Ch 4

Interpolation and Polynomial Approximation

- Interpolation means polynomial approximation.
  - given a function $f(x)$ on $[a,b]=[x_0, x_n]$
  - given $n+1$ points

$$(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$$

$$\downarrow f(x_0) \qquad \downarrow f(x_1) \qquad \downarrow f(x_n)$$

through the partition

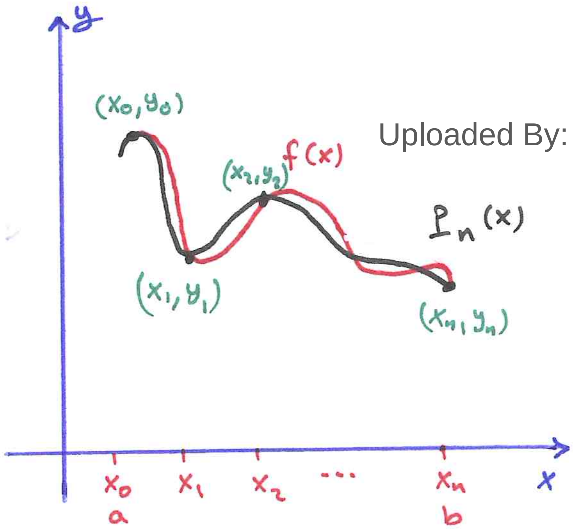
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

- We need to approximate $f(x)$ by a polynomial of order at most n passing through these $n+1$ points "nodes" on $[a,b]$.

- That is, $f(x) \approx \underbrace{P_n(x)}_{\text{is called interpolation polynomial}} + \underbrace{E_n(x)}_{\text{Truncation error}}$ on $[a,b]$

- degree $(P_n(x)) \leq n$

- $P_n(x) \approx f(x)$ on $[x_0, x_n] = [a,b]$



Th Given $n+1$ points: $(x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)$.
Then $\exists$ a unique polynomial $P_n(x)$ with degree $\leq n$ that passes through these points.

Exp Find a linear interpolating polynomial passes through (x_0, y_0) and (x_1, y_1)

- $P_1(x)$ is the interpolating polynomial of order 1 "linear" given by $P_1(x) = ax + b$
- To find a and b $\Rightarrow$ $P_1(x_0) = y_0 = ax_0 + b$

$$P_1(x_1) = y_1 = a x_1 + b$$

- Hence, $a = \dfrac{y_1 - y_0}{x_1 - x_0}$ and $b = y_0 - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right)x_0$
- Thus, $P_1(x) = \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right)x + y_0 - \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right)x_0$
- Note that $f(x) = P_1(x)$ on $[x_0, x_1]$ with no errors.

- If $(x_0, y_0) = (1,2)$ and $(x_1, y_1) = (3,4)$ then

$a = \dfrac{4-2}{3-1} = \dfrac{2}{2} = 1$ and $b = 2 - (1)(1) = 1$

$\Rightarrow P_1(x) = x + 1$

- One can write $P_1(x) = m(x - x_0) + y_0$ where

$$m = \frac{y_1 - y_0}{x_1 - x_0} \text{ is the slope.}$$

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Exp Given $(-1,6)$, $(2,9)$, $(0,3)$.

1. Find the polynomial of degree ≤ 2 that passes through these points
2. Estimate $f(1)$
3. Estimate $f'(\frac{1}{4})$
4. Estimate $\int_1^2 f(x)\,dx$

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1. • $P_2(x) = ax^2 + bx + c$ $\qquad P_2(0) = 3 \Rightarrow \boxed{c=3}$

$P_2(-1) = a - b + 3 = 6 \Rightarrow \boxed{a-b=3} \rightarrow \boxed{a=2}$

$P_2(2) = 4a + 2b + 3 = 9 \Rightarrow \boxed{2a+b=3} \rightarrow \boxed{b=-1}$

• Hence, $\boxed{P_2(x) = 2x^2 - x + 3}$

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2. $f(1) \approx P_2(1) = 2 - 1 + 3 = 4$

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3. $f'(x) \approx P_2'(x) = 4x - 1 \quad \Rightarrow f'(\frac{1}{4}) \approx P_2'(\frac{1}{4}) = 1 - 1 = 0$

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4. $\int_1^2 f(x)\,dx \approx \int_1^2 P_2(x)\,dx = \frac{2}{3}x^3 - \frac{x^2}{2} + 3x \Big|_1^2$

