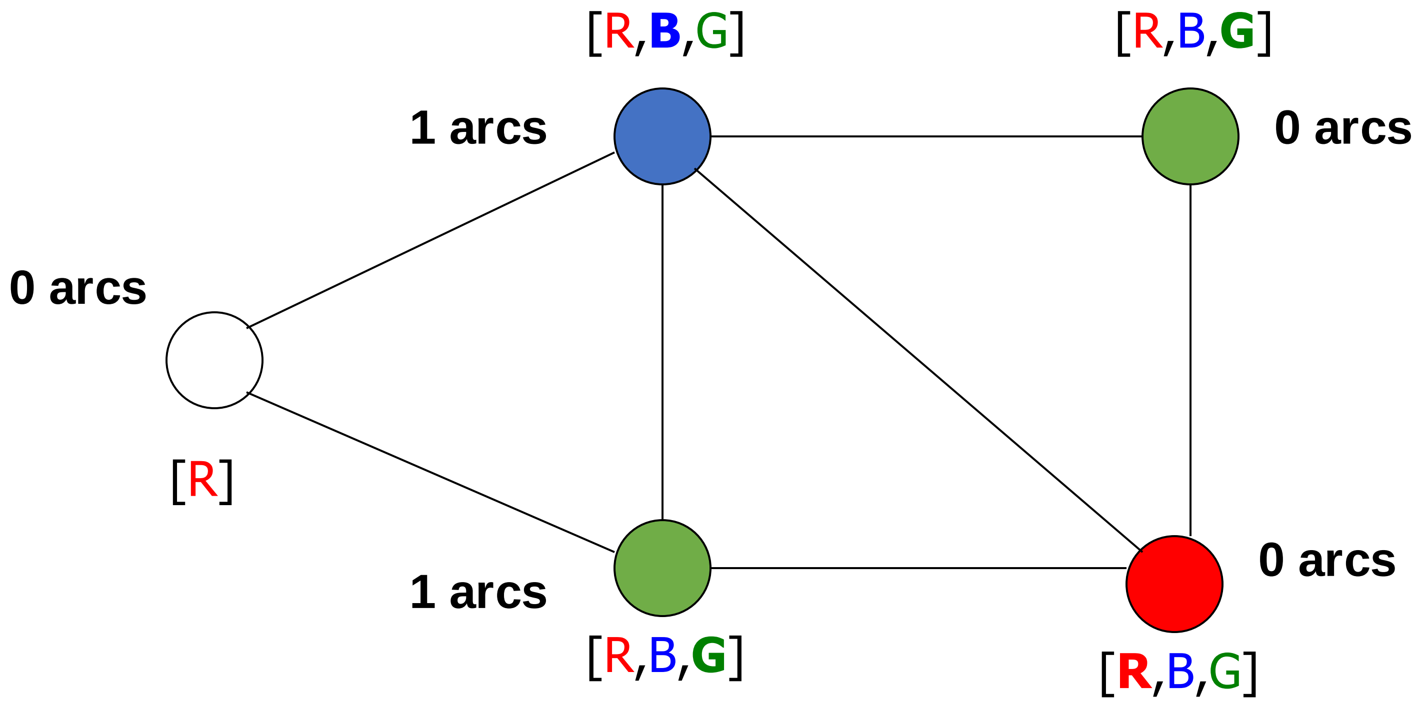

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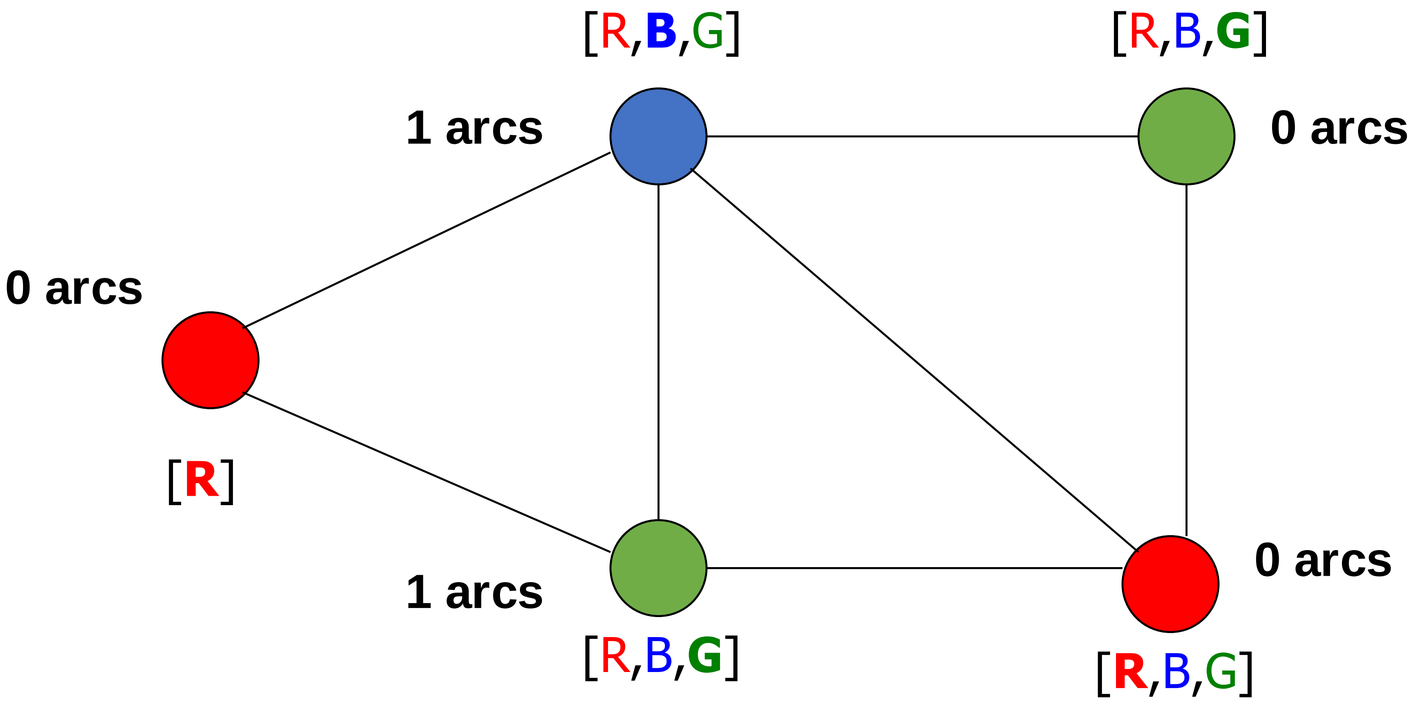
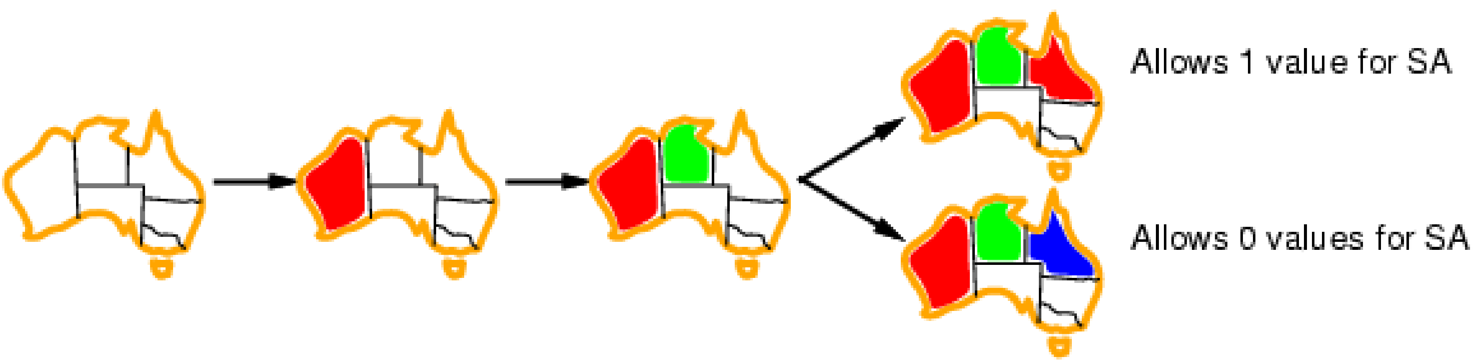


Solution !!!

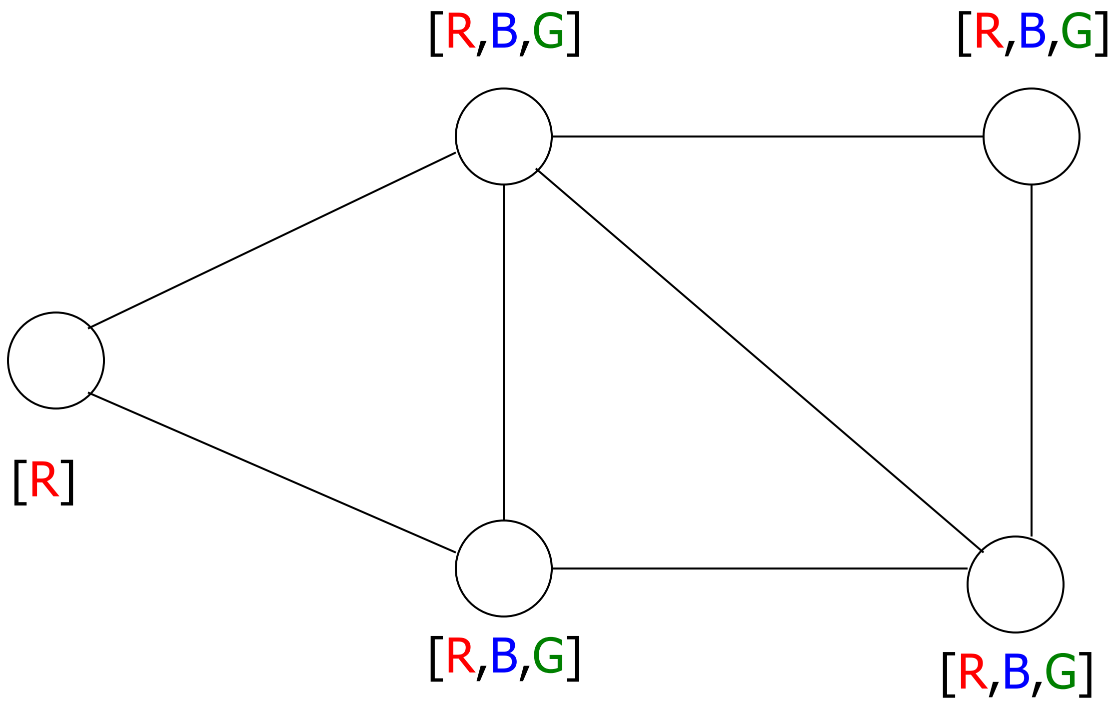
Least constraining value - LCV

- Given a variable, choose the least constraining value:
  - the one that rules out/eliminates the fewest values in the remaining variables (keeps the most)



- Combining these heuristics makes 1000 queens feasible

Example: backtracking with LCV (Least constraining value) heuristic



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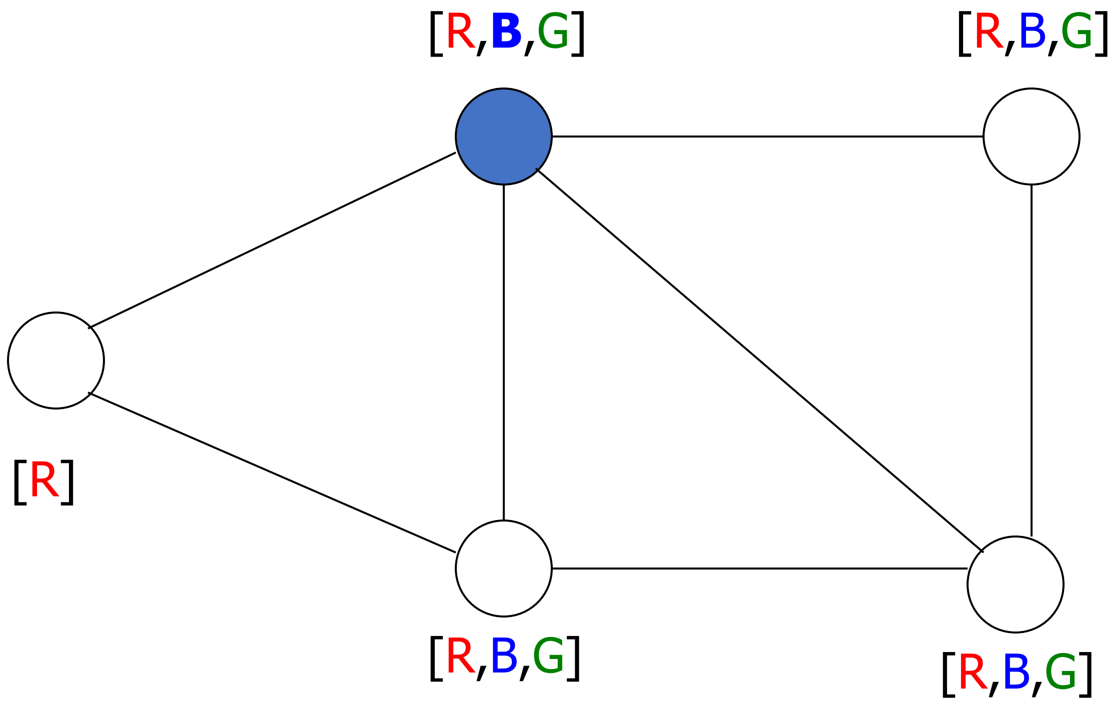

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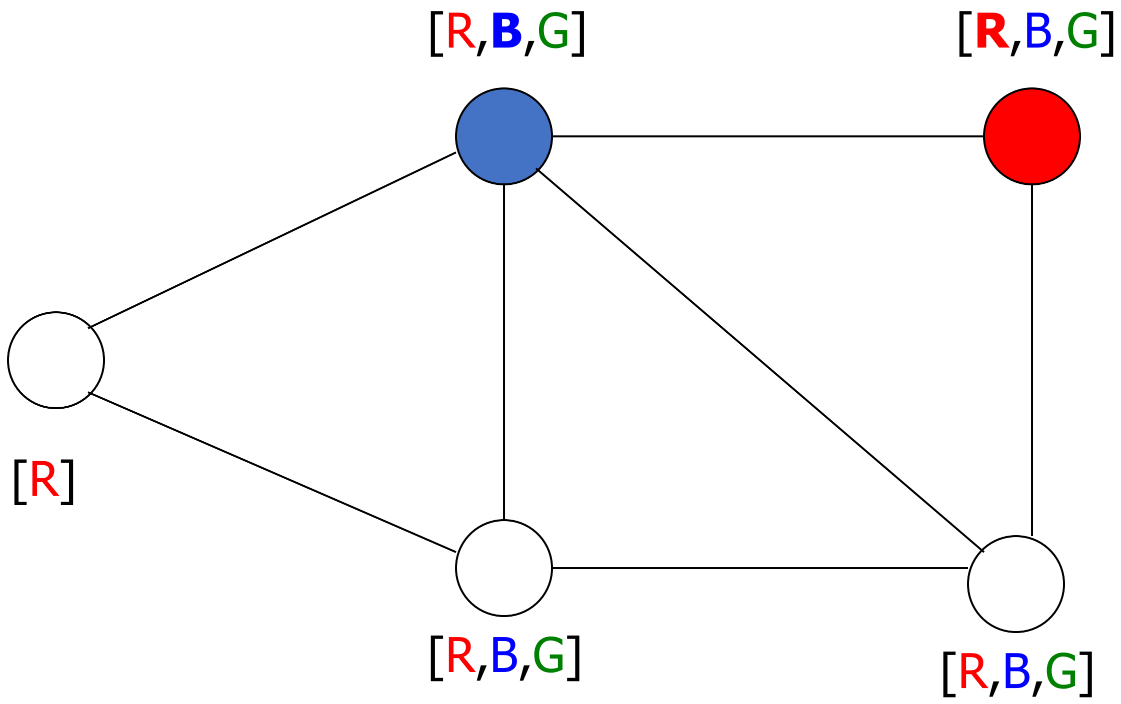

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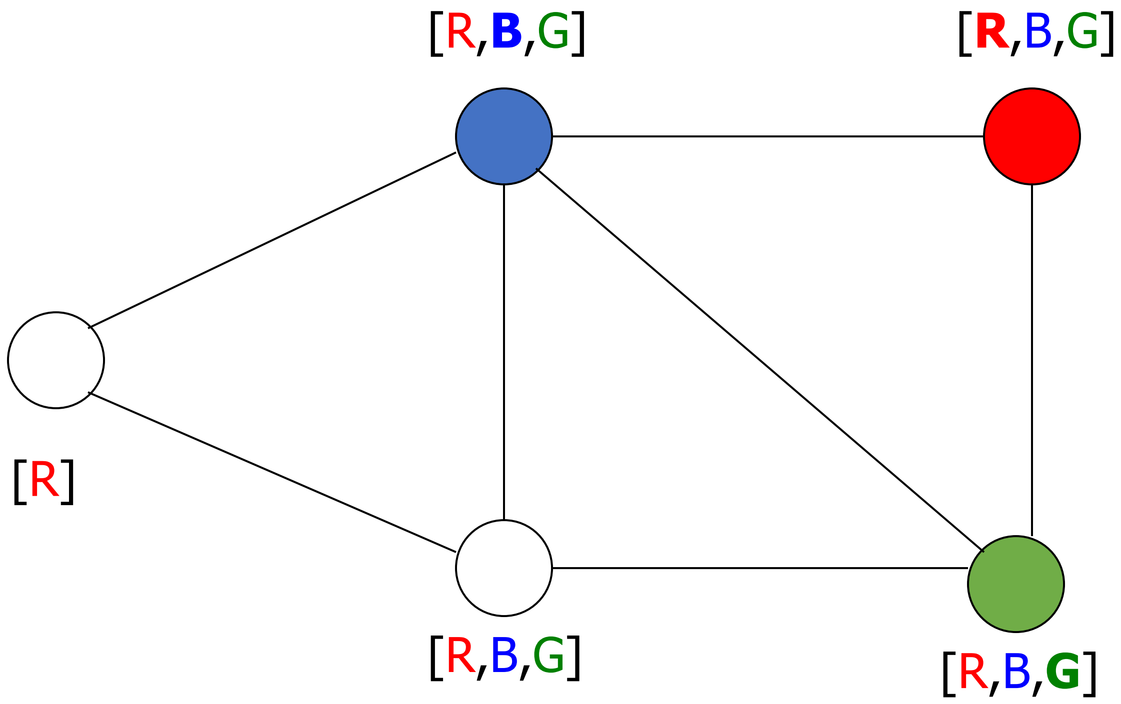

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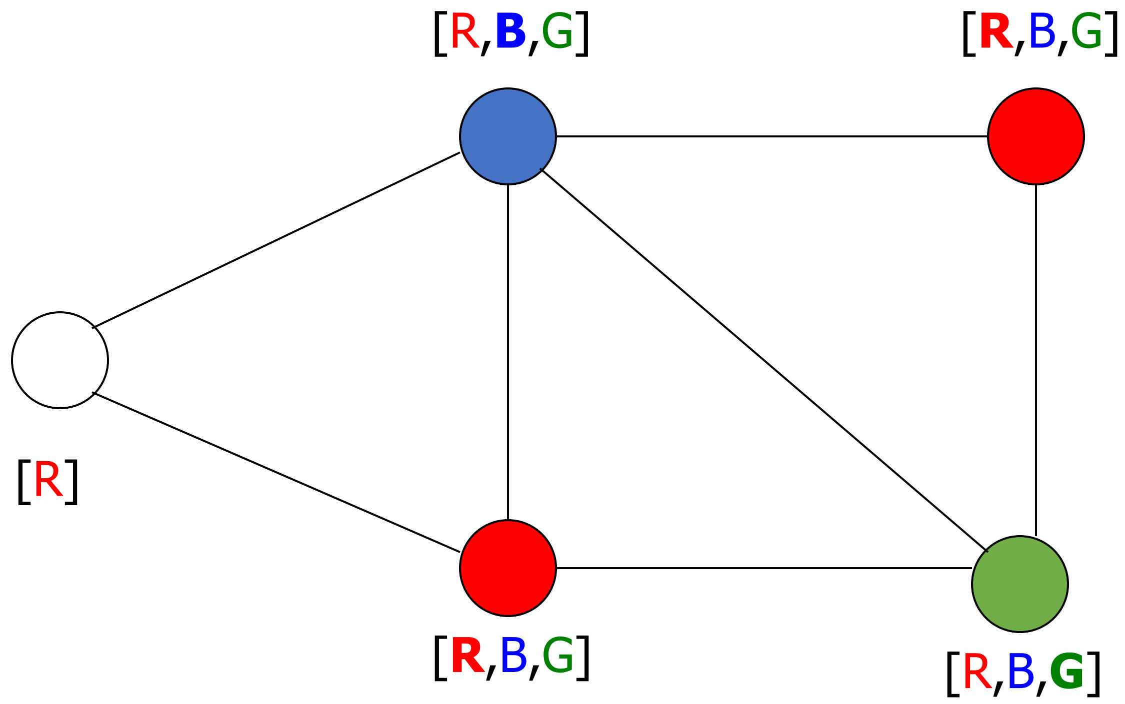

