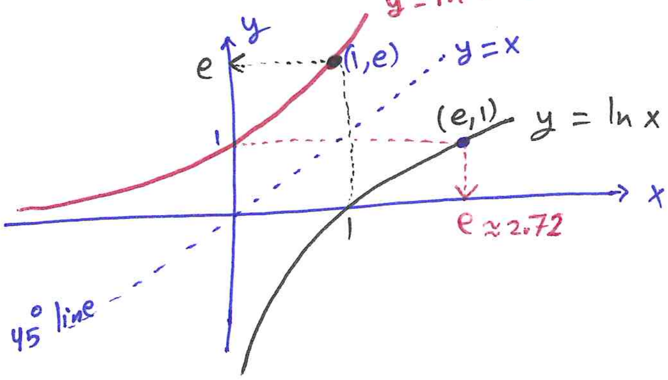
7.3 Exponential Functions

Def: For every real number x, the natural exponential function is $e^x = \exp x = \ln^{-1} x$

- $\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to \infty} e^x = \infty$
- $\ln e = 1$ and $\ln^{-1} 1 = e^1 = e$



$$e^{\ln x} = x \quad \text{for all } x > 0$$
$$\ln e^x = x \quad \text{for all } x$$

Inverse Equations for e^x and $\ln x$

Example: Solve the equation for x: [a] $e^{\ln(0.2x)} = 0.4$

$0.2x = 0.4 \Rightarrow x = 2$

[b] $e^{(\ln 0.2)x} = 0.4$

$(\ln 0.2)\, x = \ln 0.4 \Rightarrow x = \dfrac{\ln 0.4}{\ln 0.2}$

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* If $u(x)$ is differentiable function of x and $y = e^{u(x)}$,

Then $y' = \dfrac{dy}{dx} = e^{u(x)} \dfrac{du}{dx}$

Example: Find y' for [1] $y = e^x \Rightarrow y' = e^x$

[2] $y = e^{5-7x} \Rightarrow y' = -7\, e^{5-7x}$

[3] $y = e^{\cos x} \Rightarrow y' = -\sin x\, e^{\cos x}$

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* The general antiderivative of the exponential function

$$\int e^u\, du = e^u + C$$

Example: Find

$\boxed{1}$ $\displaystyle\int_{\ln 2}^{\ln 3} e^{x}\,dx = e^{x}\Big|_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3-2 = 1$ (8)

$\boxed{2}$ $\displaystyle\int 2t\,e^{-t^2}\,dt = -\int -2t\,e^{-t^2}\,dt = -e^{-t^2} + C$

$\boxed{3}$ $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr = 2\int \frac{e^{\sqrt{r}}}{2\sqrt{r}}\,dr = 2e^{\sqrt{r}} + c$

Th For all x_1, x_2, and x_3 we have

$\boxed{1}$ $e^{x_1} e^{x_2} = e^{x_1+x_2}$ $\qquad$ $\boxed{2}$ $e^{-x} = \dfrac{1}{e^{x}}$

$\boxed{3}$ $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$ $\qquad$ $\boxed{4}$ $\left[e^{x_1}\right]^{r} = e^{rx_1}$, $r \in \mathbb{Q}$

Proof $\boxed{1}$ let $y_1 = e^{x_1} \Rightarrow x_1 = \ln y_1$

let $y_2 = e^{x_2} \Rightarrow x_2 = \ln y_2$

$x_1 + x_2 = \ln y_1 + \ln y_2$

$= \ln y_1 y_2$

$e^{x_1+x_2} = e^{\ln y_1 y_2}$

$e^{x_1+x_2} = y_1 y_2 = e^{x_1} e^{x_2}$

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* The general Exp. Function a^{x}, $\underset{\text{base}}{a} > 0$ is given by

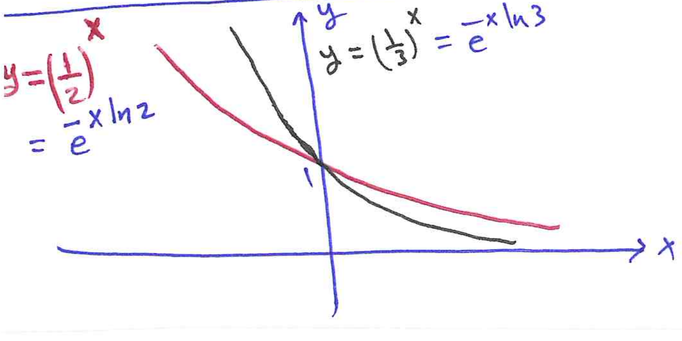
$$a^{x} = e^{x\ln a}$$

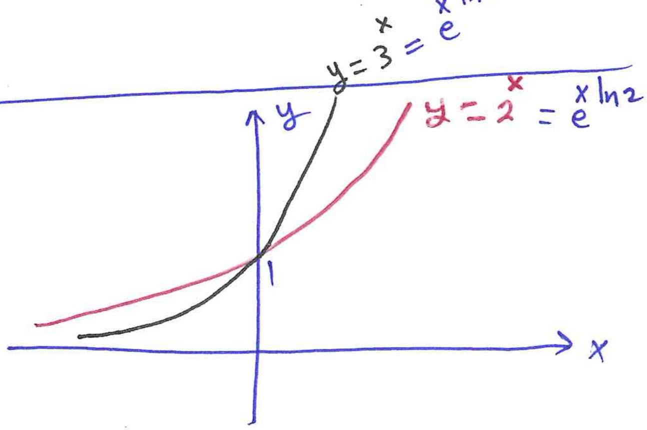
- $a = e^{\ln a} \Rightarrow a^{x} = \left(e^{\ln a}\right)^{x} = e^{x\ln a}$
- when $a = e \Rightarrow e^{x} = e^{x\ln e} = e^{x}$