$= \left(\frac{16}{3} - 2 + 6\right) - \left(\frac{2}{3} - \frac{1}{2} + 3\right)$

$= \frac{37}{6} \approx 6.17$

Remark: If $f(x)$ is given and analytic at x_0 " has continuous derivatives of all orders and can be represented as Taylor series in an interval about x_0", then we can use Taylor Polynomial Approximation to estimate $f(x)$ by a Taylor Polynomial

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Th (Taylor Polynomial Approximation)

- Assume $f \in C^{n+1}[a,b]$ and $x_0 \in [a,b]$ is fixed
- If $x \in [a,b]$, then $f(x) \simeq P_n(x) + E_n(x)$

where $P_n(x)$ is the Taylor polynomial of degree n that estimates $f(x)$ on $[a,b]$ given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

and $E_n(x)$ is the truncation error given by

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

However, f is usually not known or hard to compute. So How to find $P_n(x)$

1. Lagrange Interpolation $\Rightarrow P_n(x)$ is the Lagrange Polynomial
2. Newton Interpolation $\Rightarrow P_n(x)$ is the Newton Polynomial

1 Lagrange's Polynomial

* linear interpolation uses a line segment that passes throw two points.

Exp Find Lagrange polynomial through the points (x_0, y_0) and (x_1, y_1).

$$P_1(x) \approx f(x) = y = y_0 + m(x - x_0) \quad \text{where } m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$P_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$= y_0 + \frac{y_1}{x_1 - x_0}(x - x_0) - \frac{y_0}{x_1 - x_0}(x - x_0)$$

$$= y_0 \left[1 - \frac{x - x_0}{x_1 - x_0} \right] + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= y_0 \frac{x_1 - x}{x_1 - x_0} + y_1 \frac{x - x_0}{x_1 - x_0}$$

Hence, the Lagrange polynomial of order 1 is

$$P_1(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

$$= y_0 L_{1,0}(x) + y_1 L_{1,1}(x)$$

Lagrange coefficient polynomials

$$= \sum_{k=0}^{1} y_k L_{1,k}(x)$$

• Note that $P_1(x_0) = y_0$ and $P_1(x_1) = y_1$

Remark: In general, the Lagrange Polynomial of degree at most n passes through $n+1$ points: $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ has the form:

$$P_n(x) = \sum_{k=0}^{n} y_k L_{n,k}(x)$$

where the Lagrange coefficient polynomial based on these points:

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Exp • Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

• Lagrange polynomial of order of degree ≤ 2 is

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k}(x)$$

$$= y_0 L_{2,0}(x) + y_1 L_{2,1}(x) + y_2 L_{2,2}(x)$$

$$= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Remark Note that $L_{n,k}(x) = \dfrac{\prod_{j=0, j\neq k}^{n} (x-x_j)}{\prod_{j=0, j\neq k}^{n} (x_k-x_j)}$

Exp* Given $\overset{x_0\ y_0}{(-1,6)}, \overset{x_1\ y_1}{(2,9)}, \overset{x_2\ y_2}{(0,3)}$. $\boxed{n=2}$

Find Lagrange polynomial and estimate $f(1)$

- $$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= (6)\frac{(x-2)(x-0)}{(-1-2)(-1-0)} + (9)\frac{(x-{-1})(x-0)}{(2-{-1})(2-0)} + (3)\frac{(x-{-1})(x-2)}{(0-{-1})(0-2)}$$

$$= 2x(x-2) + \frac{3}{2}x(x+1) - \frac{3}{2}(x+1)(x-2)$$

$$= 2x^2 - 4x + \cancel{\frac{3}{2}x^2} + \frac{3}{2}x - \cancel{\frac{3}{2}x^2} + \frac{3}{2}x + 3$$

$$\boxed{P_2(x) = 2x^2 - x + 3}$$ "quadratic interpolation polynomial"

- To estimate $f(1)$, we use Lagrange polynomial:

$$f(1) \approx P_2(1) = 2 - 1 + 3 = 4$$

Exp Given $(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3) \Rightarrow P_3(x)$ is the Lagrange polynomial of degree ≤ 3 given by

$$P_3(x) = y_0 L_{3,0}(x) + y_1 L_{3,1}(x) + y_2 L_{3,2}(x) + y_3 L_{3,3}(x)$$

"cubic interpolation polynomial"

$$= y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} +$$

$$y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Def (Uniform Partition)