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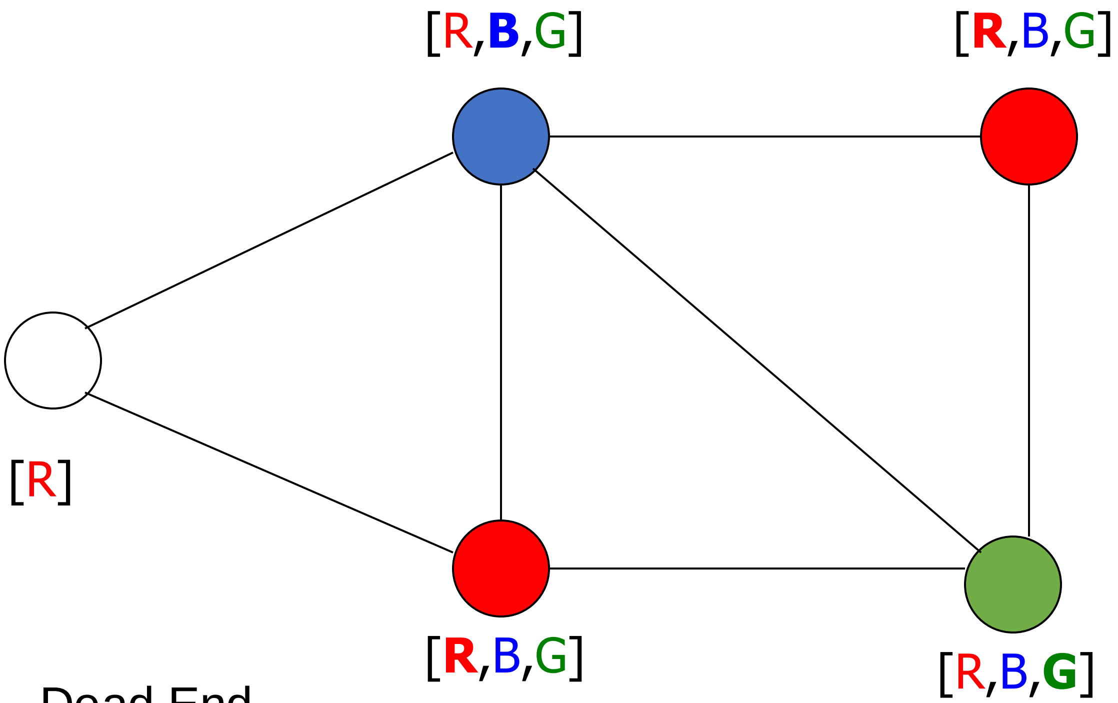
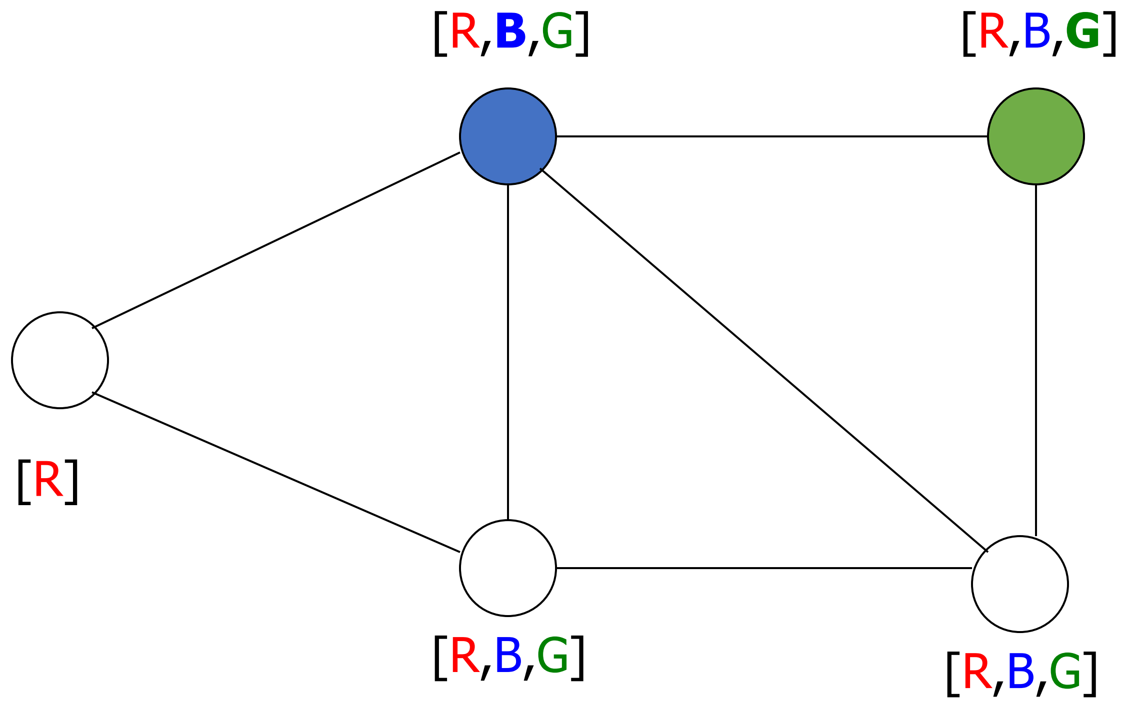


Dead End

Example: backtracking with LCV (Least constraining value) heuristic



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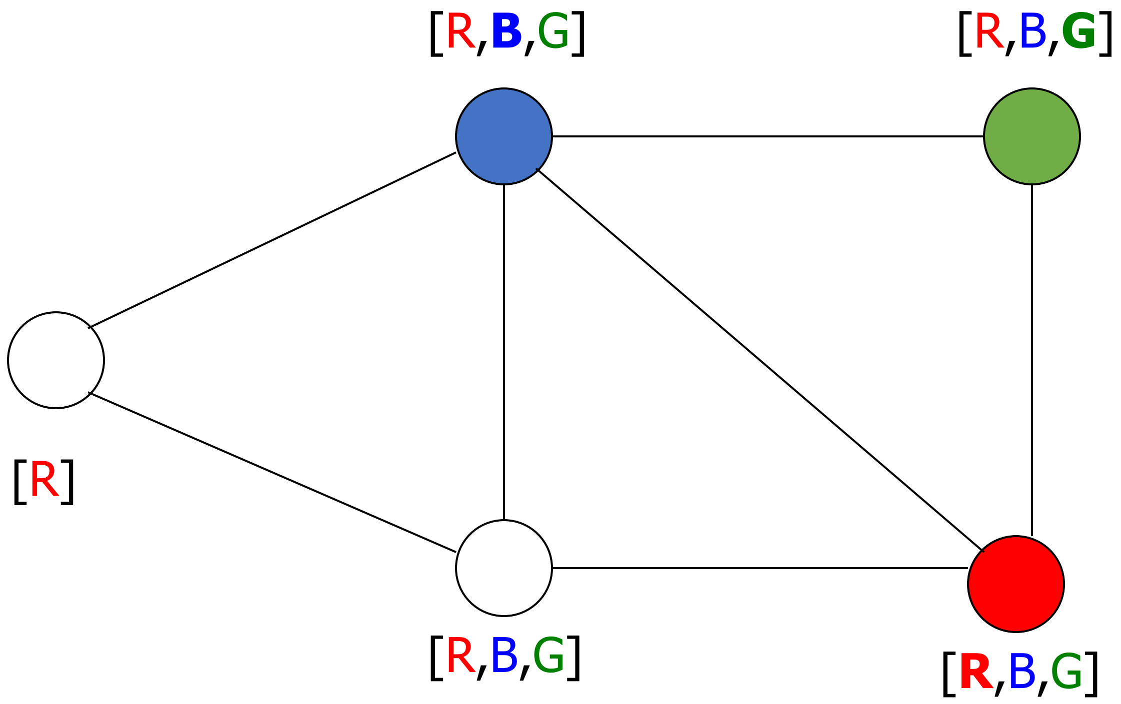

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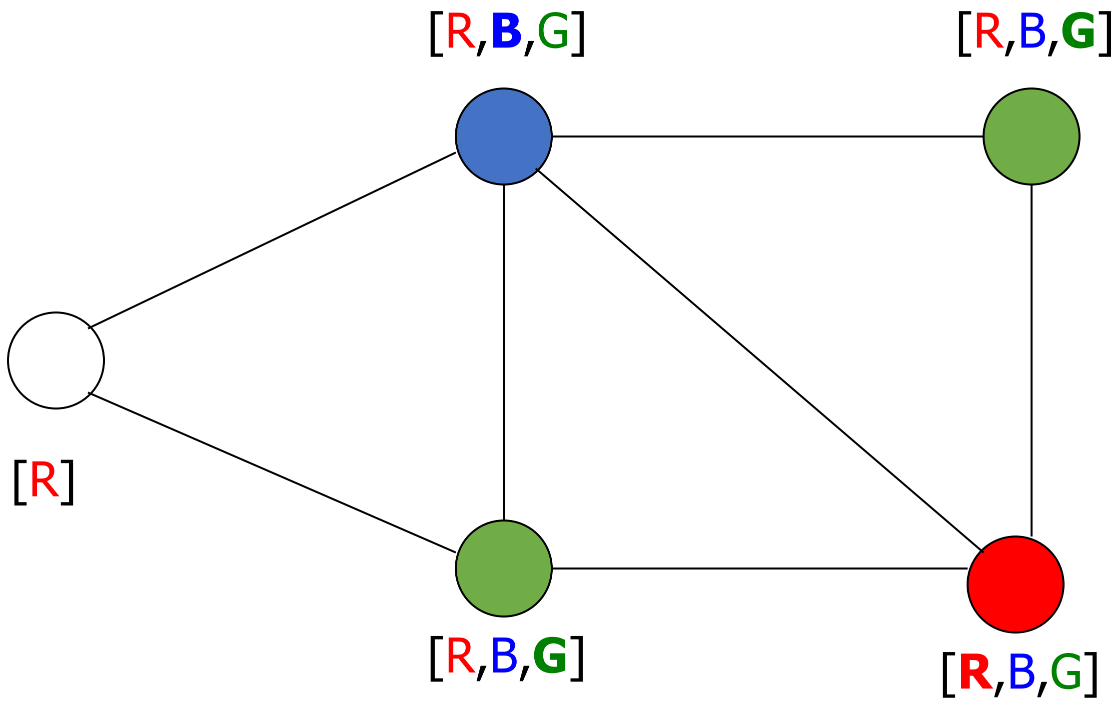

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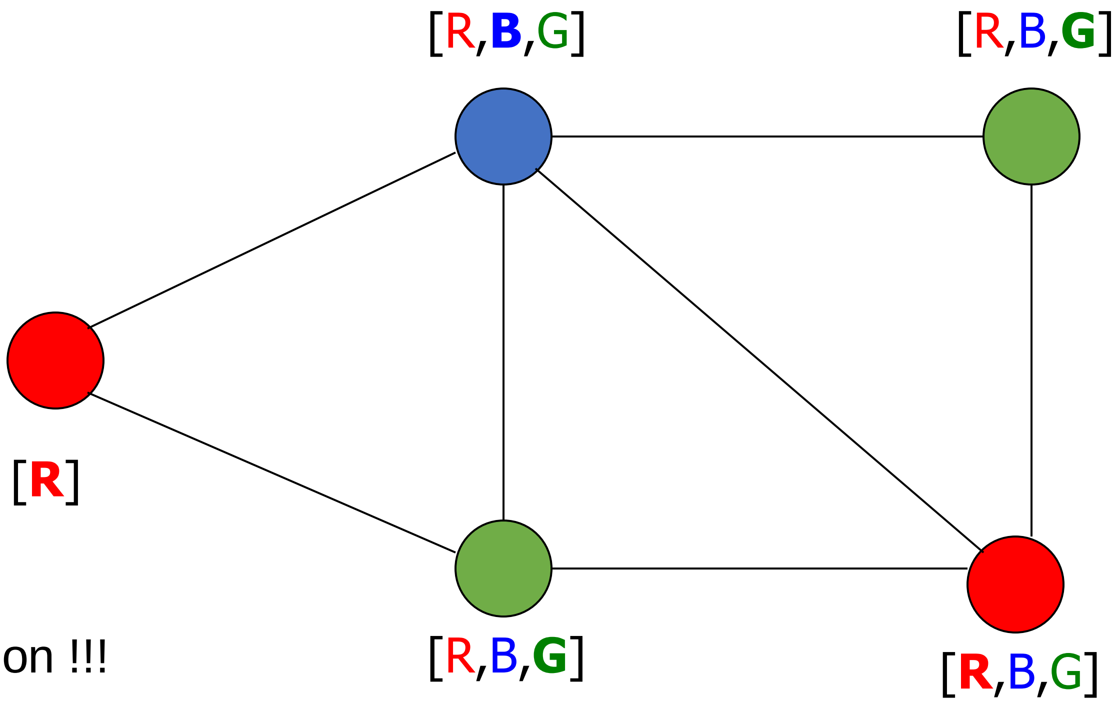


Solution !!!

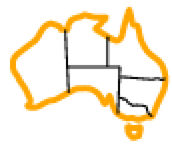
Analyzing Constraints

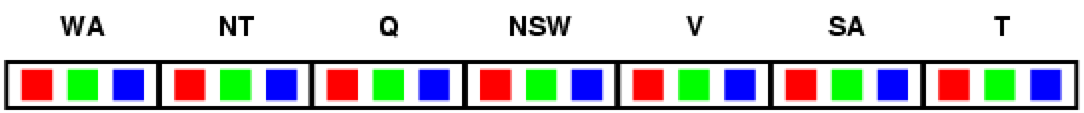
- forward checking:
  - when a value **X** is assigned to a variable, inconsistent values are eliminated for all variables connected to **X** [remove conflicting values]
  - identifies "dead" branches of the tree before they are visited

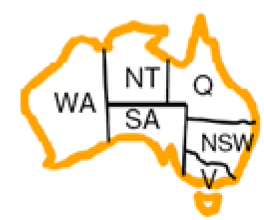
- constraint propagation:
  - analyses interdependencies between variable assignments via arc consistency
  - an arc between **X** and **Y** is consistent if for every possible value x of **X**, there is some value y of **Y** that is consistent with x
  - more powerful than forward checking, but still reasonably efficient
  - but does not reveal every possible inconsistency

Forward checking

- Idea
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values







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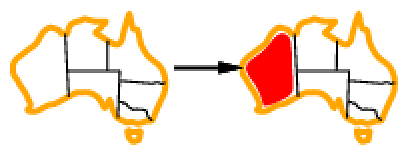

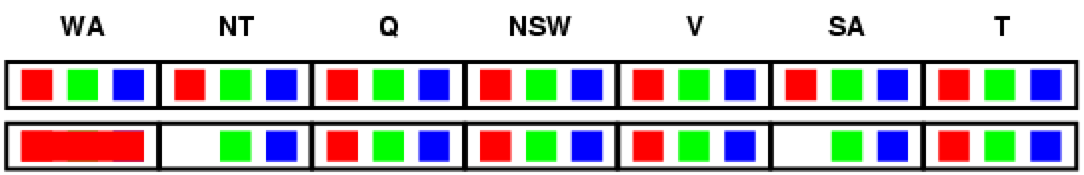

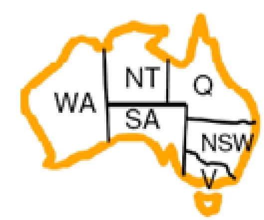

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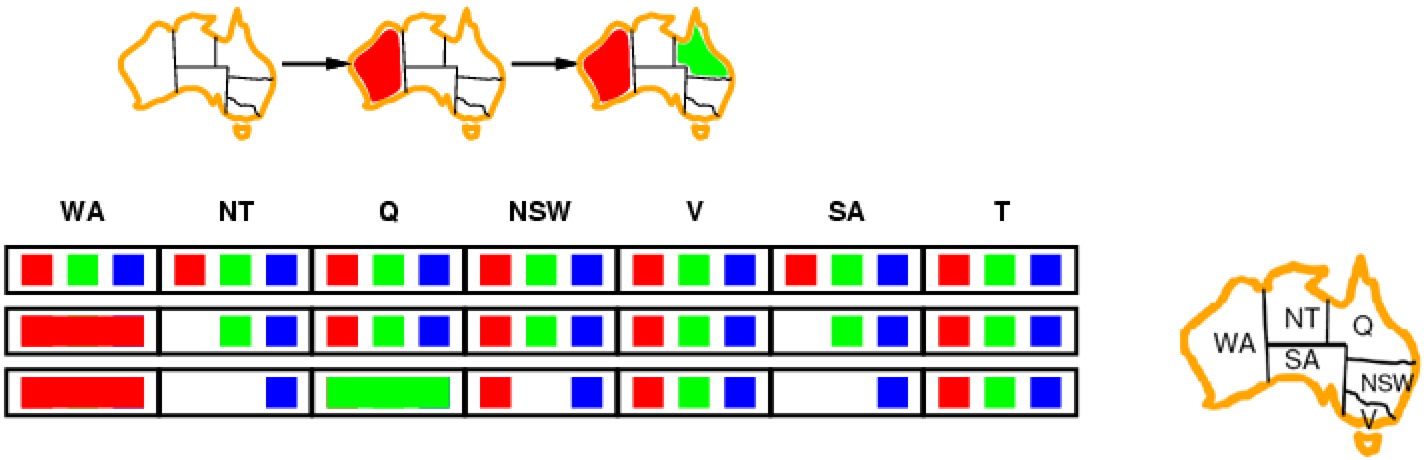

