Chapter 7.1, Problem 53E

Problem

Exercise refers to the Euler phi function, denoted ϕ, which is defined as follows: For each integer $n \geq 1$, $\phi(n)$ is the number of positive integers less than or equal to n that have no common factors with n except ±1. For example, $\phi(10) = 4$ because there are four positive integers less than or equal to 10 that have no common factors with 10 except ±1; namely, 1, 3, 7, and 9.

Exercise

Prove that there are infinitely many integers n for which $\phi(n)$ is a perfect square.

Step-by-step solution

Step 1 of 1

The objective is to prove that there are infinitely many integers *n* for which $\phi(n)$ is perfect square.

If p is prime then,

$$\phi(p^n) = p^n\left(1-\frac{1}{p}\right)$$

Put $p=2$ (Since 2 is an prime number) in equation (1), to obtain,

$$\begin{aligned}\phi(2^n) &= 2^n\left(1-\frac{1}{2}\right)\\ &= 2^n \cdot \frac{1}{2}\\ &= 2^{n-1}\\ &= \left(2^{\frac{n-1}{2}}\right)^2\end{aligned}$$

If we take any odd integer value of n say, $n = 2m+1$, then to get

$$\begin{aligned}\phi(2^{2m+1}) &= \left(2^{\frac{2m+1+1}{2}}\right)^2\\ &= \left(2^{m+1}\right)^2\end{aligned}$$

Since $m+1$ is an integer 2^{m+1} is an integer.

For different values of *m*, there is infinitely many numbers of the type 2^{2m+1} with $\phi(2^{2m+1}) = (2^{2m+1})^2$, which is perfect square.