- The partition of $[a,b] = [x_0, x_n]$ is uniform if the nodes $x_0, x_1, x_2, \dots, x_n$ are equally spaced.
- That is, $x_k = x_0 + hk$ for $k = 0, 1, 2, \dots, n$

Exp Consider $y = f(x) = \cos x$ on $[0, 1.2]$

1. Find Lagrange Polynomial of order 2 using equally spaced nodes. (use 3 digits)

- Nodes: $x_0 = 0$, $x_1 = 0.6$, $x_2 = 1.2$ which is $n+1 = 2+1 = 3$
- Points: (x_0, y_0), (x_1, y_1), (x_2, y_2)

  $(0, 1)$, $(0.6, 0.825)$, $(1.2, 0.362)$

- $$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= (1)\frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)} + (0.825)\frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)} + (0.362)\frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}$$

$$= 1.39(x-0.6)(x-1.2) - 2.29x(x-1.2) + 0.503x(x-0.6)$$

$$= -0.397x^2 + 0.246x + 0.698$$

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2. Find Lagrange Polynomial of order 3 using uniform partition.

- Nodes: $x_0 = 0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$ which is $n+1 = 3+1 = 4$
- Points: (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)

  $(0, 1)$, $(0.4, 0.921)$, $(0.8, 0.697)$, $(1.2, 0.362)$

$$P_3(x) = y_0 L_{3,0}(x) + y_1 L_{3,1}(x) + y_2 L_{3,2}(x) + y_3 L_{3,3}(x)$$

$$= (1)\frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)} + (0.921)\frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)} +$$

$$(0.697)\frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)} + (0.362)\frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}$$

$$= -2.60(x-0.4)(x-0.8)(x-1.2) + 7.20\,x\,(x-0.8)(x-1.2)$$

$$-5.44\,x\,(x-0.4)(x-1.2) + 0.944\,x\,(x-0.4)(x-0.8)$$

$$= 0.100\,x^3 - 0.580\,x^2 + 0.0300\,x + 0.998$$

Exp H.W Let $f(x) = e^x$ on $[1, 4]$ (use 3 digits)

1. Find Lagrange Polynomial of order 1 using equally space nodes.
2. Find Lagrange Polynomial of order 2 using uniform partition.
3. Find Lagrange Polynomial of order 3 using uniform partition.

Remark Now we study the second method " Newton Polynomial " then we study the error term since the error of these two interpolations is the same.

Newton's Polynomial

Th (Newton's Polynomial)

- Given $x_0, x_1, x_2, \dots, x_n$ $n+1$ distinct numbers in $[a,b]$.
- Then, $\exists$ a unique polynomial $P_n(x)$ " Called Newton's Polynomial " of degree at most n s.t $f(x_i) = P_n(x_i)$ for $i=1,2,\dots,n$.
- Furthermore, Newton's Polynomial is given by

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots$$
$$+ a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

where the coefficients of Newton's Polynomial are given by the divided differences: $a_k = f[x_0, x_1, \dots, x_k]$ for $k=0,1,\dots,n$.

- That is:

$a_0 = f[x_0] = f(x_0) = y_0$: zero divided differences

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \quad \text{First divided differences}$$
$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \quad 2^{nd} \text{ D.D}$$

$$= \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

$\vdots$

Divided Difference Table for $y = f(x)$

x_k	$f[x_k]=y_k$	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
x_0	$y_0 = a_0$			
x_1	y_1	$f[x_0,x_1]=a_1$		
x_2	y_2	$f[x_1,x_2]$	$f[x_0,x_1,x_2]=a_2$	
x_3	y_3	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]=a_3$
x_4	y_4	$f[x_3,x_4]$	$f[x_2,x_3,x_4]$	$f[x_1,x_2,x_3,x_4]$

Exp* Given $\overset{x_0}{(-1},\overset{y_0}{6)}, \overset{x_1}{(2},\overset{y_1}{9)}, \overset{x_2}{(0},\overset{y_2}{3)}$. Find Newton Polynomial.