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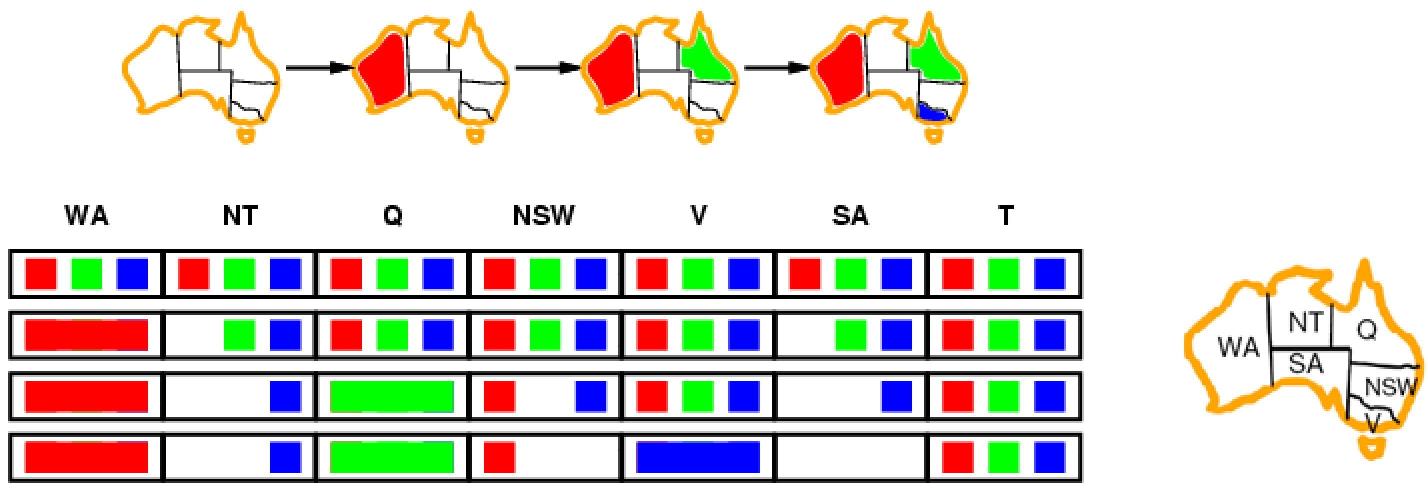
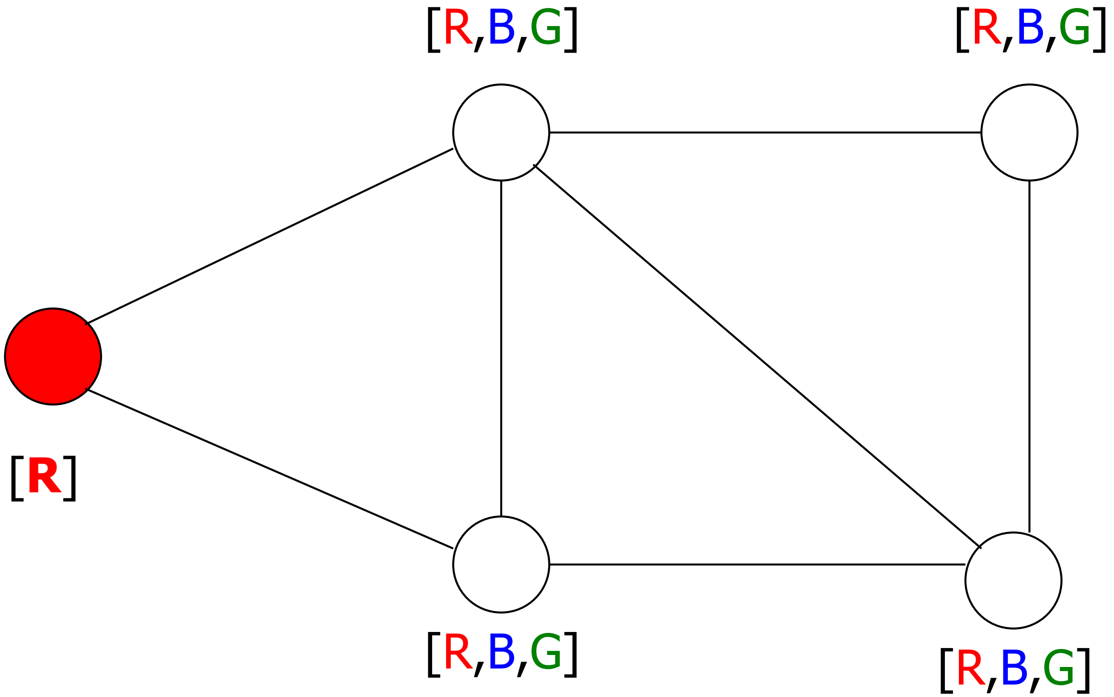
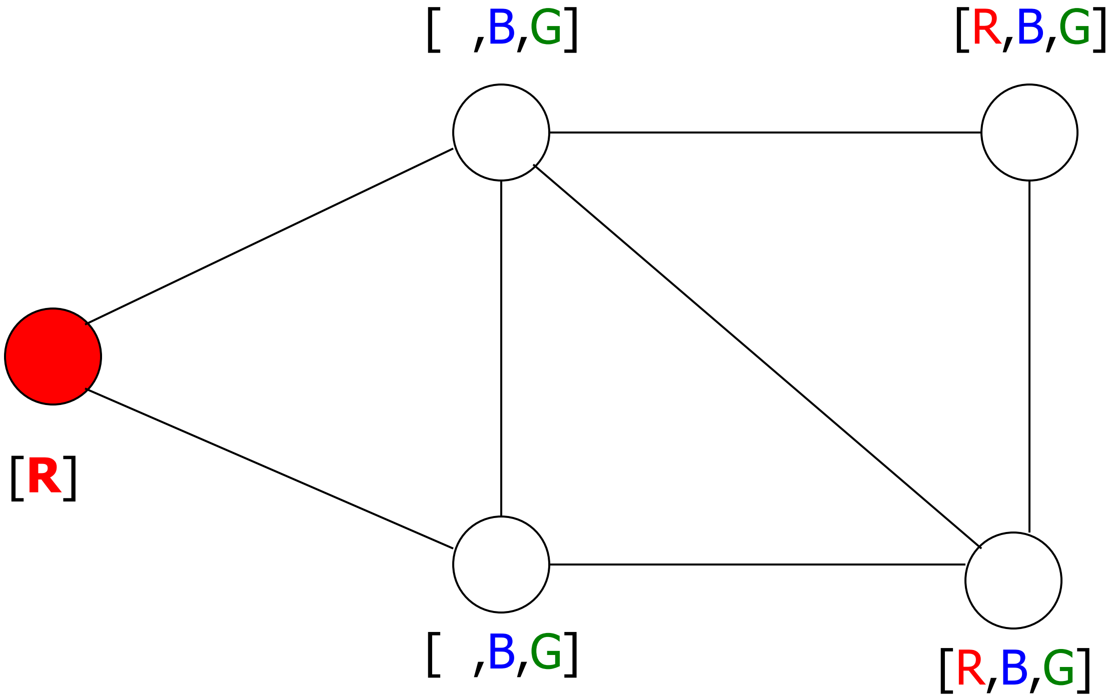
- Idea
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



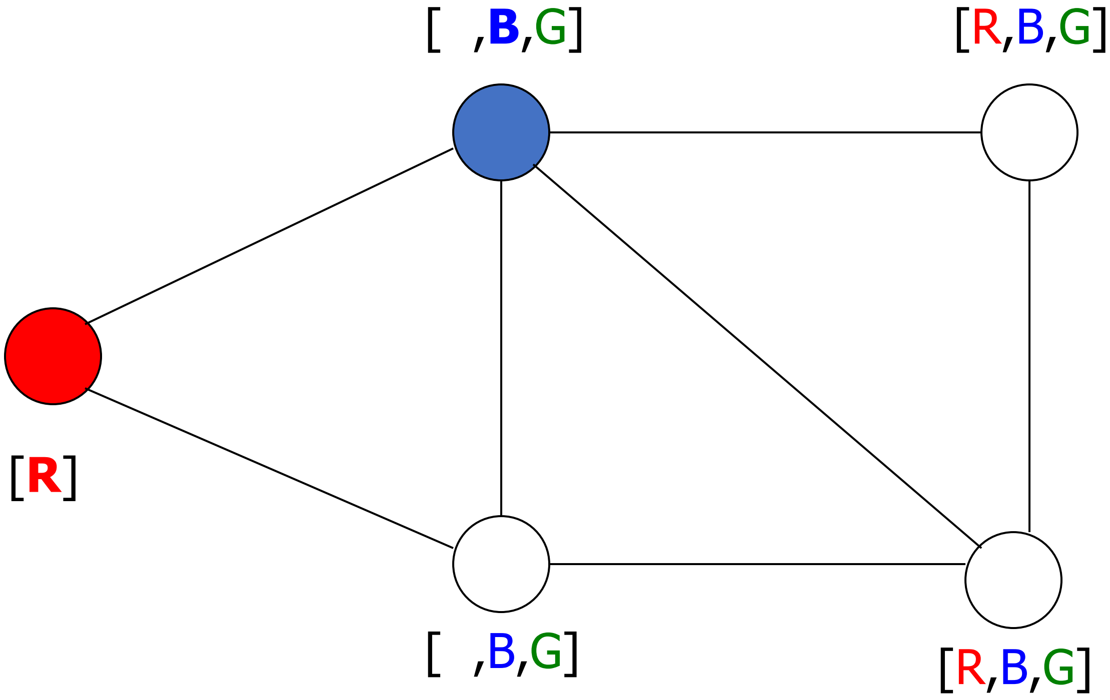
Forward Checking Example



Forward Checking Example



Forward Checking Example



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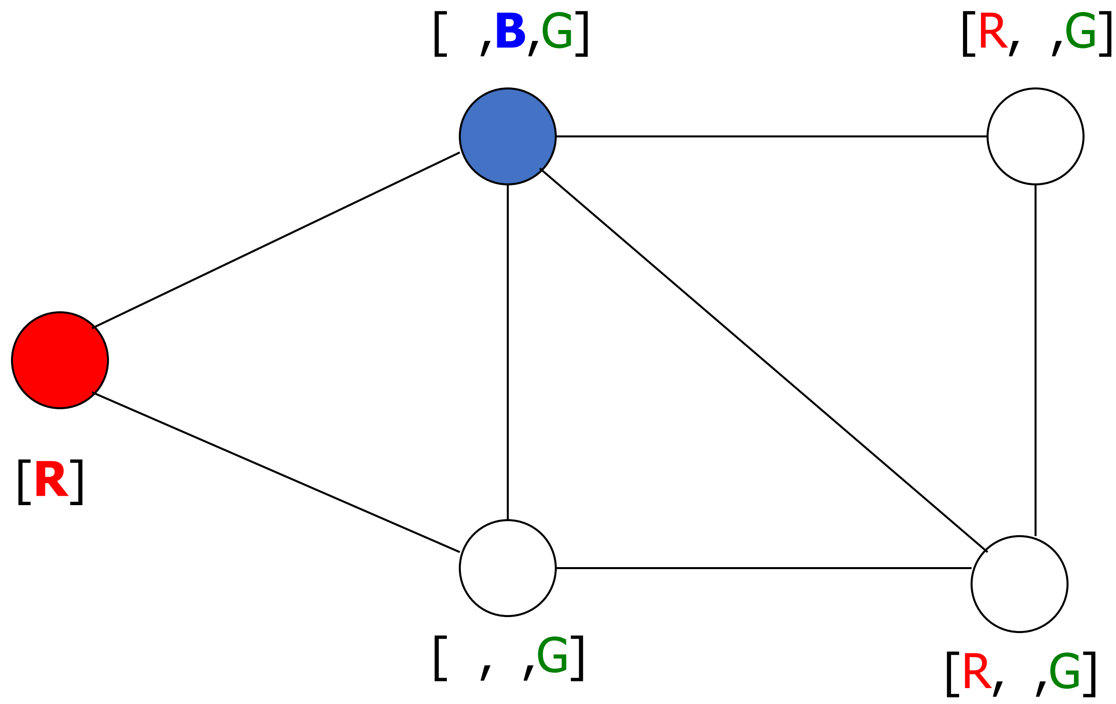

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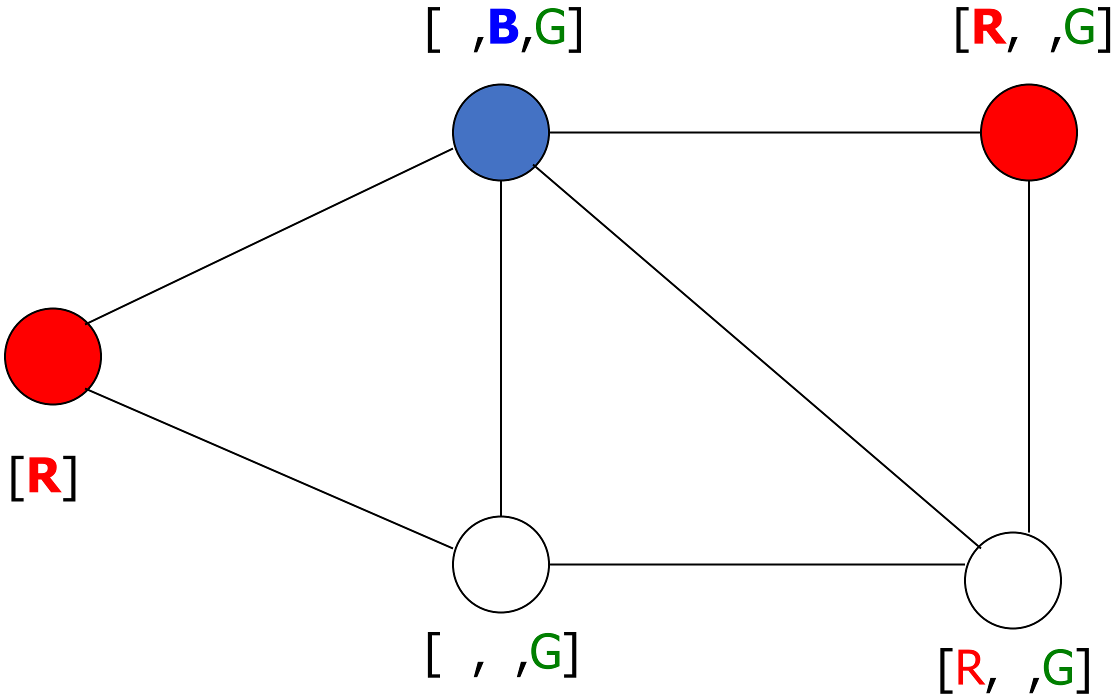

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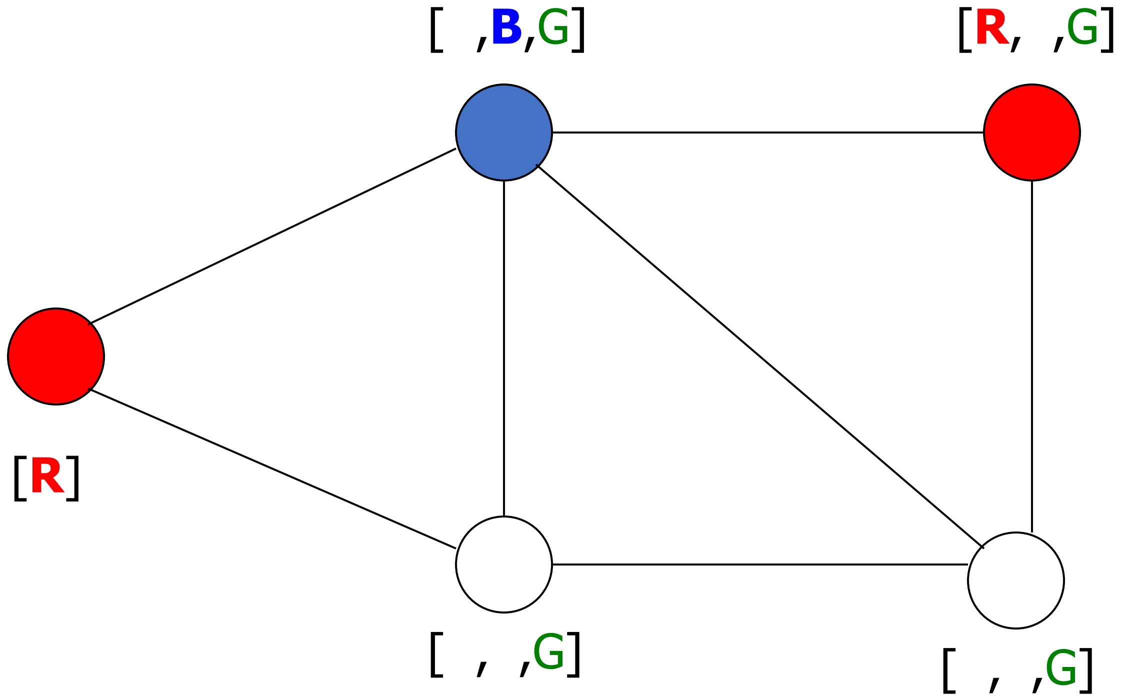

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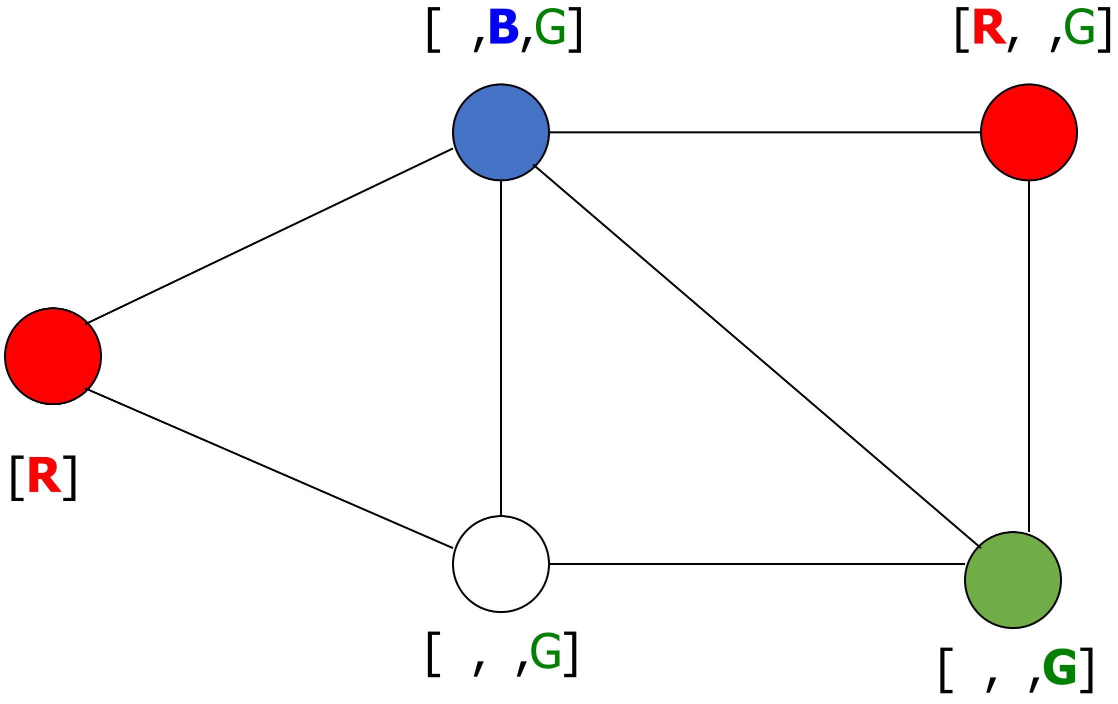

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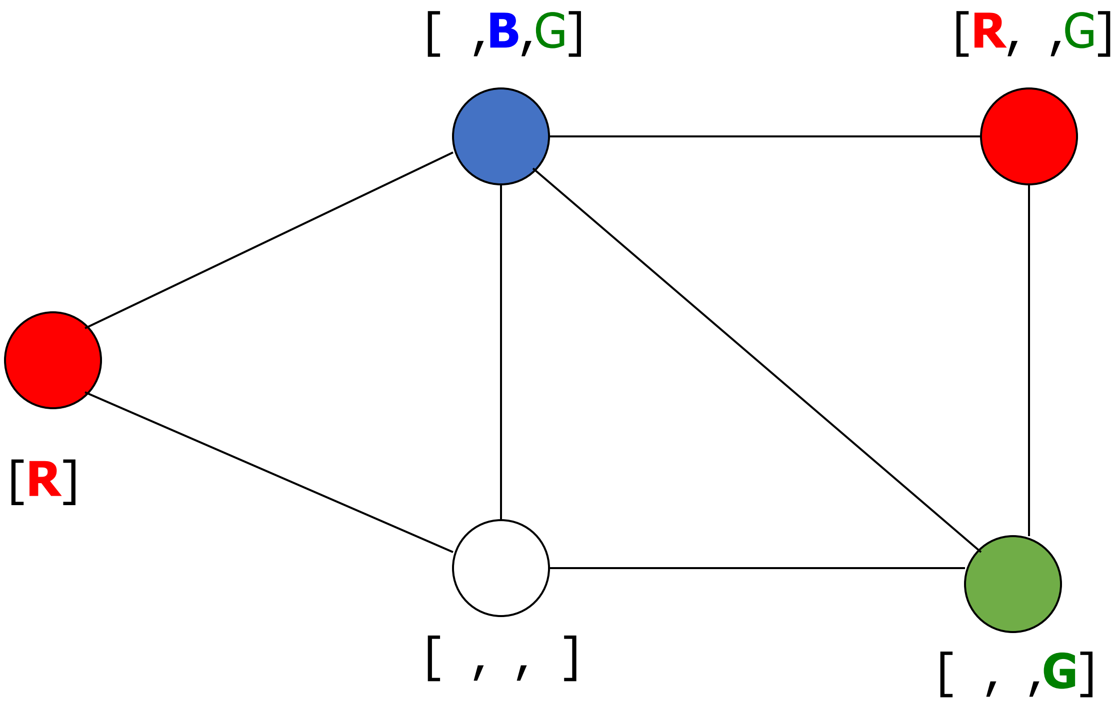

