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MATH1321

Calculus 2

Chapter 10.10

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التشابتر، هاد تلخيص دكتور
عبدالرحيم موسى وضفت عليه
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$\boxed{10.10}$ The Binomial Series and Applications of Taylor Series

* The Binomial series of $f(x) = (1+x)^m$ is "using Taylor series expa."

$$(1+x)^m = 1 + \sum_{K=1}^{\infty} \binom{m}{K} x^K \quad , \quad |x| < 1 \quad , \quad m \text{ is constant}$$

where the series converges absolutely.

where $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2}$

$$\binom{m}{K} = \frac{m(m-1)(m-2)\cdots(m-K+1)}{K!} \quad \text{for } K \geq 3$$

Powers and Roots)

Exp Find the first four terms of the binomial series for

$\boxed{1}$ $(1+x)^{\frac{1}{2}} = 1 + \sum_{K=1}^{\infty} \binom{\frac{1}{2}}{K} x^K = 1 + \frac{x}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} + \cdots$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

$\boxed{2}$ $(1 - \frac{x}{2})^{-2} = 1 + \sum_{K=1}^{\infty} \binom{-2}{K} \left(\frac{-x}{2}\right)^K$

$$= 1 + x + \frac{(-2)(-3)\left(-\frac{x}{2}\right)^2}{2} + \frac{(-2)(-3)(-4)\left(-\frac{x}{2}\right)^3}{3!} + \cdots$$

$$= 1 + x + \frac{3x^2}{4} + \frac{x^3}{2} + \cdots$$

3rd example in page 7

Approximating Nonelementary Integrals)

Exp Use series to estimate the following integrals with an error of magnitude less than 10^{-3}

$\boxed{1}$ $\int_0^{0.1} \frac{dx}{\sqrt{1+x^4}} = \int_0^{0.1} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{0.1} \left(1 + \sum_{K=1}^{\infty} \binom{-\frac{1}{2}}{K} x^{4K}\right) dx$

$$= \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \cdots\right) dx = \left[x - \frac{x^5}{10} + \frac{3x^9}{9(8)} - \cdots\right]_0^{0.1}$$

$\approx x \Big|_0^{0.1} \approx 0.1$ with error $|E| \leq \frac{(0.1)^5}{10} \approx 0.000001$.

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$\boxed{2}$ $\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.2} \frac{1}{x}(e^{-x}-1)\,dx$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$= \int_0^{0.2} \frac{1}{x}\left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \cancel{-1}\right) dx$

$= \int_0^{0.2} \left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \dots\right) dx$

$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{4(4!)} - \dots\right]_0^{0.2} \approx -(0.2) + \frac{(0.2)^2}{4} - \frac{(0.2)^3}{18} \approx -0.19044$

with error $|E| \leq \frac{(0.2)^4}{96} \approx 0.00002$

Arctangents: Remember that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

section 10.7

with Leibniz's formula:

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots + \frac{(-1)^n}{2n+1} + \dots$$

- Note that $\tan^{-1} x = \int \frac{dx}{1+x^2}$

$$\Rightarrow \frac{1}{1+x^2} = \frac{d}{dx}\tan^{-1} x = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$$

- Integrate both sides from 0 to x $\Rightarrow$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \underbrace{\int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\,dt}_{R_n(x)}$$

where $|R_n(x)| \leq \int_0^{|x|} t^{2n+2}\,dt = \frac{|x|^{2n+3}}{2n+3} \xrightarrow[n\to\infty]{} 0$ as if $|x| \leq 1$

- If $|x| \leq 1$, then $\lim_{n\to\infty} R_n(x) = 0$ and so

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad , \quad |x| \leq 1 .$$

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Exp Find Taylor series of $f(x) = \ln x$ at $x=1$ [82.1]

$f(x) = \ln x \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(1) = 1$

$f''(x) = -x^{-2} \Rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4} \Rightarrow f^{(4)}(1) = -6$

$\vdots \qquad \vdots$

$$\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 - \frac{6}{24}(x-1)^4 + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

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Indeterminate forms

Exp Use series to evaluate the limits:

1. $\lim_{x\to 1} \frac{\ln x}{x-1}$

Taylor Series of $\ln x$ about $x=1$ is

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots$$

$$= \lim_{x\to 1} \left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1$$

2. $$\lim_{x\to 0} \frac{e^x - (1+x)}{x^2} = \lim_{x\to 0} \frac{1}{x^2}\left(e^x - 1 - x\right)$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left(\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \cancel{-1} \cancel{-x}\right)$$

$$= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \cdots\right) = \frac{1}{2}$$

3. $$\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y\to 0} \frac{1}{y^3}\left[\cancel{y} - \left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right)\right]$$

$$= \lim_{y\to 0} \left[\frac{1}{3} - \frac{y^2}{5} + \cdots\right] = \frac{1}{3}$$

Euler's Formula

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$

Recall that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \rightarrow \text{Euler's formula}$$

θ is polar angle.

- Any complex number has the form $a + bi$, $a, b \in \mathbb{R}$.

Exp $e^{i\pi} = \cos\pi + i\sin\pi = -1$

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31) $\sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$

$= 1 + \binom{\frac{1}{3}}{1} x^1 + \binom{\frac{1}{3}}{2} x^2 + \binom{\frac{1}{3}}{3} x^3 + \dots$

$= 1 + \frac{x}{3} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2} x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!} x^3 + \dots$

$= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} + \dots$

Ex. Find poly. to approximates $F(x) = \int_0^x \sin t^2 \, dt$ on $[0,1]$? with error less than 10^{-3}.

Maclaurine of $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$

$F(x) = \int_0^x \sin t^2 \, dt = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \dots\right) dt$

$= \left(\frac{t^3}{3} - \frac{t^7}{(7)3!} + \frac{t^{11}}{(11)5!} - \frac{t^{15}}{(15)7!} + \dots\right)\Big|_0^x$

$= \frac{x^3}{3} - \frac{x^7}{(7)3!} + \frac{x^{11}}{(11)5!} - \frac{x^{15}}{(15)7!} + \dots - 0$

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$F(x) \approx \frac{x^3}{3}$ with error $< \left|\frac{x^7}{(7)3!}\right| < \underbrace{\frac{1}{(7)3!}}_{x \in [0,1]} \approx 0.0238 > 0.001$

$F(x) \approx \frac{x^3}{3} - \frac{x^7}{(7)3!}$ with error $< \frac{x^{11}}{(11)5!} < \frac{1}{(11)5!} \approx 0.000757 < 0.001$

Hence, $P_7(x) = \frac{x^3}{3} - \frac{x^7}{(7)3!} \approx F(x) = \int_0^x \sin t^2 \, dt$ on $[0,1]$ #

$P_{11}(x)$ and $P_{15}(x)$ ... works with less error and better approx.

TABLE 10.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

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