• Newton Polynomial is $P_2(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$

$$= 6 + a_1(x+1) + a_2(x+1)(x-2)$$

• $a_1 = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{9-6}{2+1} = 1$

$$a_2 = \frac{\dfrac{y_2-y_1}{x_2-x_1} - \dfrac{y_1-y_0}{x_1-x_0}}{x_2 - x_0} = \frac{\dfrac{3-9}{0-2} - \dfrac{9-6}{2+1}}{0+1} = 3 - 1 = 2$$

• Hence, $P_2(x) = 6 + x+1 + 2(x+1)(x-2)$

$$= 2x^2 - x + 3$$

Exp Given $\underset{x_0}{(-2}, \underset{y_0}{-12)}, \underset{x_1}{(-1}, \underset{y_1}{-4)}, \underset{x_2}{(1}, \underset{y_2}{0)}, \underset{x_3}{(2}, \underset{y_3}{8)}$

- Construct the Divided Difference table
- Find Newton's Polynomial
- Estimate $f(0)$.

x_K	y_K	1^{st} D.D	2^{nd} D.D	3^{rd} D.D
-2	$a_0 = -12$			
-1	-4	$a_1 = 8$		
1	0	$f[x_1,x_2] = 2$	$a_2 = -2$	
2	8	$f[x_2,x_3] = 8$	$f[x_1,x_2,x_3] = 2$	$a_3 = 1$

$f[x_2,x_3] = \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{8-0}{2-1} = 8$

- $a_0 = f[x_0] = y_0 = -12$

- $a_1 = f[x_0,x_1] = f[-2,-1] = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{-4+12}{-1+2} = 8$

- $a_2 = f[x_0,x_1,x_2] = f[-2,-1,1] = \dfrac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = \dfrac{\frac{0+4}{1+1} - 8}{1+2} = -2$

- $a_3 = f[x_0,x_1,x_2,x_3]$

$$= \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0} = \frac{\dfrac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} + 2}{2+2}$$

$$= \frac{\frac{8-2}{2+1} + 2}{4} = \frac{2+2}{4} = 1$$

- Newton's Polynomial is

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$= -12 + 8(x+2) - 2(x+2)(x+1) + (x+2)(x+1)(x-1)$$

$$= x^3 + x - 2$$

- $f(0) \approx P_3(0) = 0 + 0 - 2 = -2$

Error Terms and Error Bounds

- **Error Term $E_n(x)$:** • Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
  - Given $P_n(x)$ interpolating Polynomial.
    → $P_n(x)$ can be Lagrange or Newton Polynomial that approximates $f(x)$

- The Error term $E_n(x)$ is the same for Lagrange and Newton approximation $P_n(x)$

- And $E_n(x)$ is similar to the error term for Taylor polynomial except the factor $(x - x_0)^{n+1}$ is replaced by the product

$$(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$$

- This is because the interpolation $P_n(x)$ is exact " $P_n(x) = f(x)$ " at each $n+1$ nodes x_k $\Rightarrow$

$$E_n(x_k) = f(x_k) - P_n(x_k) = y_k - y_k = 0$$

Th (Error Term)

- Assume $f \in C^{n+1}[a,b]$ and $x_0, x_1, \dots, x_n \in [a,b]$ are $n+1$ nodes.

- Then $f(x) = P_n(x) + E_n(x)$ where $E_n(x)$ is the error term given by
$$E_n(x) = \frac{(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)\, f^{(n+1)}(c)}{(n+1)!}$$
for some $c = c(x)$ lies in $[a,b]$.

$$P_n(x) = y_0 L_{n,0}(x) + y_1 L_{n,1}(x) + \cdots + y_n L_{n,n}(x) \quad \text{Lagrange Polynomial} \quad \underline{\text{or}}$$

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}) \quad \text{Newton Polynomial}$$

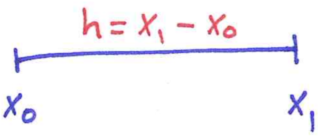
How to find an upper bound for the Error Term $E_n(x)$?

- That is, we need to find some constant s.t $|E_n(x)| \leq$ constant.
- Finding the upper bound depends on whether the nodes are equally spaced (Uniform Partition) or not (not uniform partition).