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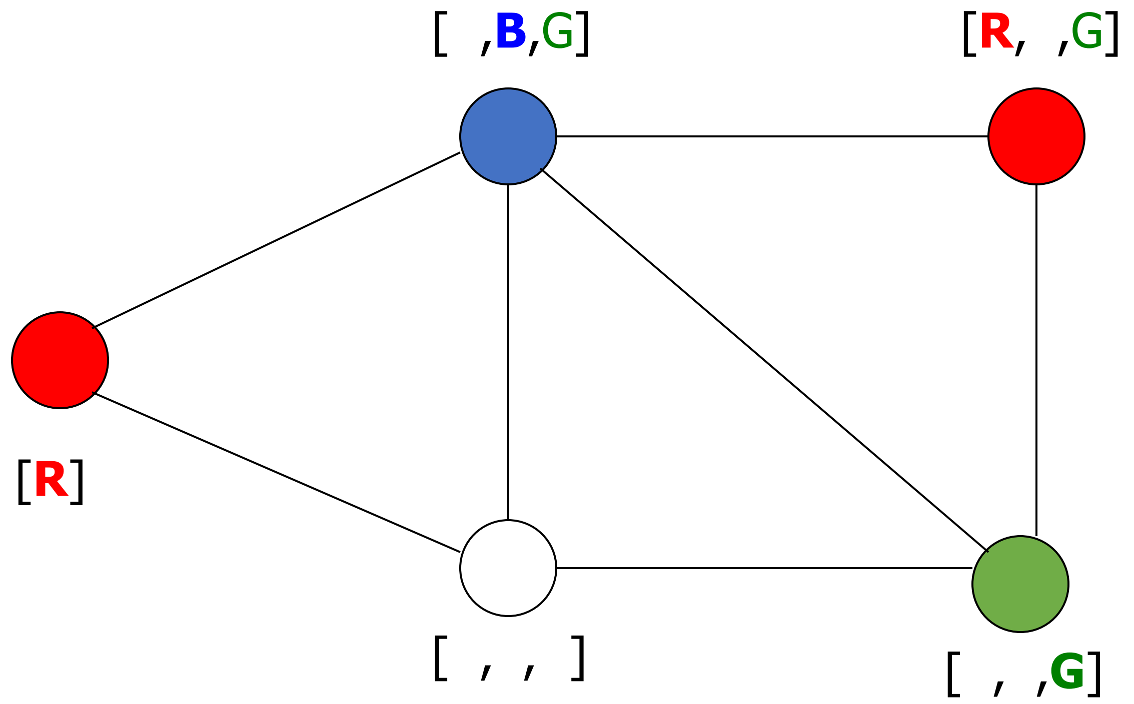
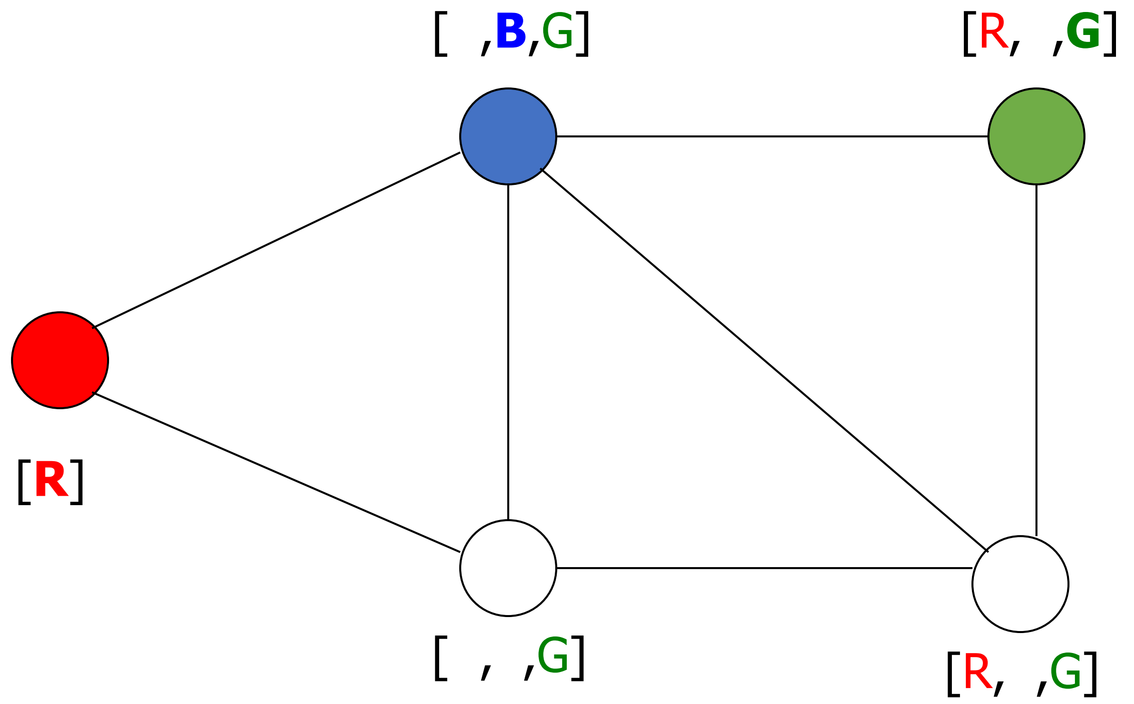


Dead End

Forward Checking Example



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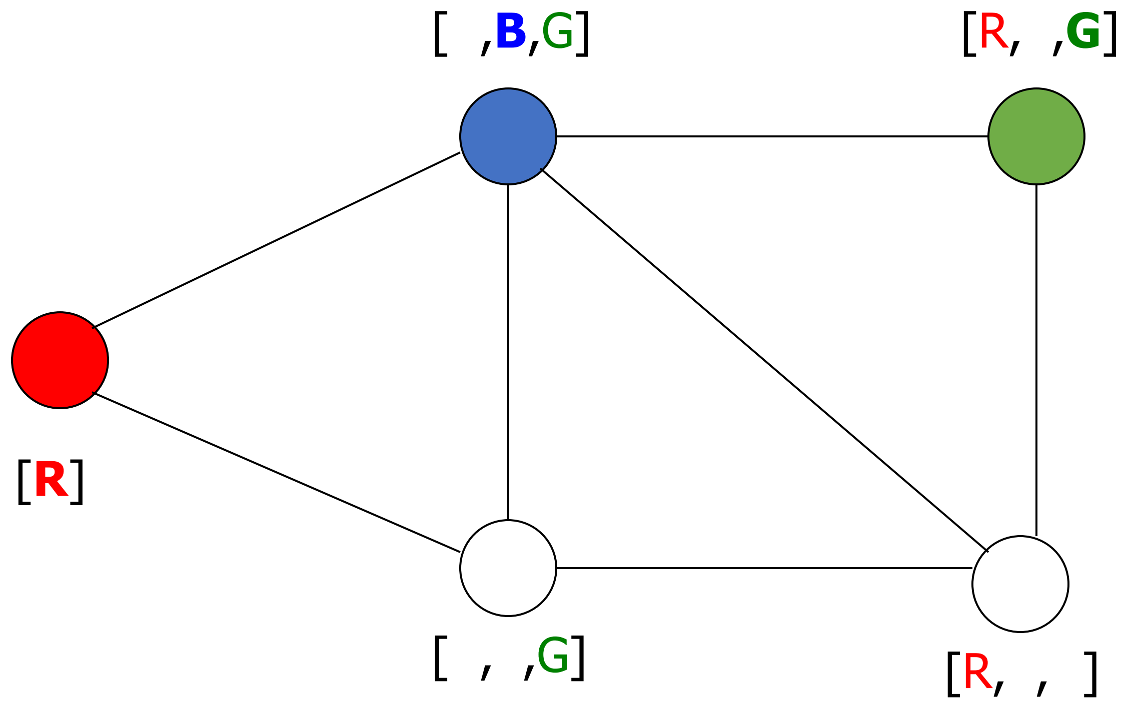

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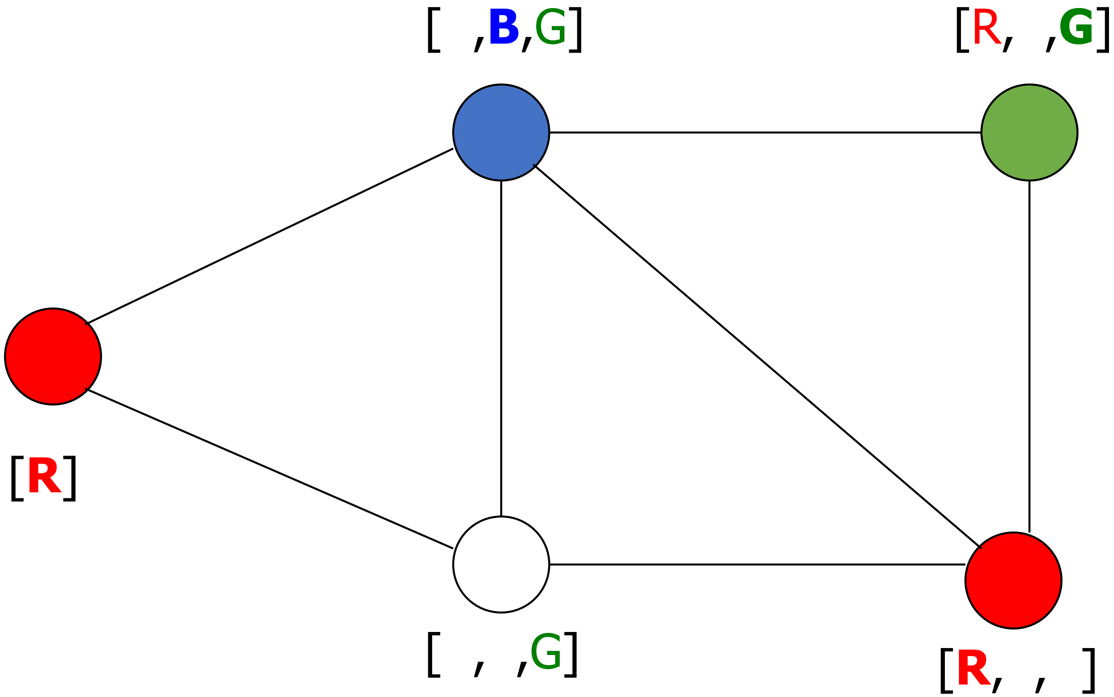
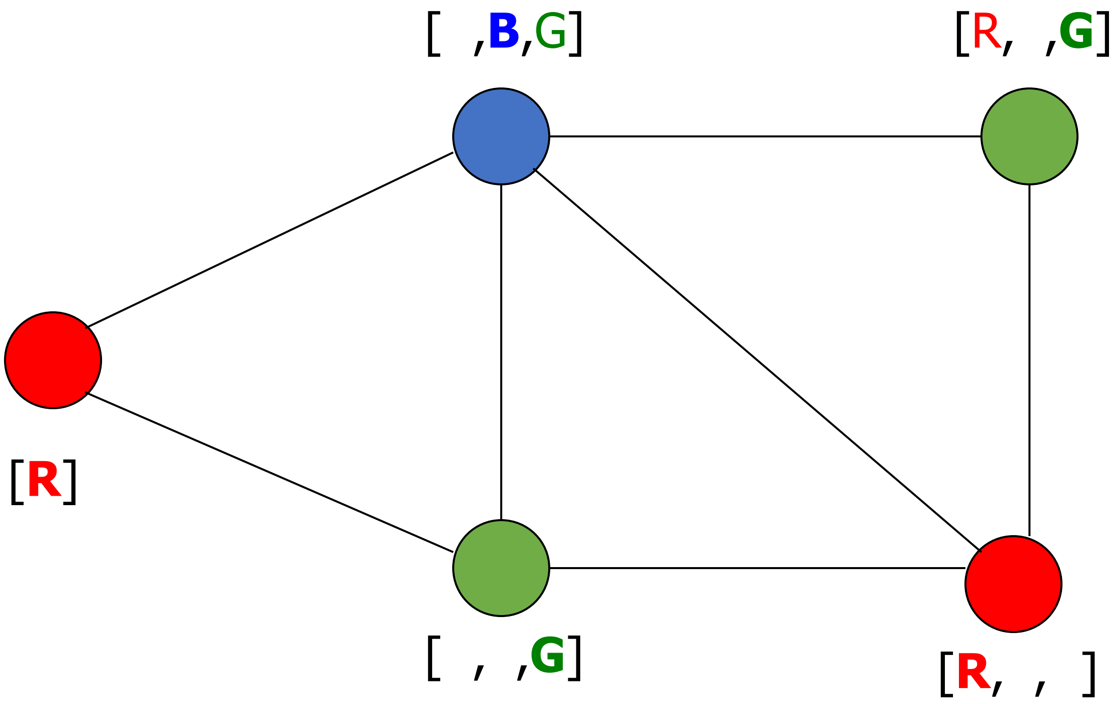
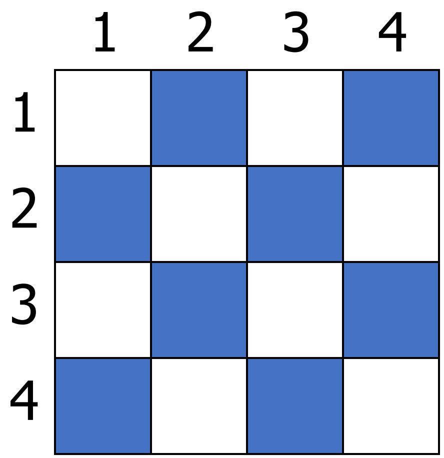


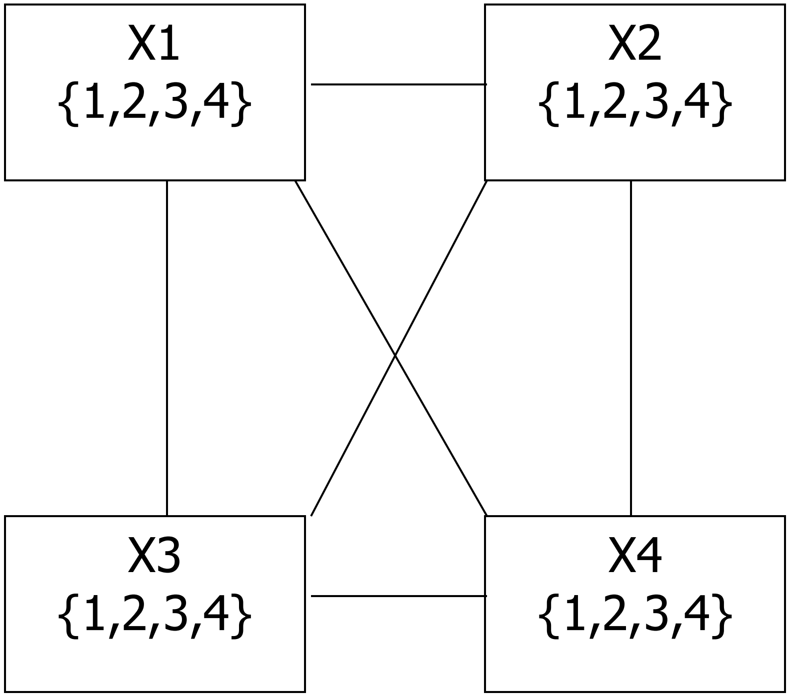
Forward Checking Example



Solution !!!

Example: 4-Queens Problem





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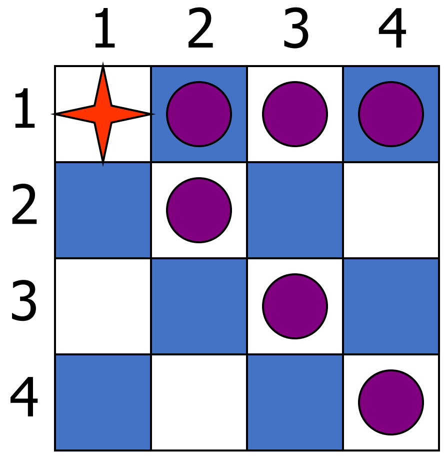

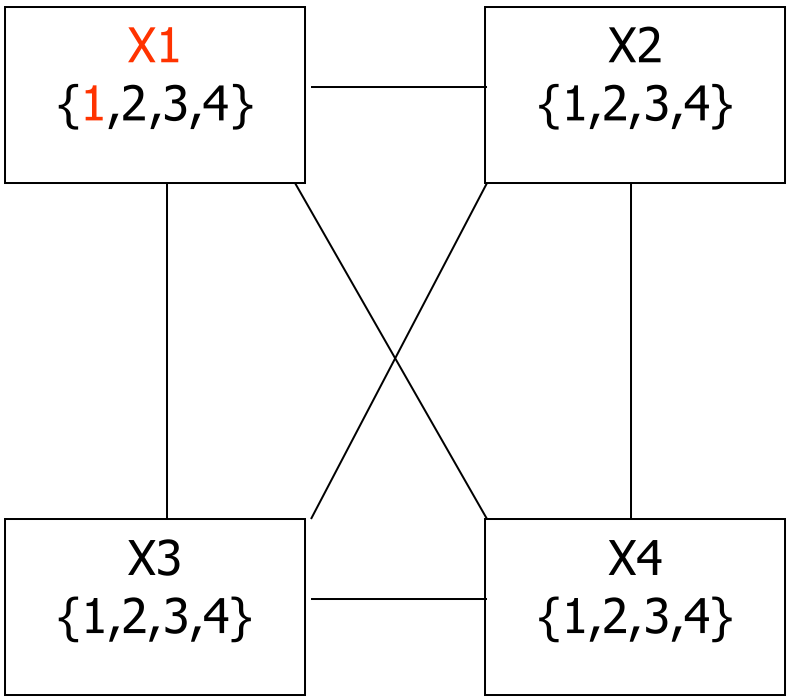

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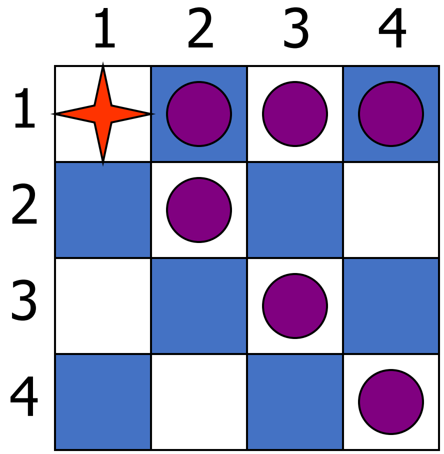

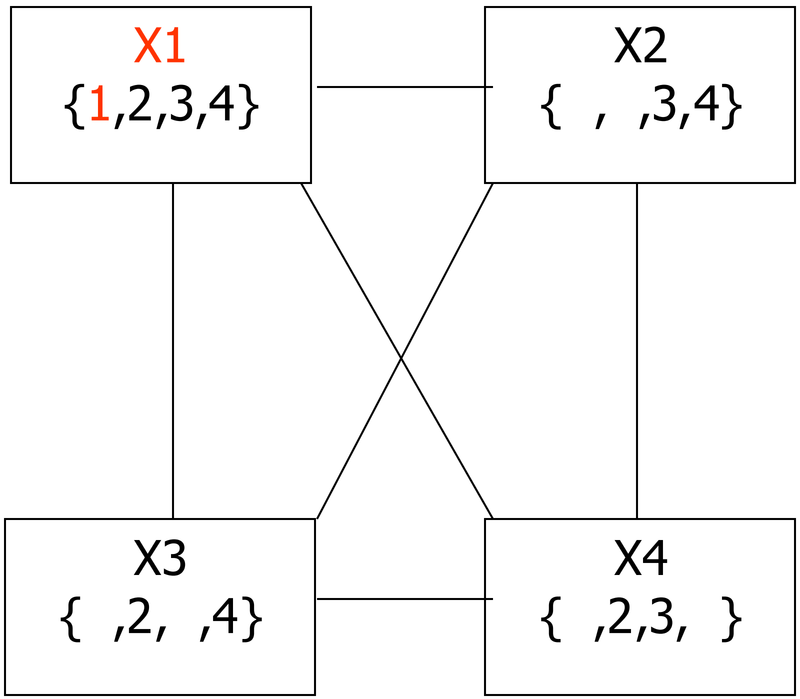

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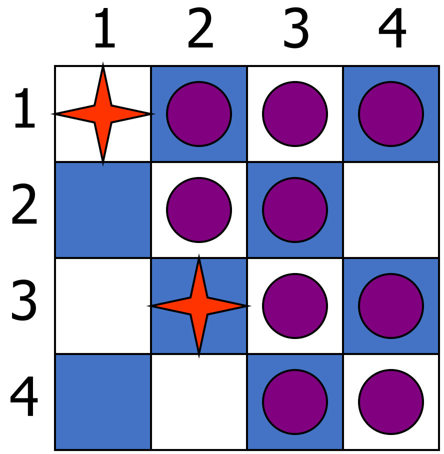