* If $y = a^{x}$, then $y' = a^{x}\ln a$ $\quad$ i.e $y' = \ln a\, e^{x\ln a} = \ln a\, a^{x}$

* If $y = a^{u(x)}$, then $y' = a^{u(x)} \ln a\, u'(x)$

* $\displaystyle\int a^{u}\,du = \frac{a^{u}}{\ln a} + C$





Example Find y' for

1. $y = 5^x \Rightarrow y' = 5^x \ln 5$
2. $y = 5^{\sqrt{x}} \Rightarrow y' = 5^{\sqrt{x}} \ln 5 \, \frac{1}{2\sqrt{x}}$
3. $y = x^{\pi} \Rightarrow y' = \pi \, x^{\pi - 1}$
4. $y = 2^{\sin 3t} \ln 2 \Rightarrow y' = 3 \, 2^{\sin 3t} (\ln 2)^2 \cos 3t$

Example Find

1. $\int 7^{\sec\theta} \ln 7 \sec\theta \tan\theta \, d\theta = 7^{\sec\theta} + C$

take $u = \sec\theta \Rightarrow du = \sec\theta \tan\theta \, d\theta$

$$\int 7^u \ln 7 \, du = \ln 7 \int 7^u \, du = \ln 7 \frac{7^u}{\ln 7} + C = 7^u + C$$
$$= 7^{\sec\theta} + C$$

2. $\int 7^x \, dx = \frac{7^x}{\ln 7} + C$

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* For $x > 0$, we have $x^n = e^{n \ln x}$, $n \in \mathbb{R}$

If $y = x^n$, then $y' = e^{n \ln x} \frac{n}{x} = x^n \frac{n}{x} = n x^{n-1}$

Example Find $f'(x)$ if $f(x) = x^x$

$$f(x) = x^x = e^{x \ln x}$$
$$f'(x) = e^{x \ln x} \left(x \frac{1}{x} + \ln x \right)$$
$$= x^x (1 + \ln x)$$

Th $e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$ "The number e as a limit"

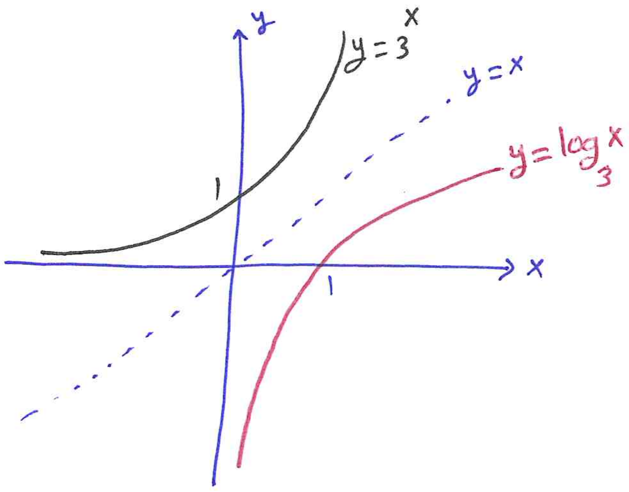
Proof: let $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$ with $f'(1) = 1$

But $f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x}$

$$= \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}$$

$$1 = \ln \left[\lim_{x \to 0} (1+x)^{\frac{1}{x}} \right] \Leftrightarrow e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

* Inverse Equations for a^x and $\log_a x$ "more general" (10)



$a^{\log_a x} = x$ for $x > 0$

$\log_a a^x = x$ for all x

since when $a = e \Rightarrow$

$e^{\log_e x} = e^{\ln x} = x$ ✓

$\log_e e^x = \ln e^x = x$ ✓

$$\log_a x = \frac{\ln x}{\ln a}$$

So $\log_e x = \frac{\ln x}{\ln e} = \ln x$

For any $x > 0$ and $y > 0$:

$\boxed{1}$ $\log_a xy = \log_a x + \log_a y$ "Product Rule"

$\boxed{2}$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ "Quotient Rule"

$\boxed{3}$ $\log_a \frac{1}{y} = -\log_a y$ "Reciprocal Rule"

$\boxed{4}$ $\log_a x^y = y \log_a x$ "Power Rule"

* If $y = \log_a x$, then $y' = \frac{1}{\ln a} \frac{1}{x}$

* If $y = \log_a u(x)$, then $y' = \frac{1}{\ln a} \frac{u'(x)}{u(x)}$

Example: Find y' for $\boxed{1}$ $y = \log_4 x^2 \Rightarrow y' = \frac{1}{\ln 4} \frac{2x}{x^2} = \frac{1}{x \ln 2}$

$\boxed{2}$ $y = \log_2 (8 x^{\ln 2}) \Rightarrow y' = \frac{1}{\ln 2} \frac{8 \ln 2 \, x^{\ln 2 - 1}}{8 x^{\ln 2}} = \frac{1}{x}$

$\boxed{3}$ $y = \int_0^{\log_4 x} 2 \ln 2 \, 4^t \, dt = \cancel{2 \ln 2} \, 4^{\log_4 x} \left(\frac{1}{\cancel{\ln 4}} \frac{1}{x} \right) = \frac{x}{x} = 1$