Th (Upper Bound of the Error Term for Interpolation – Uniform Partition)

1. Given $(x_0, y_0), (x_1, y_1)$ " $n=1$ "

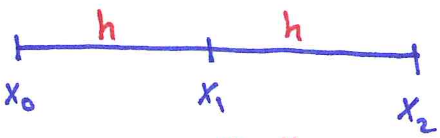
with $E_1(x) = \dfrac{(x-x_0)(x-x_1) f''(c)}{2!}$



Then $|E_1(x)| \leq \dfrac{h^2 M_2}{8}$ where $M_2 = \underset{x_0 \leq x \leq x_1}{\text{Max}} |f''(x)|$

2. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ " $n=2$ "

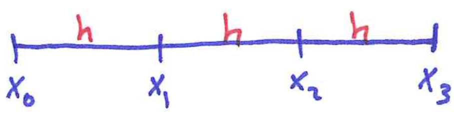
with $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2) f'''(c)}{3!}$



Then $|E_2(x)| \leq \dfrac{h^3 M_3}{9\sqrt{3}}$ where $M_3 = \underset{x_0 \leq x \leq x_2}{\text{Max}} |f'''(x)|$ $\quad h = \dfrac{x_2 - x_0}{2}$

3. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$ " $n=3$ "

with $E_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3) f^{(4)}(c)}{4!}$



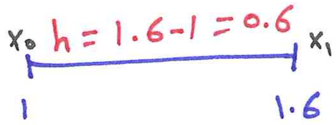
Then $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$ where $M_4 = \underset{x_0 \leq x \leq x_3}{\text{Max}} |f^{(4)}(x)|$ $\quad h = \dfrac{x_3 - x_0}{3}$

Exp Given $f(x) = \ln(x+2)$ on $[1, 1.6]$

Find an upper bound for E_1, E_2, E_3 using uniform partition.

$\boxed{1}$ • $f'(x) = \dfrac{1}{x+2} \Rightarrow f''(x) = \dfrac{-1}{(x+2)^2} \Rightarrow |f''(x)| = \dfrac{1}{(x+2)^2}$

• The upper bound of E_1 is



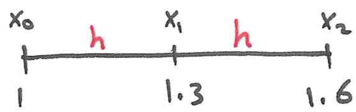
$$|E_1(x)| \leq \frac{h^2 M_2}{8} \quad \text{where} \quad M_2 = \max_{1 \leq x \leq 1.6} |f''(x)|$$

• $|f''|$ is decreasing $\Rightarrow |f''(x)| \leq \dfrac{1}{(1+2)^2} = \dfrac{1}{9} = M_2$

• Hence, $|E_1(x)| \leq \dfrac{(0.6)^2\left(\frac{1}{9}\right)}{8} = 0.005$

$\boxed{2}$ • The upper bound of E_2 is $|E_2(x)| \leq \dfrac{h^3 M_3}{9\sqrt{3}}$

• $M_3 = \max\limits_{1 \leq x \leq 1.6} |f'''(x)| \Rightarrow f'''(x) = \dfrac{2}{(x+2)^3}$



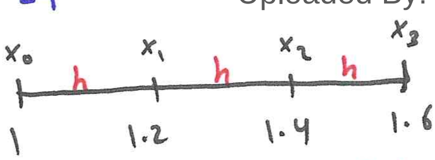
$h = \dfrac{1.6-1}{2} = 0.3$

• $|f'''(x)|$ is decreasing $\Rightarrow |f'''(x)| \leq \left|\dfrac{2}{(1+2)^3}\right| = \dfrac{2}{27} = M_3$

• Hence, $|E_2(x)| \leq \dfrac{(0.3)^3\left(\frac{2}{27}\right)}{9\sqrt{3}} = \dfrac{0.002}{9\sqrt{3}} = 0.000128$

$\boxed{3}$ • The upper bound of E_3 is $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$

• $M_4 = \max\limits_{1 \leq x \leq 1.6} |f^{(4)}(x)| \Rightarrow f^{(4)}(x) = \dfrac{-6}{(x+2)^4}$



• $|f^{(4)}(x)| = \dfrac{6}{(x+2)^4} \leq \dfrac{6}{(1+2)^4} = \dfrac{6}{81} = \dfrac{2}{27} = M_4$

$h = \dfrac{1.6-1}{3} = 0.2$

• Hence, $|E_3(x)| \leq \dfrac{(0.2)^4\left(\frac{2}{27}\right)}{24} = 0.00000494$

Upper Bound of the Error Term For interpolation - Non Uniform Partition

- Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ "$n=2$"

  with $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\, f'''(c)}{3!}$

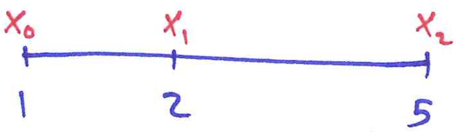
- Then $|E_2(x)| \leq \dfrac{|\phi_2(x)|\ |f'''(c)|}{6}$ where

  $|\phi_2(x)|$ is an upper bound of $\phi_2(x) = (x-x_0)(x-x_1)(x-x_2)$ and $|f'''(c)|$ is an upper bound of $f'''(x)$.