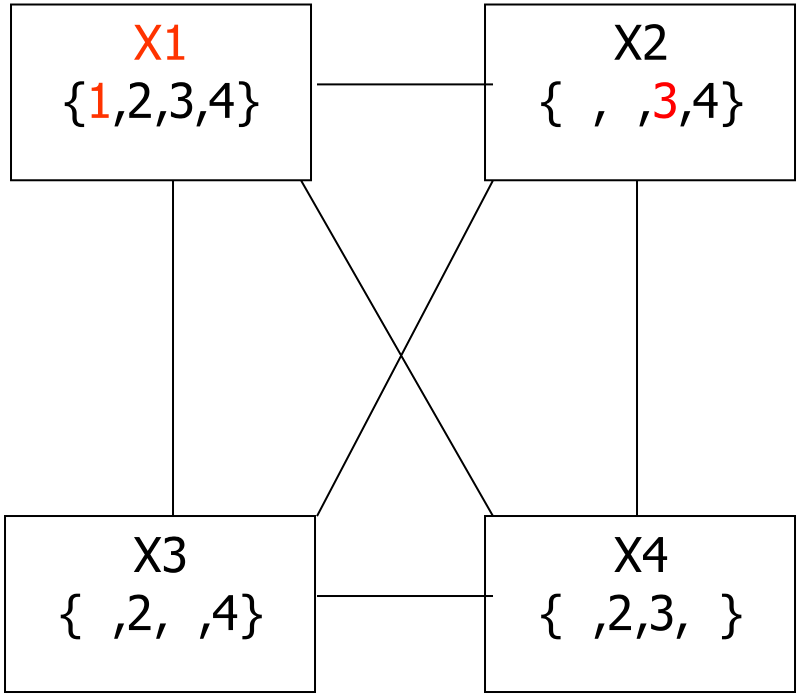

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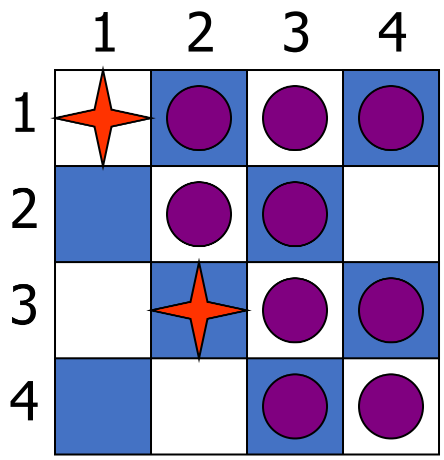
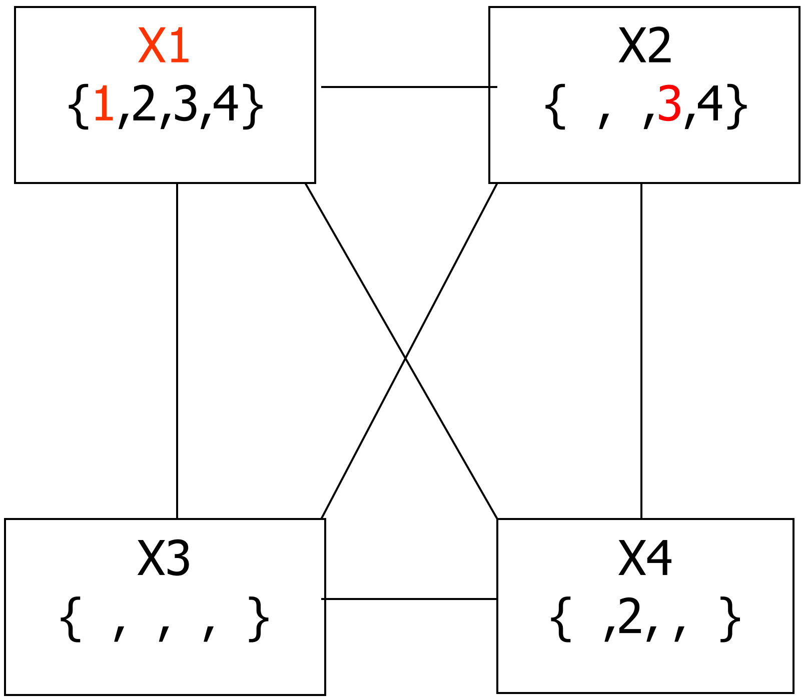
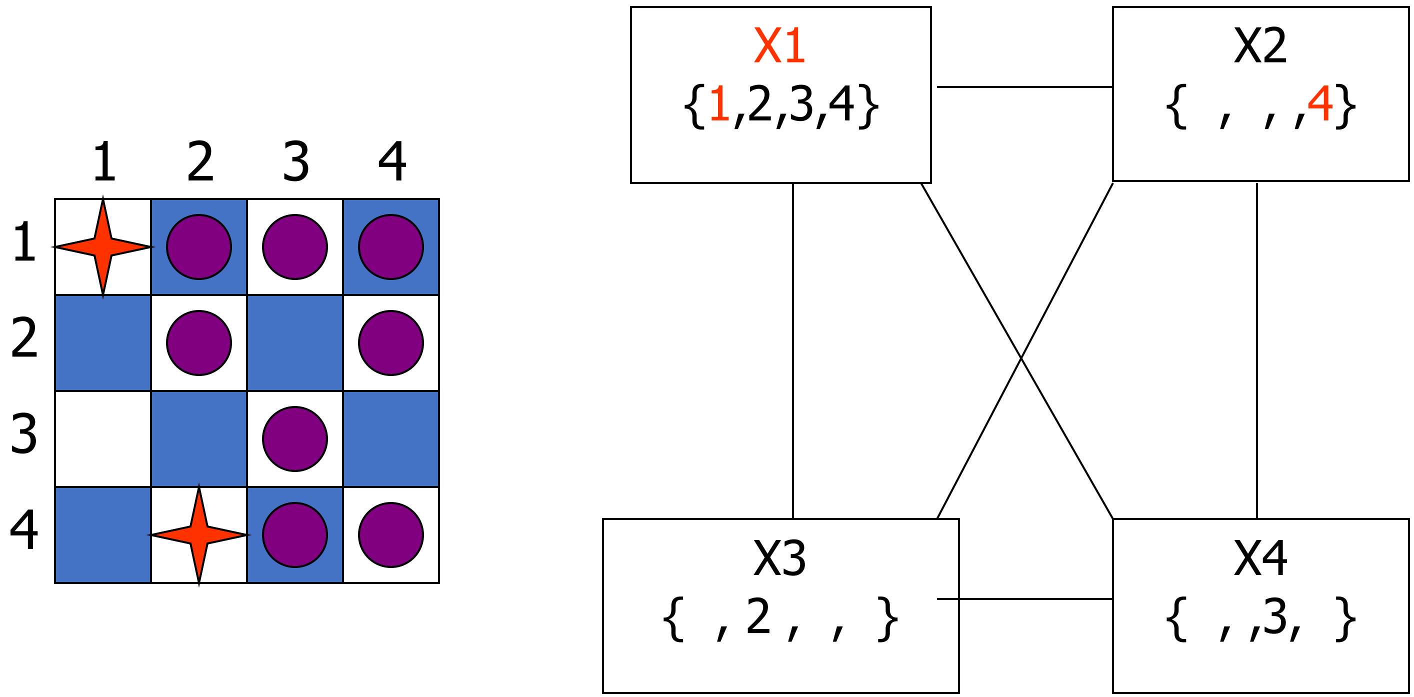




Dead End → Backtrack

Example: 4-Queens Problem



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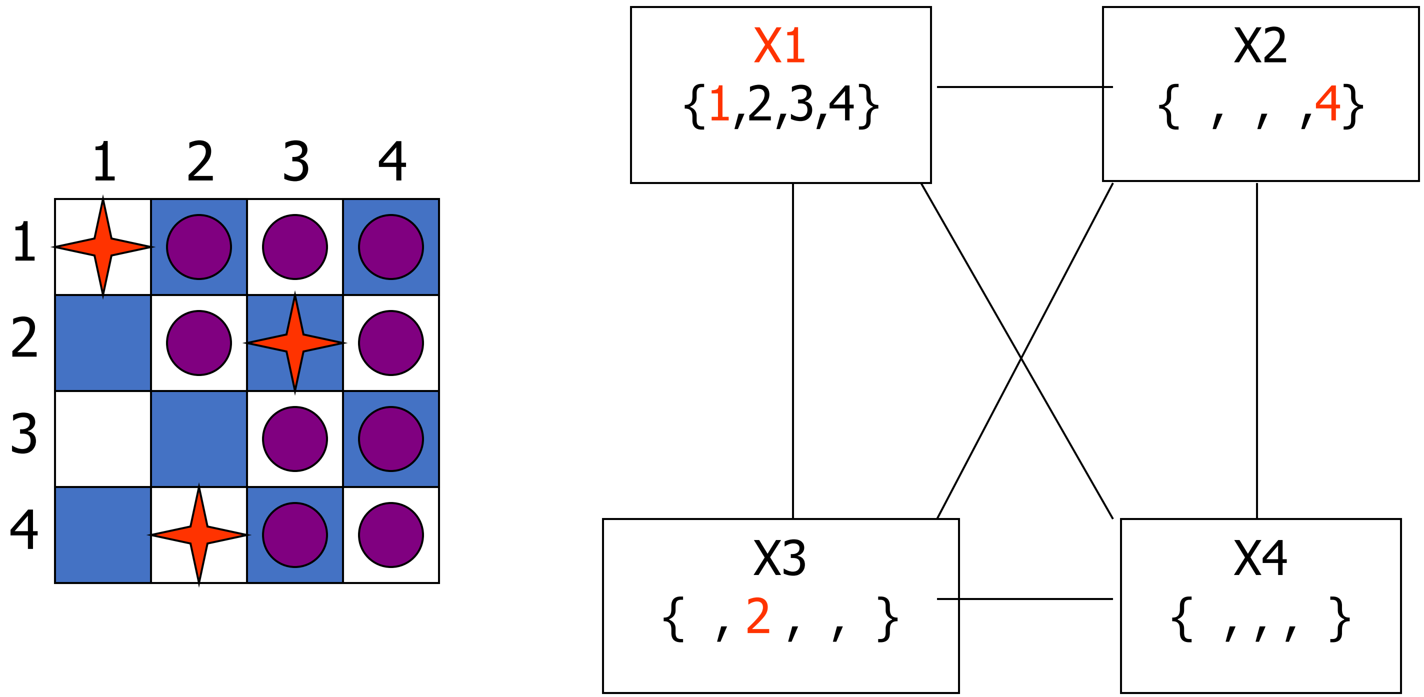

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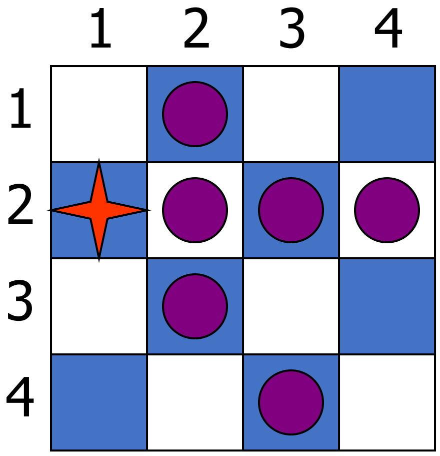

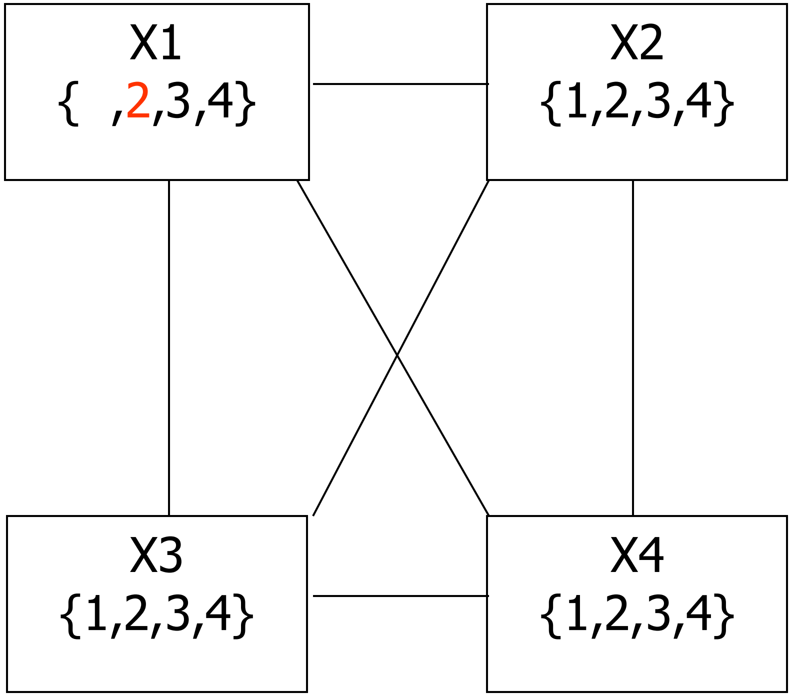

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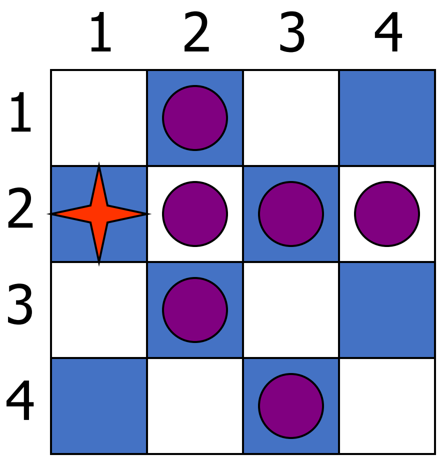

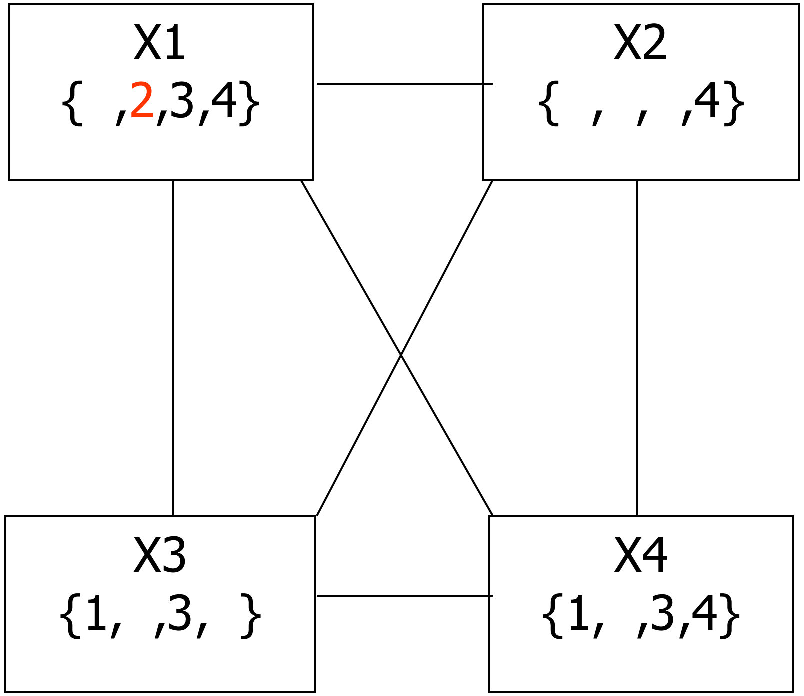

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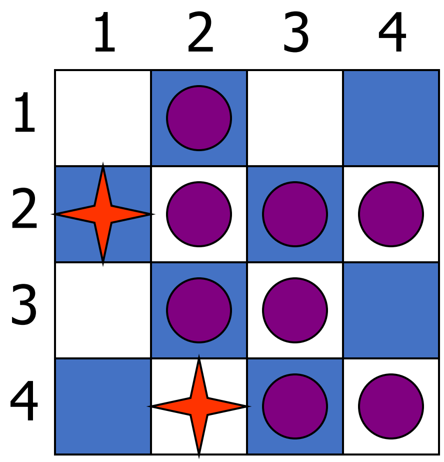

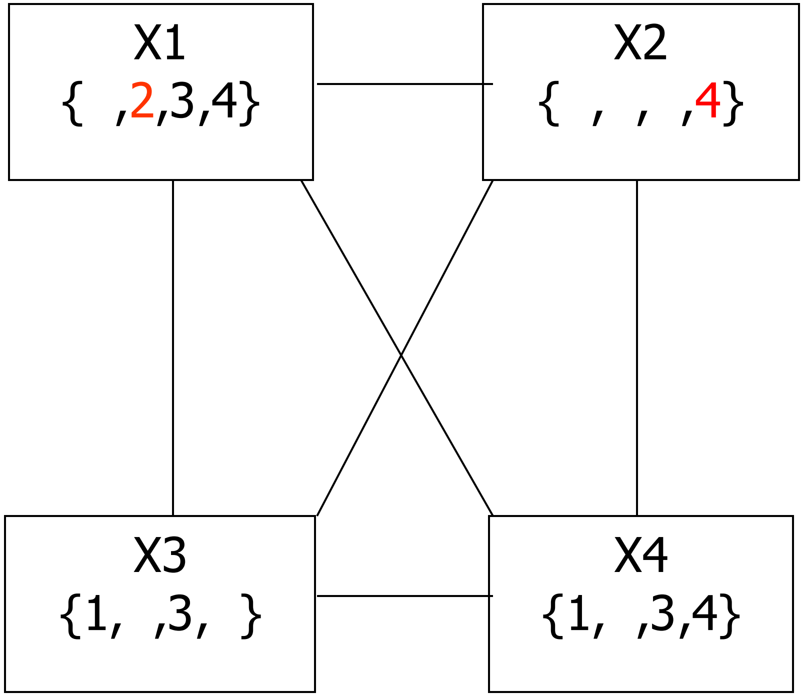

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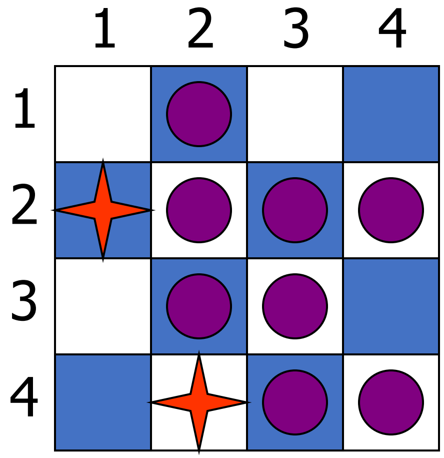

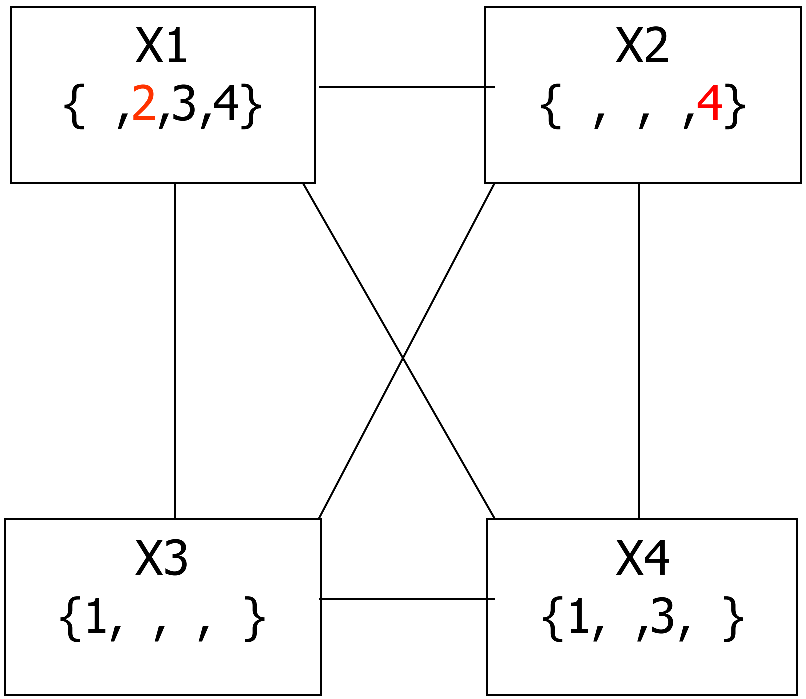

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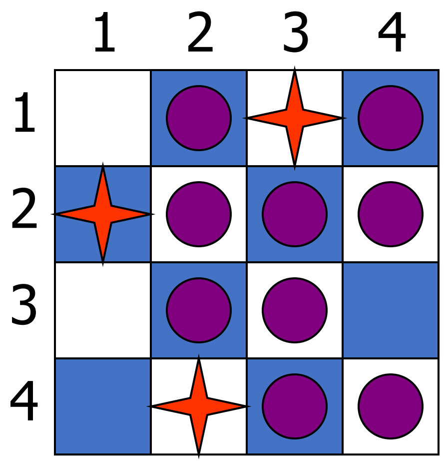

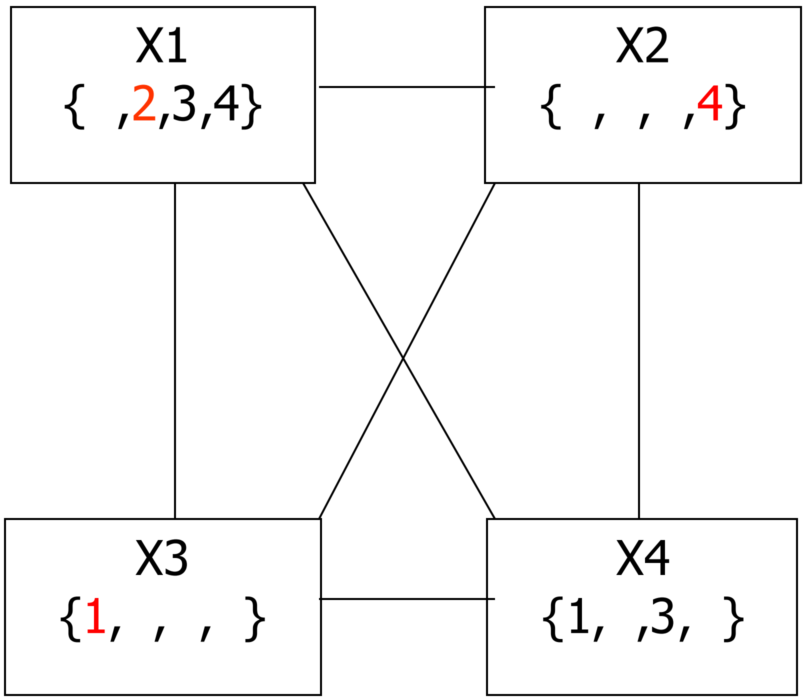

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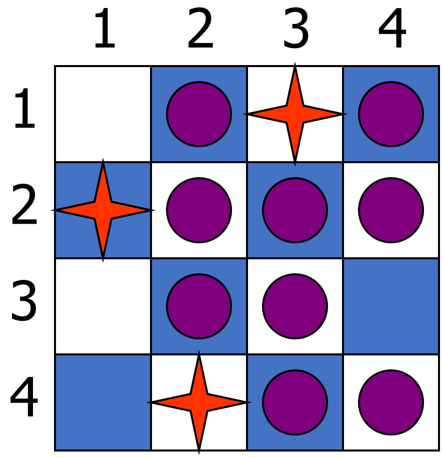
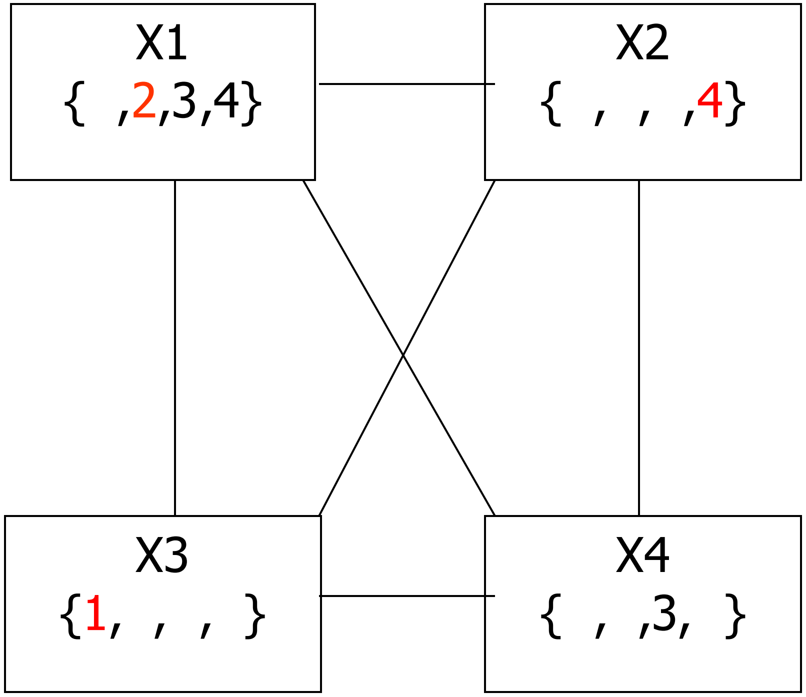
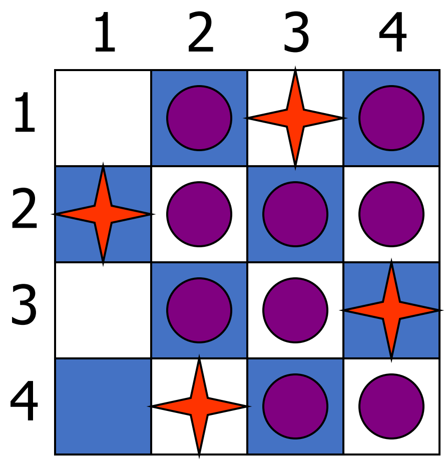
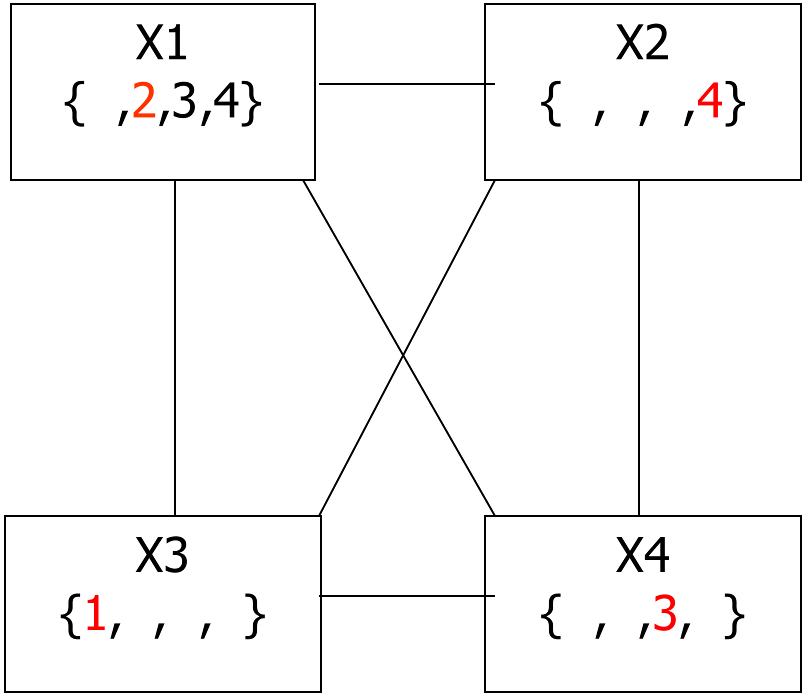




Example: 4-Queens Problem

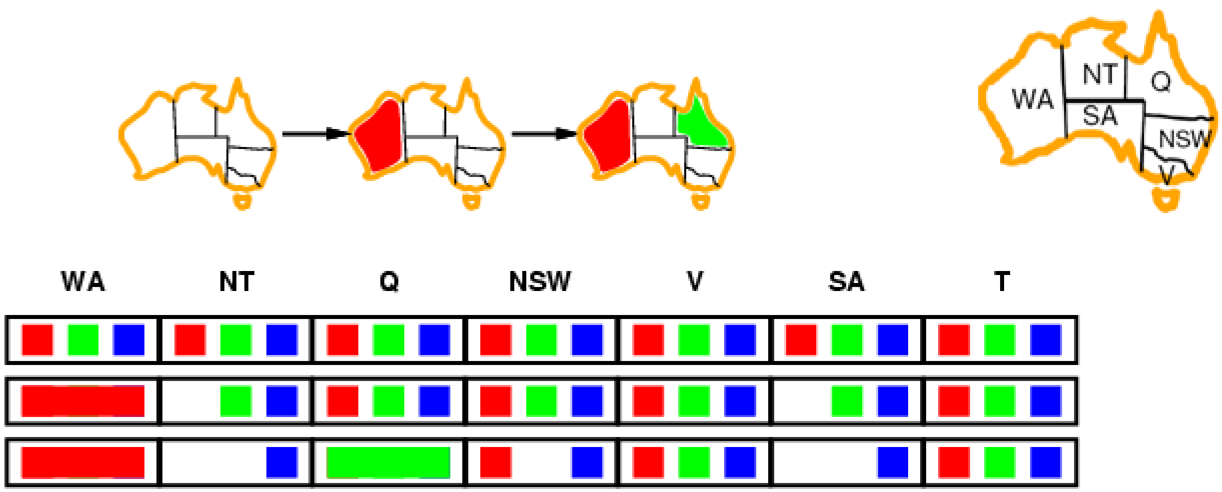




Solution !!!!

Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!

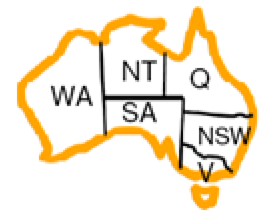
- Constraint propagation repeatedly enforces constraints locally

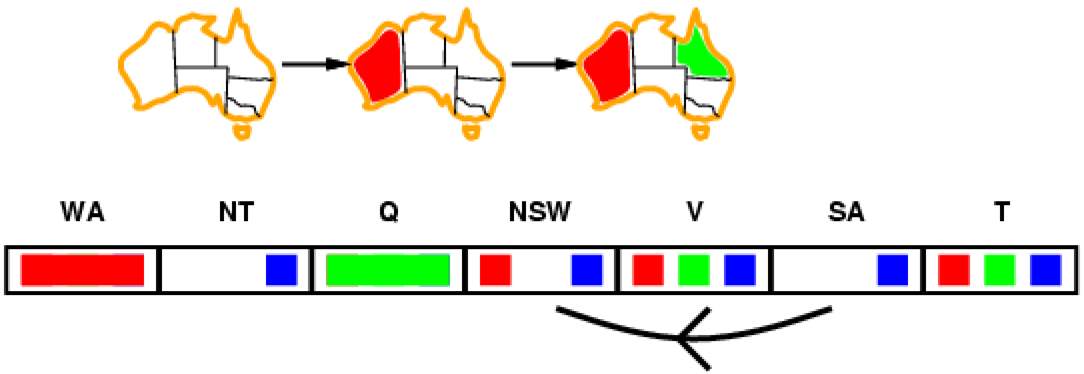
Arc consistency

- Simplest form of propagation makes each arc consistent

- X → Y is consistent iff

  for every value x of X there is some allowed y





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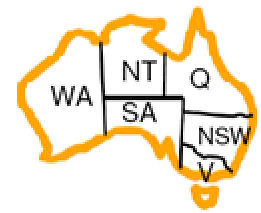

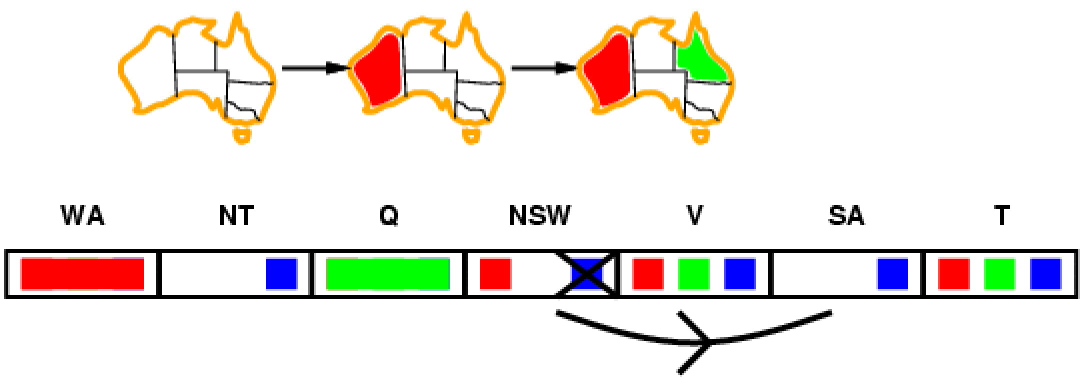

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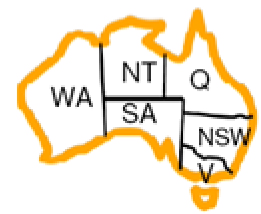

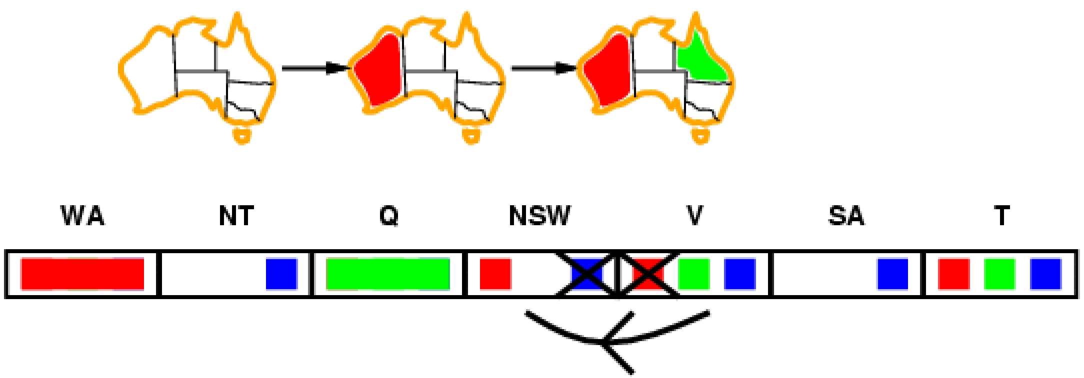

- If X loses a value, neighbors of X need to be rechecked

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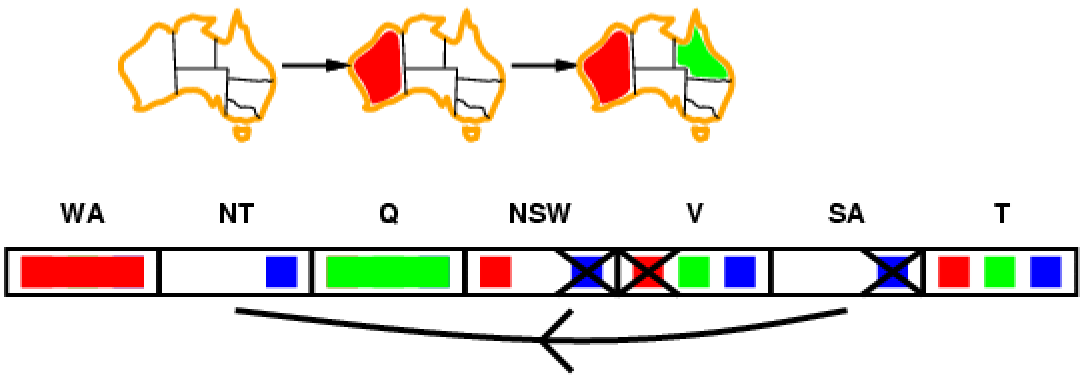
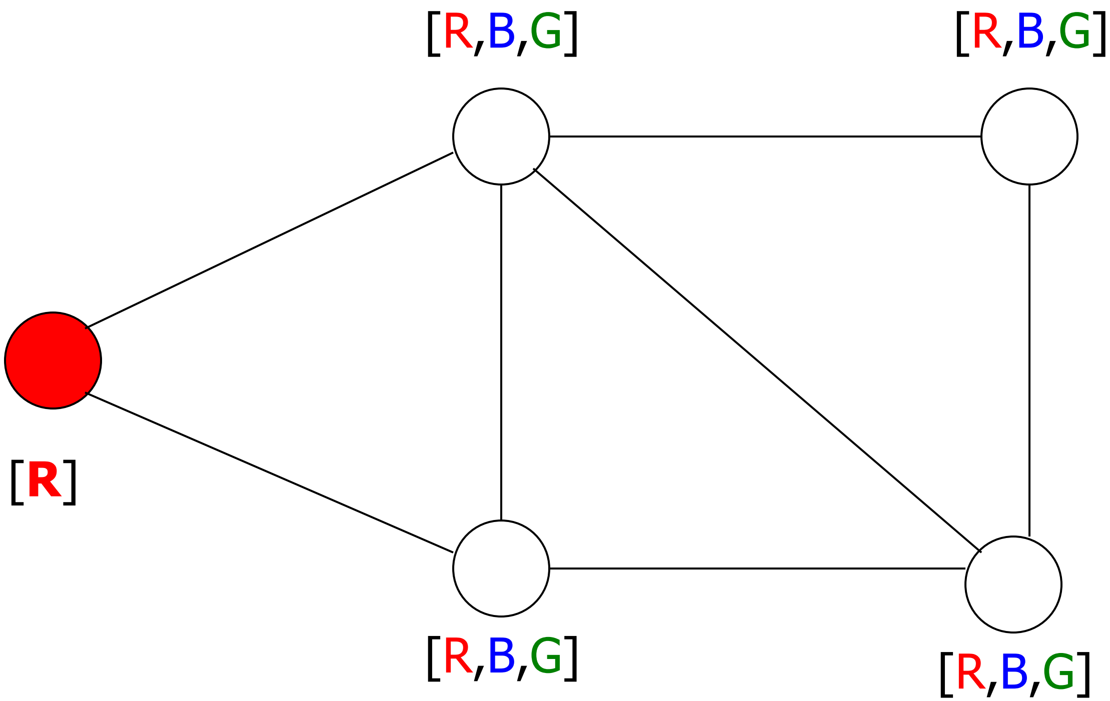


- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc Consistency: AC3



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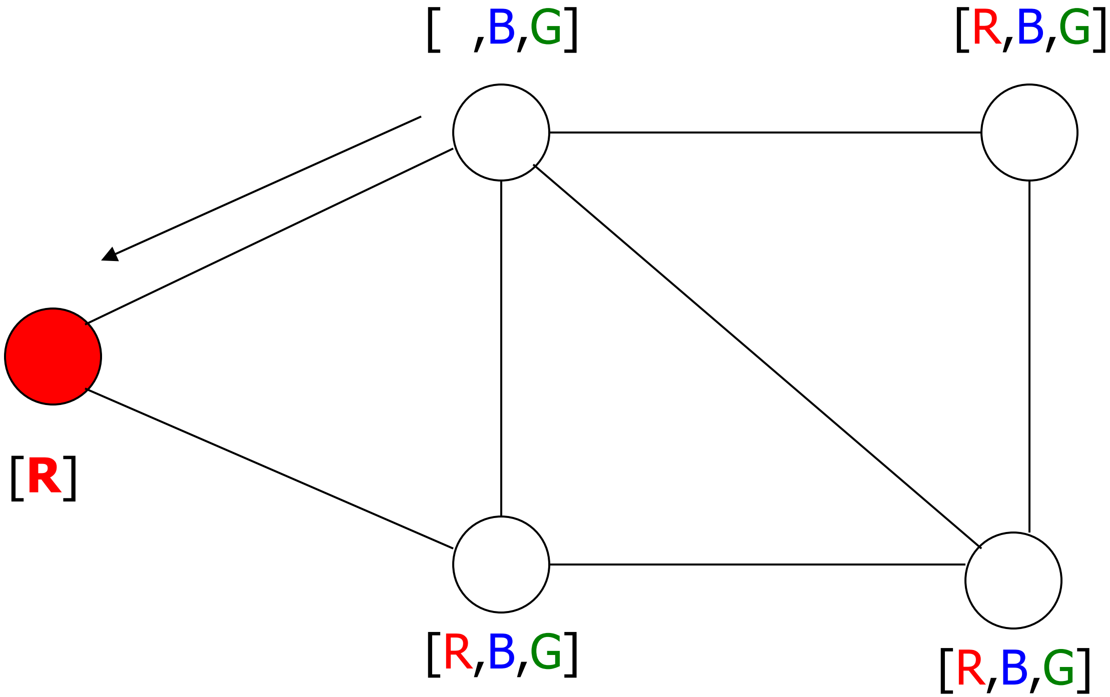