Exp Given $f(x) = x^2 - \dfrac{2}{x}$ and $(1, f(1)), (2, f(2)), (5, f(5))$. Find an upper bound for the error term of interpolation.

- $n=2$
- The upper bound of error is

  $|E_2(x)| \leq \dfrac{|\phi_2(x)|\ |f'''(c)|}{6}$



Not uniform

- $f'(x) = 2x + \dfrac{2}{x^2} \Rightarrow f''(x) = 2 - \dfrac{4}{x^3} \Rightarrow$

  $f'''(x) = \dfrac{12}{x^4} \Rightarrow |f'''(x)| = \dfrac{12}{x^4} \leq 12 = M_3$

- $\phi_2(x) = (x-1)(x-2)(x-5) = (x^2-3x+2)(x-5)$

  $\phi_2'(x) = (x^2-3x+2)(1) + (x-5)(2x-3) = 3x^2 - 16x + 17 = 0$

  $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{16 \pm \sqrt{(16)^2 - 4(3)(7)}}{6} \Rightarrow x_1 = 3.8685$ or $x_2 = 1.4682$

  $|\phi(x_1)| = \boxed{6.06}$ Max and $|\phi(x_2)| = 3.783$, $|\phi(1)| = |\phi(5)| = 0$ end points

- Hence, $|E_2(x)| \leq \dfrac{(6.06)(12)}{6} = 12.12$

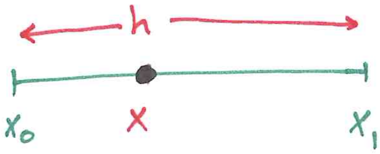
Proof of Th$^{\underline{m}}$

$\boxed{\text{I}}$ $\quad |E_1(x)| \le \dfrac{h^2 M_2}{8}$ where $M_2 = \max\limits_{x_0 \le x \le x_1} |f''(x)|$

- The Error Term is $E_1(x) = \dfrac{(x-x_0)(x-x_1)\, f''(c)}{2} \Rightarrow$

$$|E_1(x)| = \frac{|\phi_1(x)|\,|f''(c)|}{2} \quad \text{where } |f''(c)| \le M_2$$

- Now we need to find an upper bound for $|\phi_1(x)|$.
  - $\phi_1(x) = (x-x_0)(x-x_1)$ using change of variables



  - Let $\boxed{x - x_0 = t}$
  - We have

$$x_1 = x_0 + h$$
$$-x_1 = -x_0 - h$$
$$x - x_1 = x - x_0 - h$$
$$\boxed{x - x_1 = t - h}$$

$$x_0 \le x \le x_1$$
$$0 \le x - x_0 \le x_1 - x_0$$
$$\boxed{0 \le t \le h}$$

- $\phi_1(x) = \phi_1(x_0 + t) = t(t-h) = t^2 - ht = \phi_1(t)$

$\phi_1' = 2t - h = 0 \Leftrightarrow t = \dfrac{h}{2}$ critical point

- $\left|\phi\left(\frac{h}{2}\right)\right| = \left|\dfrac{h^2}{4} - \dfrac{h^2}{2}\right| = \boxed{\dfrac{h^2}{4}}$ Max since $|\phi(0)| = |\phi(h)| = 0$ end points

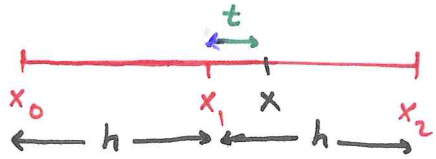
- Hence, $|E_1(x)| = \dfrac{|\phi_1(x)|\,|f''(c)|}{2} \le \dfrac{\frac{h^2}{4} M_2}{2} = \dfrac{h^2 M_2}{8}$

$\boxed{2}$ $|E_2(x)| \le \dfrac{h^3 M_3}{9\sqrt{3}}$ where $M_3 = \max\limits_{x_0 \le x \le x_2} |f'''(x)|$

- The Error Term is $E_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)\, f'''(c)}{3!} \Rightarrow$

$$|E_2(x)| = \frac{|\phi_2(x)|\, |f'''(c)|}{6} \quad \text{where } |f'''(c)| \le M_3$$