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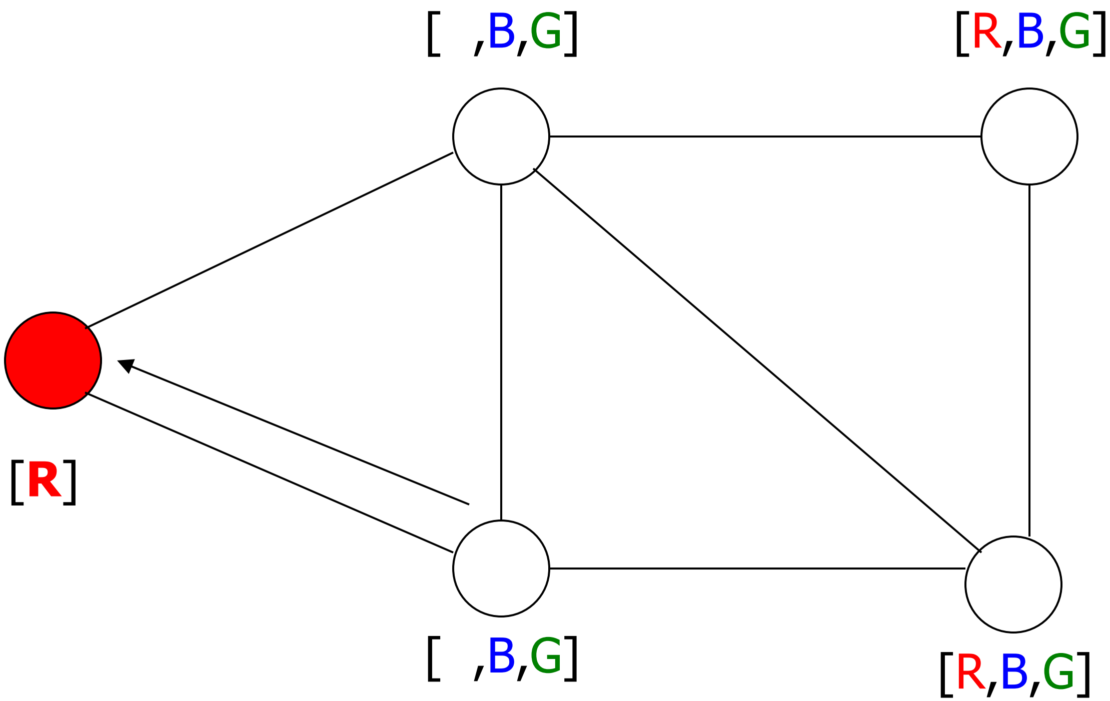

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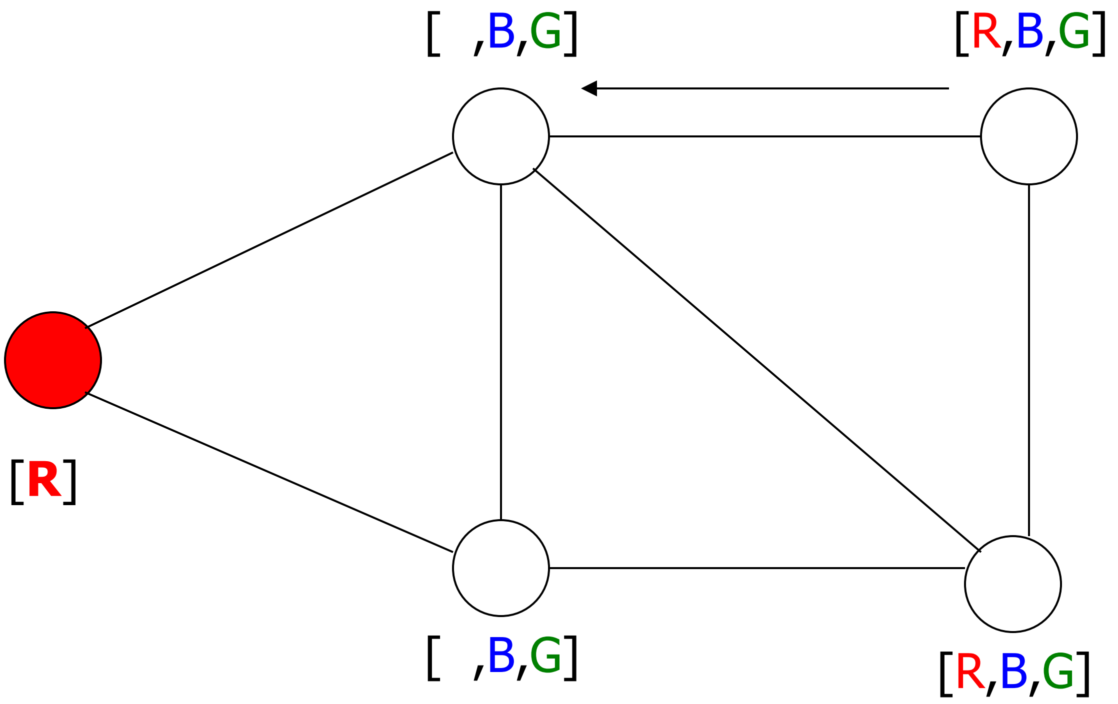

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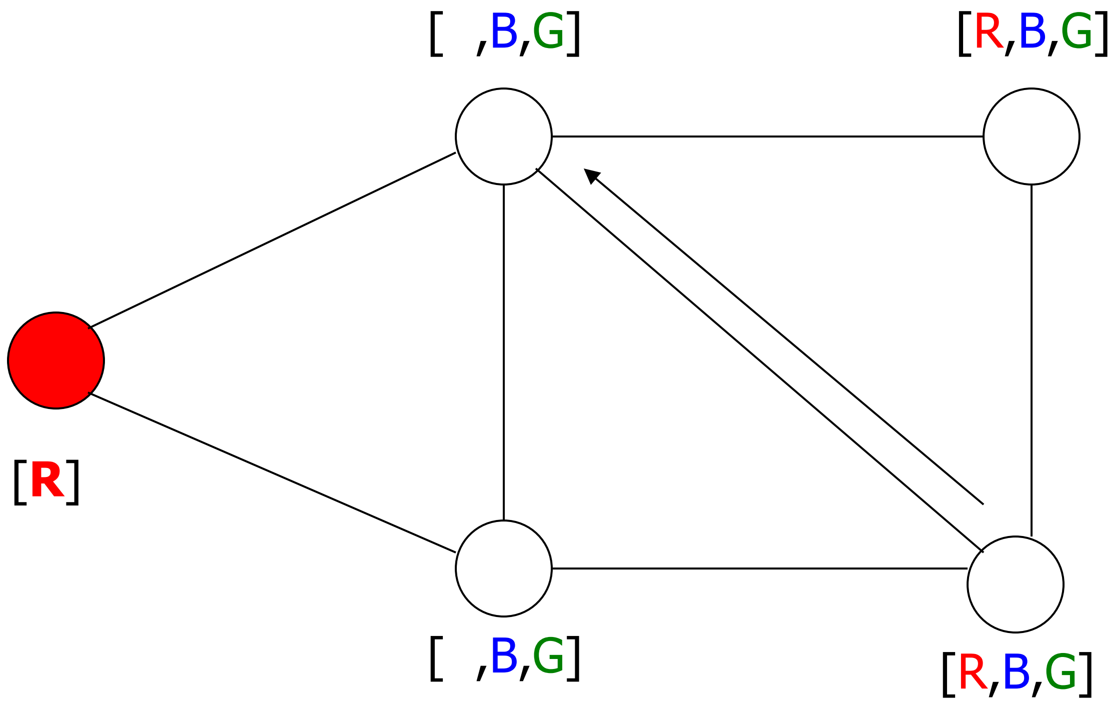

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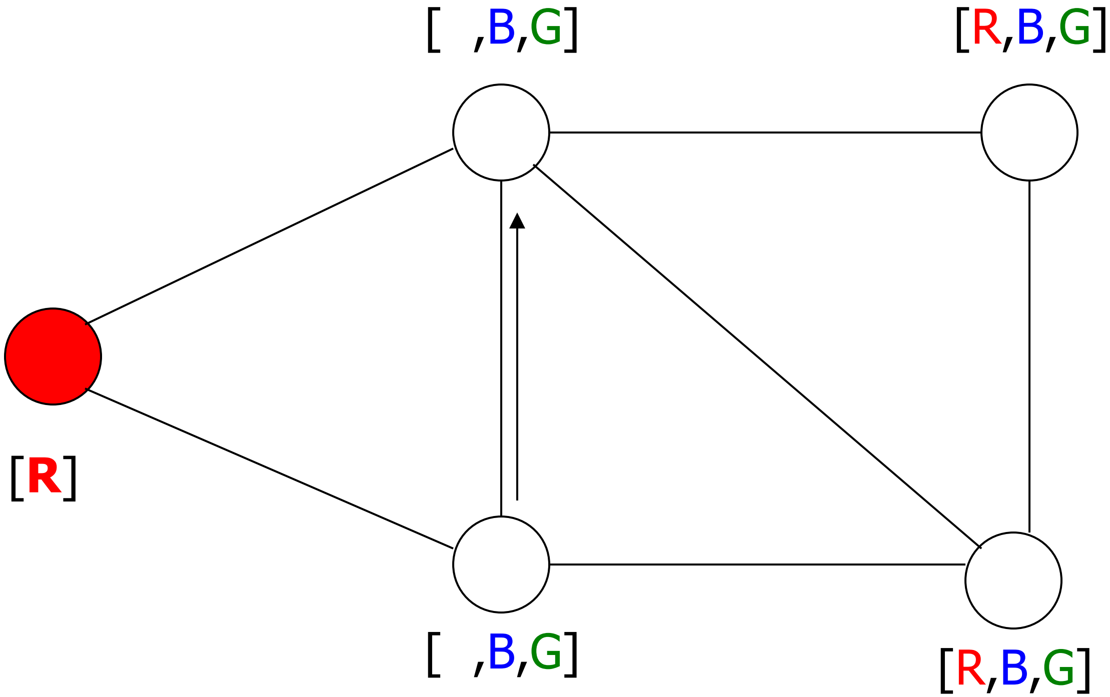

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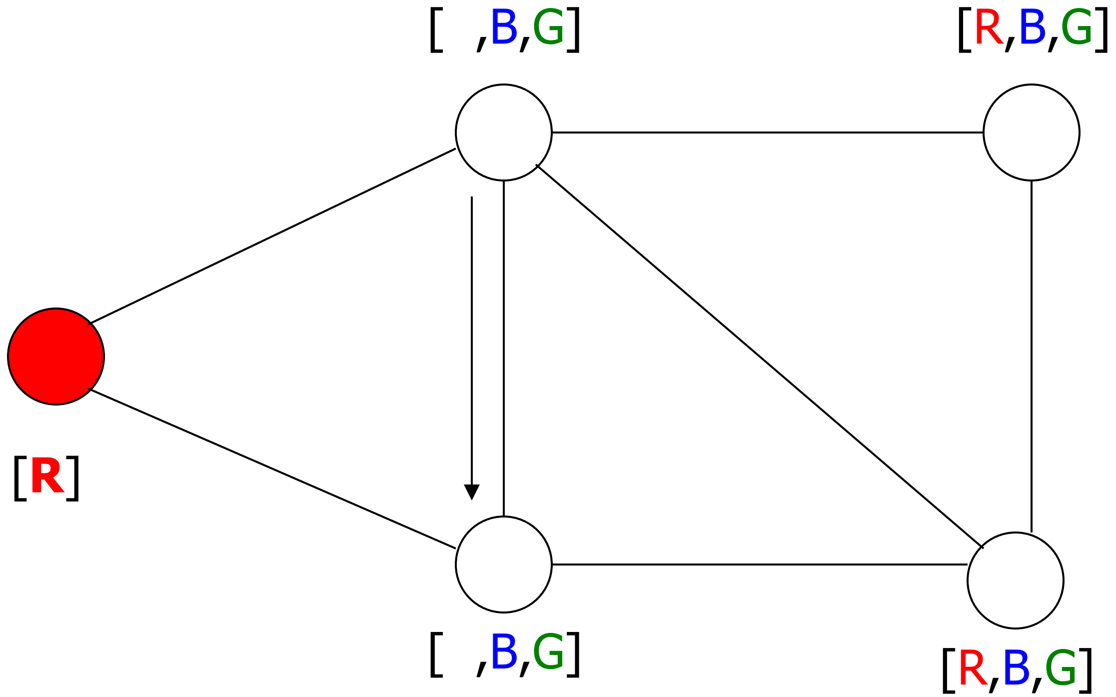

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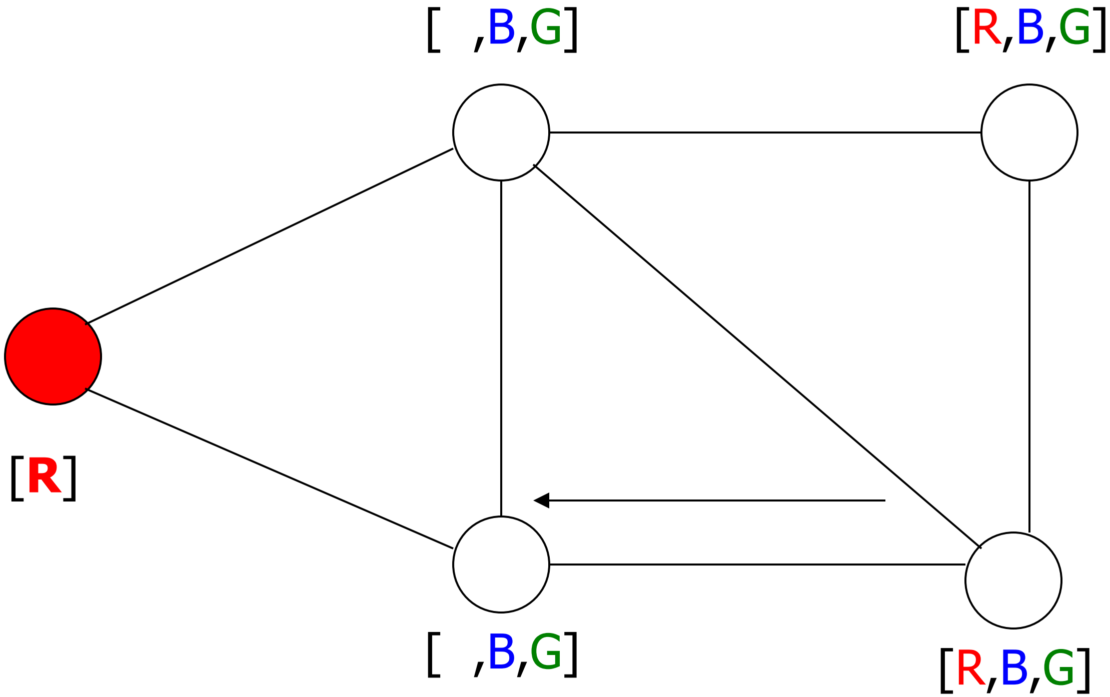

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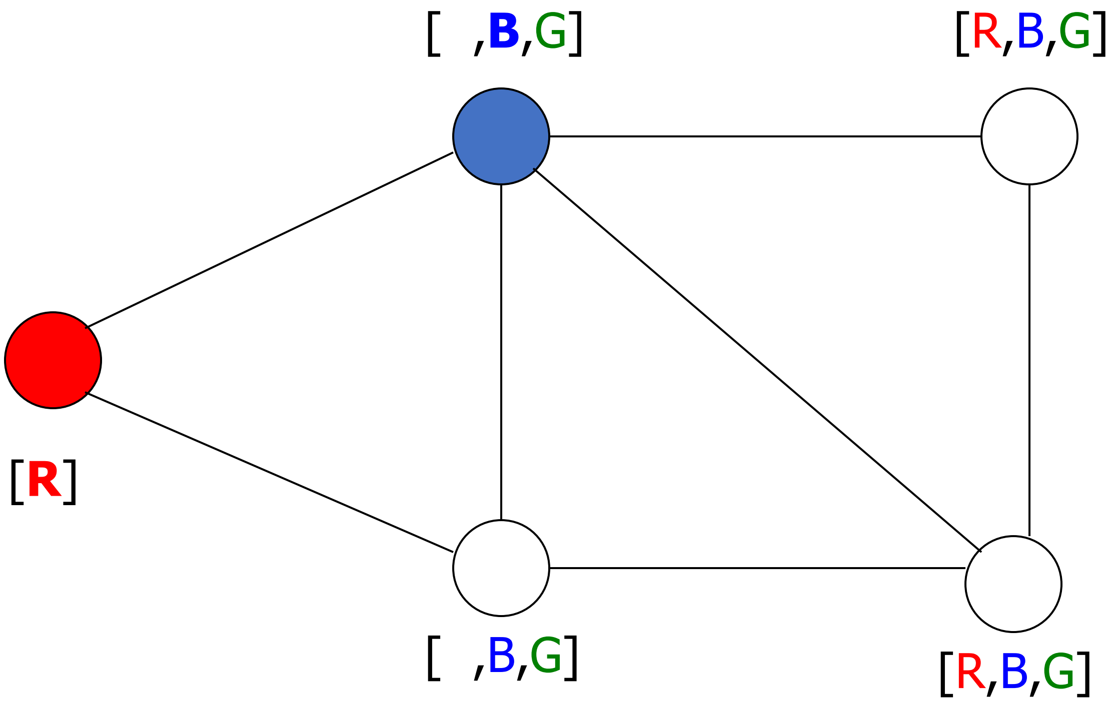

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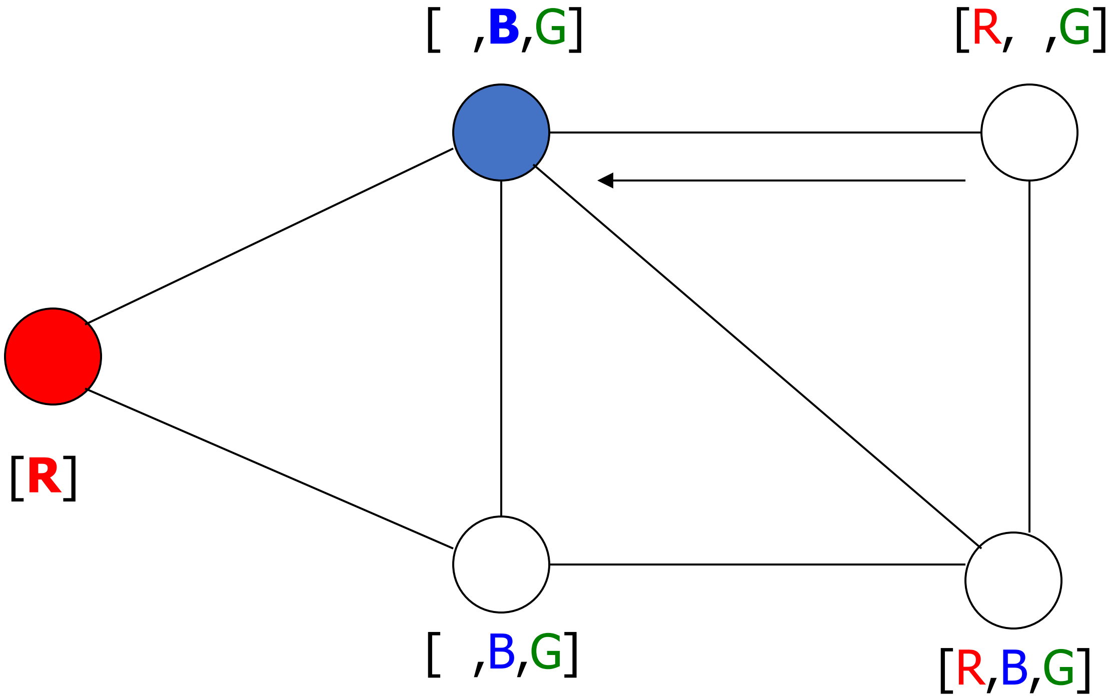

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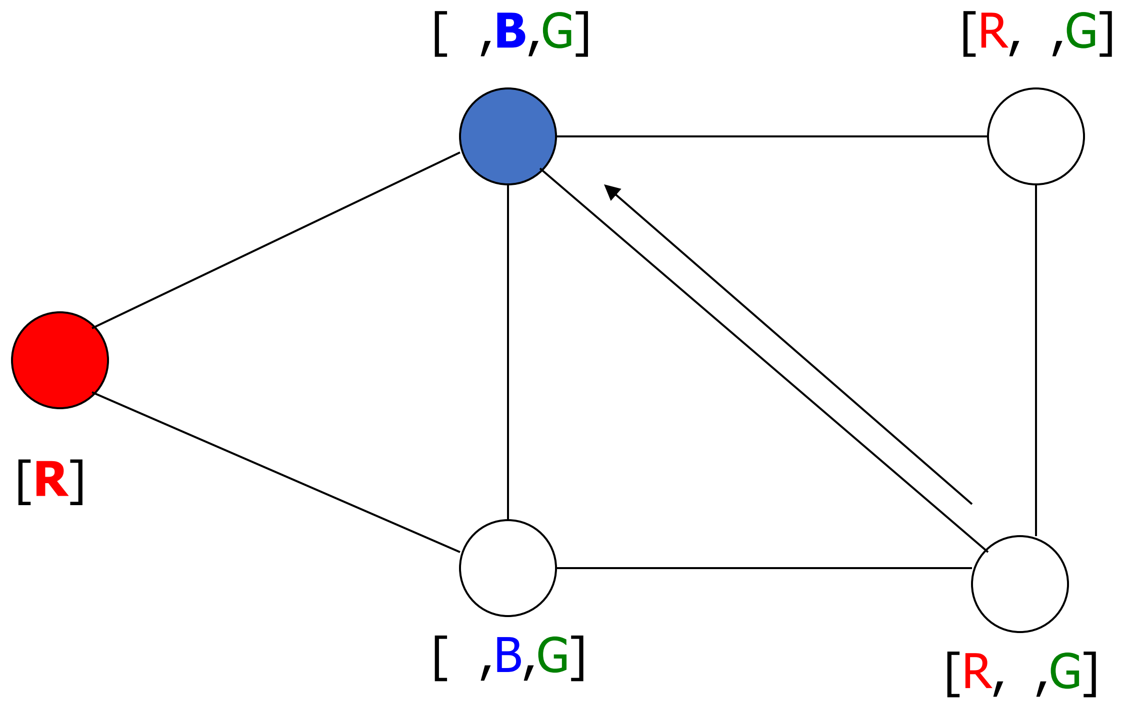

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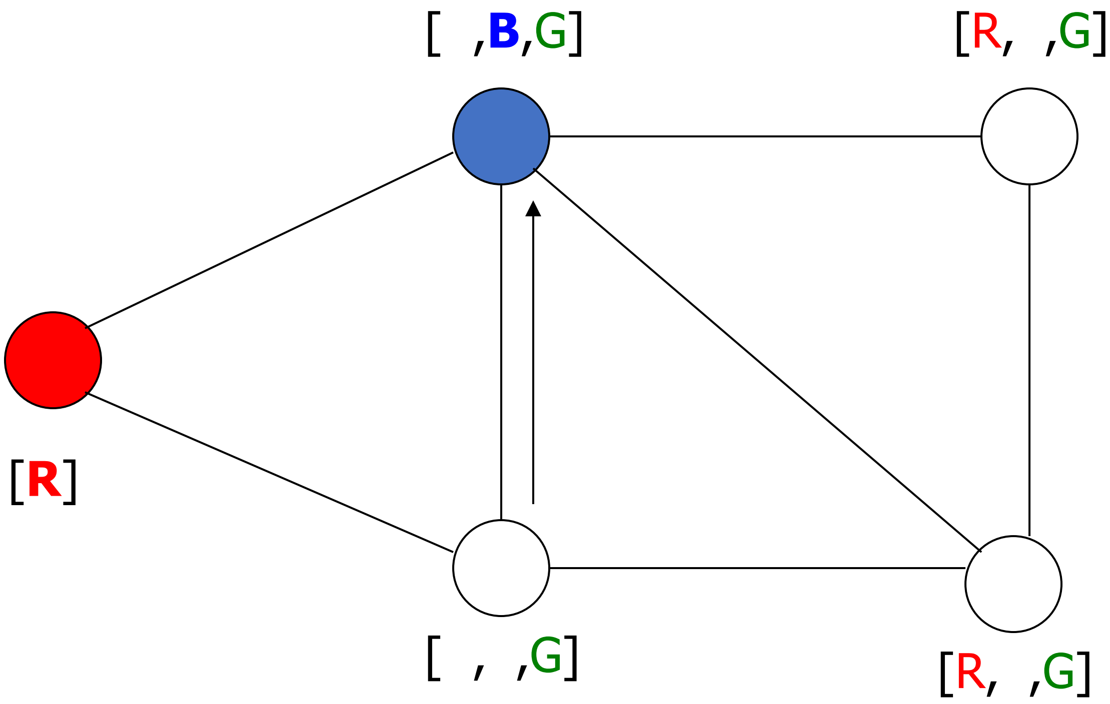

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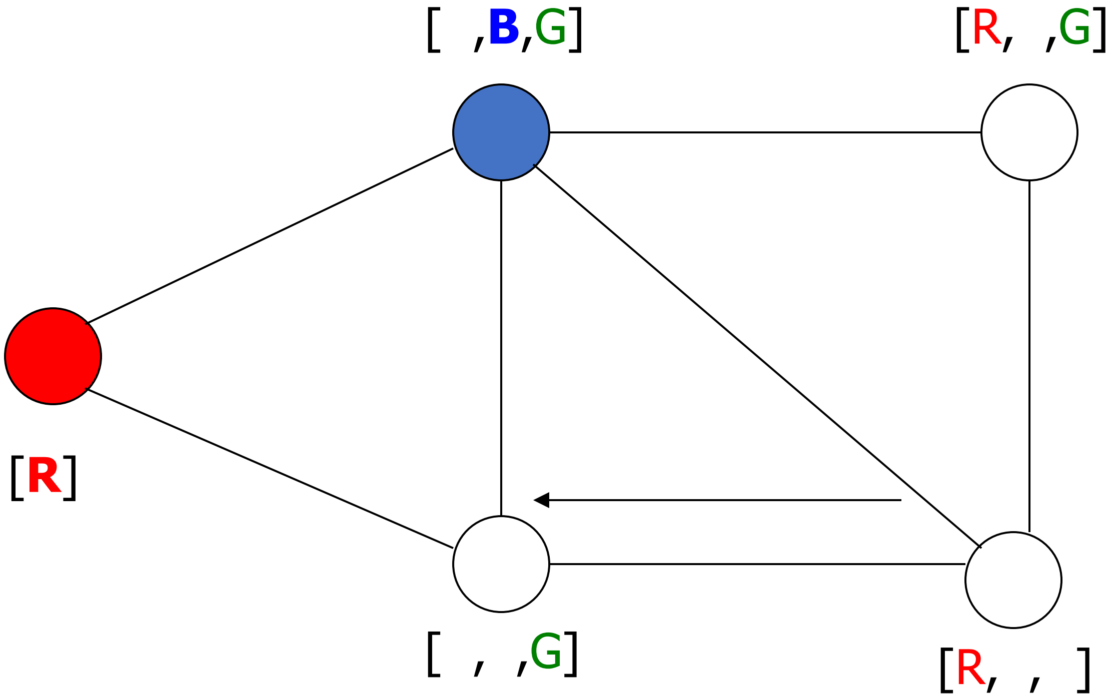

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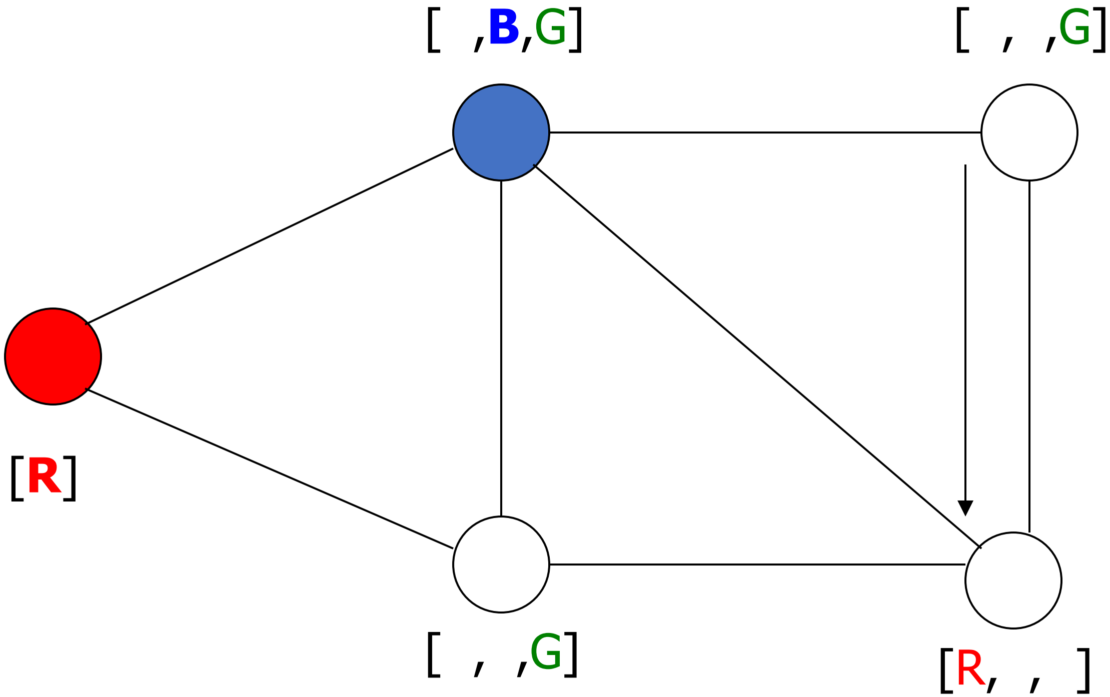

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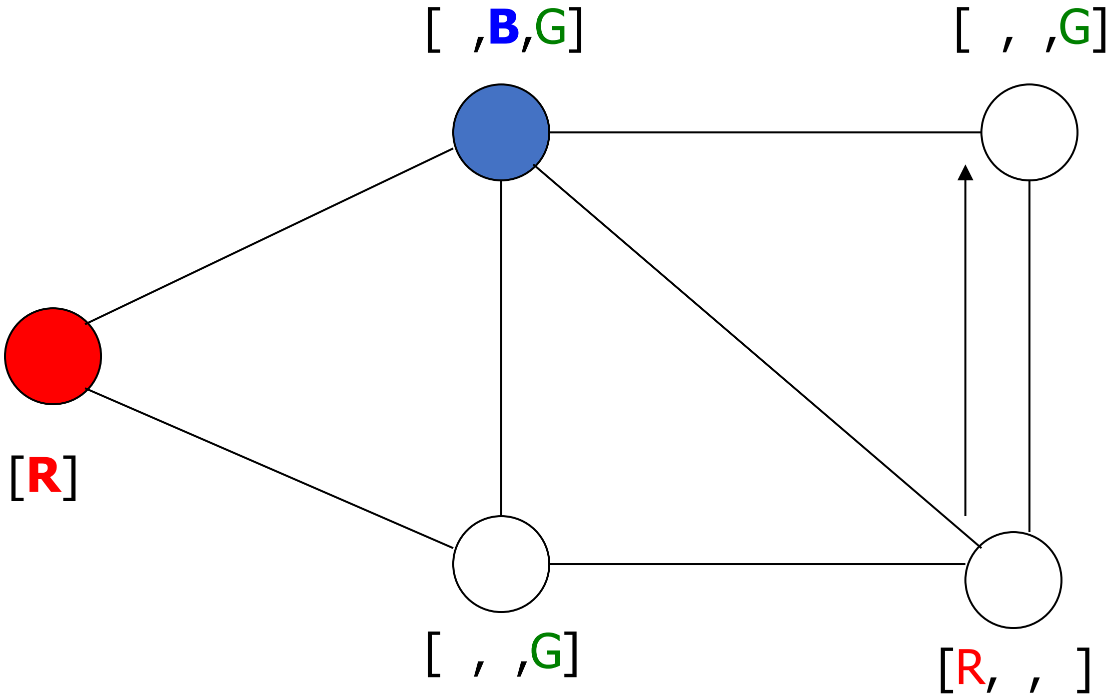

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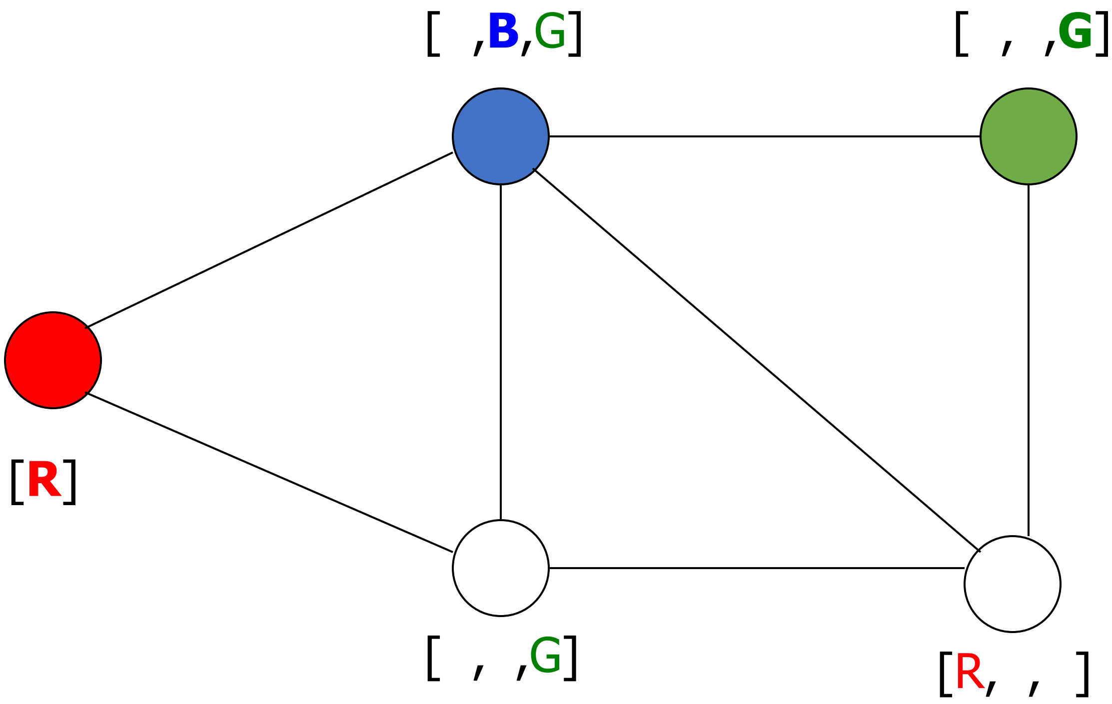

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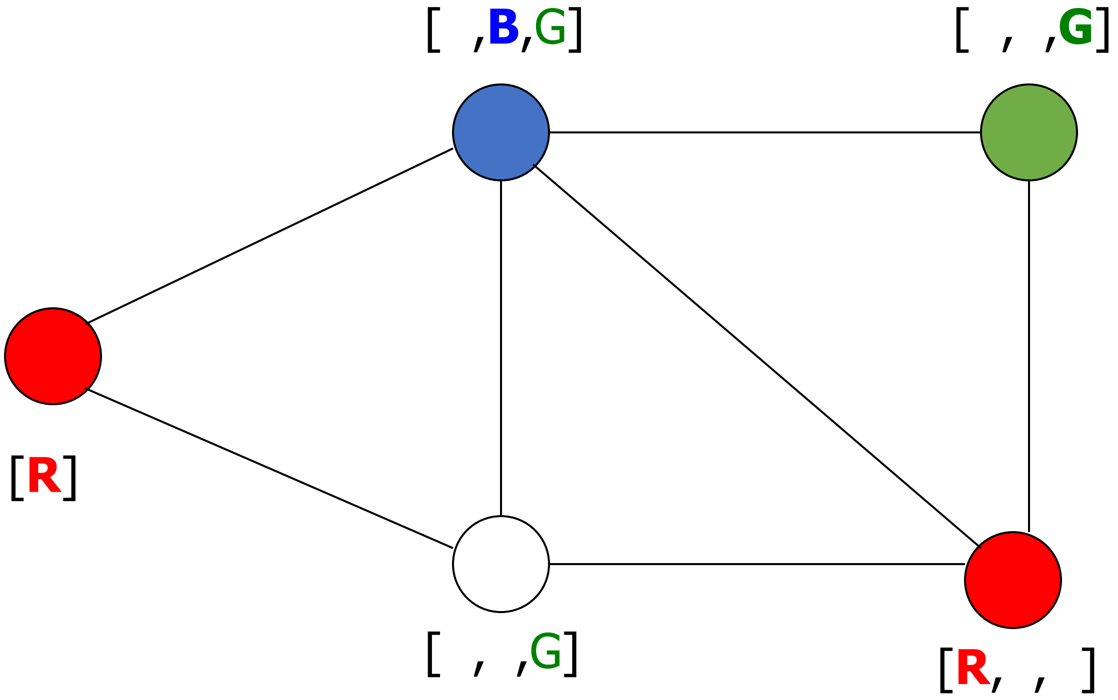

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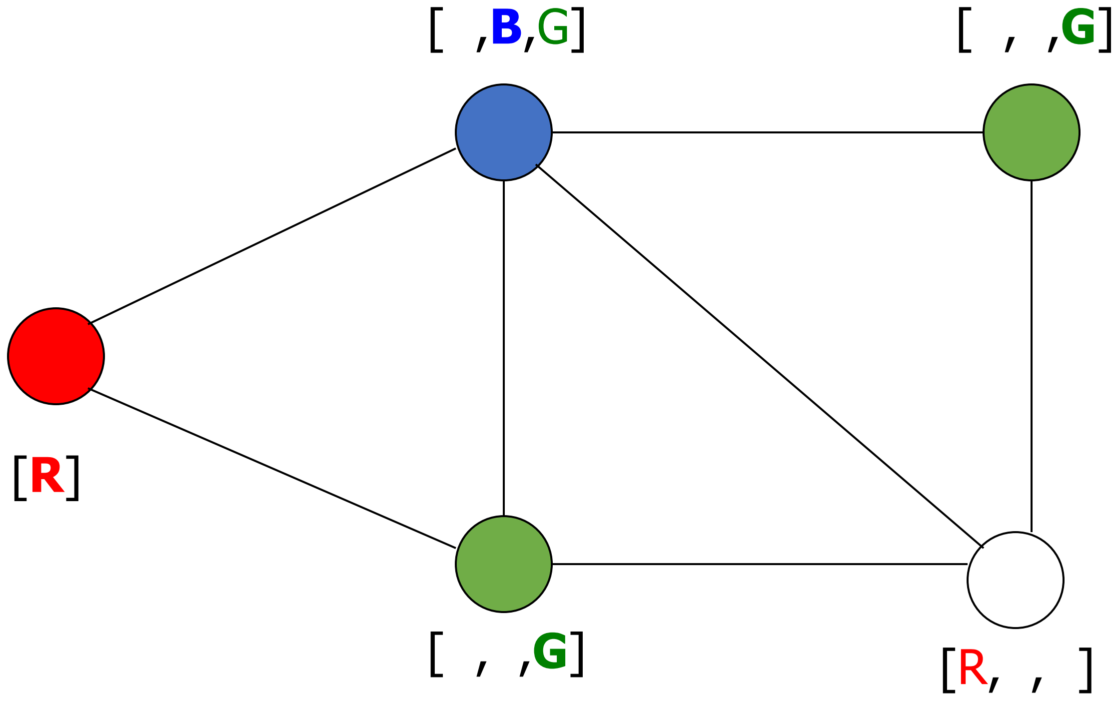


Solution !!!

Local Search and CSP

- local search (iterative improvement) is frequently used for constraint satisfaction problems
  - values are assigned to all variables
  - modification operators move the configuration towards a solution
- often called heuristic repair methods
  - repair inconsistencies in the current configuration
- simple strategy: min-conflicts
  - minimizes the number of conflicts with other variables
  - solves many problems very quickly
  - million-queens problem in less than 50 steps
- can be run as online algorithm
  - use the current state as new initial state

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

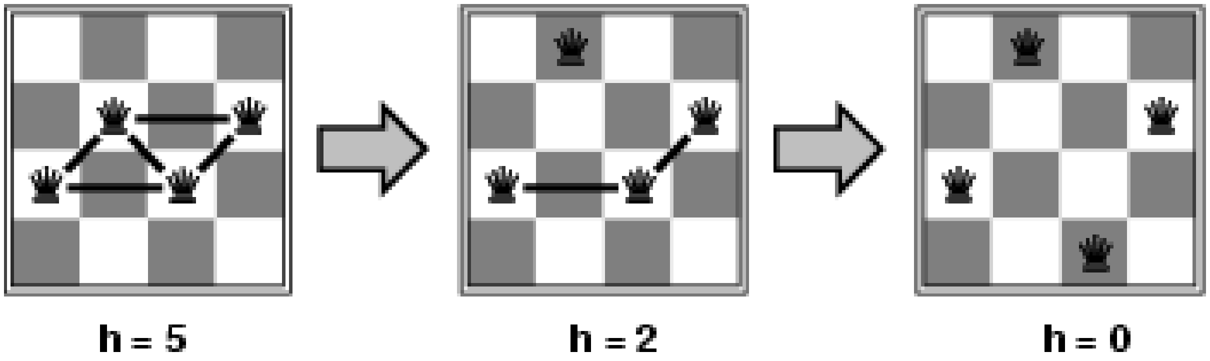
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values

- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with h(n) = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks



- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)