- Now we need to find an upper bound for $|\phi_2(x)|$.
  - $\phi_2(x) = (x-x_0)(x-x_1)(x-x_2)$
  - Using the change of variable:



$$x - x_1 = t$$
$$x - x_0 = t + h$$
$$x - x_2 = t - h$$

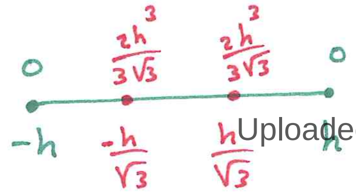
$$x_0 \le x \le x_2$$
$$x_0 - x_1 \le x - x_1 \le x_2 - x_1$$
$$-h \le t \le h$$

- $\phi_2(x) = \phi_2(x_1 + t) = (t+h)(t)(t-h)$

$$= t(t^2 - h^2)$$
$$= t^3 - t h^2$$
$$= \phi(t)$$

$$\phi_2' = 3t^2 - h^2 = 0 \iff t = \pm \frac{h}{\sqrt{3}} \quad \text{critical points}$$

- $\left|\phi_2\left(\frac{h}{\sqrt{3}}\right)\right| = \left|\dfrac{h^3}{3\sqrt{3}} - \dfrac{h^3}{\sqrt{3}}\right| = \dfrac{2h^3}{3\sqrt{3}}$



$\left|\phi_2\left(\frac{-h}{\sqrt{3}}\right)\right| = \left|\dfrac{-h^3}{3\sqrt{3}} + \dfrac{h^3}{\sqrt{3}}\right| = \dfrac{2h^3}{3\sqrt{3}}$

- Hence, $|E_2(x)| = \dfrac{|\phi_2(x)|\, |f'''(c)|}{6} \le \dfrac{\frac{2h^3}{3\sqrt{3}} M_3}{6} = \dfrac{h^3 M_3}{9\sqrt{3}}$

$\boxed{3}$ $|E_3(x)| \leq \dfrac{h^4 M_4}{24}$ where $M_4 = \underset{x_0 \leq x \leq x_3}{Max} |f^{(4)}(x)|$

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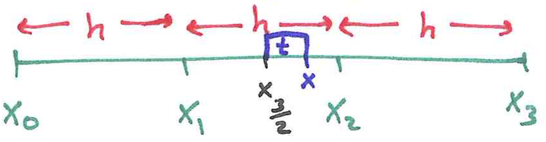
- The Error Term is $E_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c)$ $\Rightarrow$

$|E_3(x)| = \dfrac{|\phi_3(x)|\,|f^{(4)}(c)|}{24}$ where $|f^{(4)}(c)| \leq M_4$

- Now we need to find an upper bound for $|\phi_3(x)|$.
  - $\phi_3(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$



  - Using the change of variable:

$$x - x_0 = t + \tfrac{3}{2}h$$
$$x - x_1 = t + \tfrac{h}{2}$$
$$x - x_2 = t - \tfrac{h}{2}$$
$$x - x_3 = t - \tfrac{3}{2}h$$

$x_{\frac{3}{2}} = x_0 + \frac{3}{2}h$

$x_0 \leq x \leq x_3$

$x_0 - x_{\frac{3}{2}} \leq x - x_{\frac{3}{2}} \leq x_3 - x_{\frac{3}{2}}$

$-\frac{3}{2}h \leq t \leq \frac{3}{2}h$

- $\phi_3(x) = (t + \frac{3}{2}h)(t + \frac{h}{2})(t - \frac{h}{2})(t - \frac{3}{2}h)$

$= (t^2 - \frac{9}{4}h)(t^2 - \frac{h^2}{4}) = t^4 - \frac{5}{2}h^2t^2 + \frac{9h^4}{16} = \phi_3(t)$

- $\phi_3' = 4t^3 - 5h^2 t = 0 \Leftrightarrow t(4t^2 - 5h^2) = 0 \Leftrightarrow t = 0$ or $t = \pm\frac{\sqrt{5}}{2}h$

- $|\phi_3(0)| = \frac{9h^4}{16}$, $|\phi_3(\pm\frac{\sqrt{5}}{2}h)| = h^4$ Max

- Hence $|E_3(x)| = \dfrac{|\phi_3(x)|\,|f^{(4)}(c)|}{24}$

$$\leq \frac{h^4 M_4}{24}$$