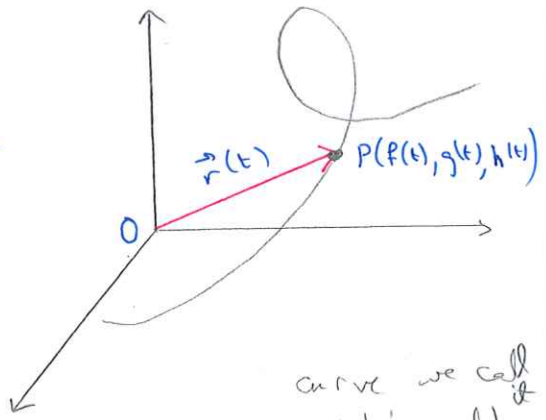
Chapter 13

Vector_Valued Functions and Motion in Space.

13.1 Curves in Space and Their Tangents:

Assume a particle moves in space along a Curve C during a time Interval I



- The position of the particle at any time is determined by the position vector:

vector function

function of time

$$\underset{\overrightarrow{OP}}{\vec{r}(t)} := f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}.$$

curve we call it

↳ particle's path

$(x, y, z) = (f(t), g(t), h(t))$

- The parametrization of the Curve C is:

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

- $f(t)$, $g(t)$, $h(t)$ are called Componend functions.

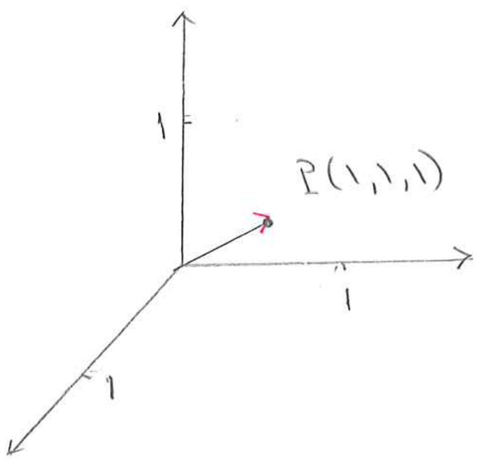
Parametrization:

1. $x = t$, $y = t$, $z = t$

   $0 \le t \le 1$.

   Initial point at $t = 0$

   Terminal pt at $t = 1$



2. $x = t-1$, $y = t-1$, $z = t-1$

   $1 \le t \le 2$.

Note: The domain of the position vector $\vec{r}(t)$ is the Intersection of the domains of $f(t)$, $g(t)$ & $h(t)$.

Note: $\vec{r}(t)$ is called a vector-valued function; that assigns a vector in space to each element in the domain D.

Def: The vector $\vec{C} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$, where c_1, c_2, c_3 are constants is called Constant vector.

Limits and Continuity:

Def: Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D, and $\vec{L}$ a vector, then:

$$\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} f(t)\vec{i} + \lim_{t \to t_0} g(t)\vec{j} + \lim_{t \to t_0} h(t)\vec{k} = \vec{L}$$

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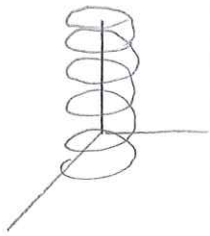
Example: Let $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + 2t\,\vec{k}$.

Find $\lim_{t \to \frac{\pi}{2}} \vec{r}(t)$.

$$\lim_{t \to \frac{\pi}{2}} \vec{r}(t) = \lim_{t \to \frac{\pi}{2}} (\cos t)\vec{i} + \lim_{t \to \frac{\pi}{2}} (\sin t)\vec{j} + \lim_{t \to \frac{\pi}{2}} 2t\,\vec{k}$$

$$= 0\vec{i} + 1\vec{j} + \pi\,\vec{k} = \vec{j} + \pi\vec{k}.$$



Example: $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + \lfloor t \rfloor \vec{k}$ ← greatest Integer function. or "floor fun."

$\vec{r}$ is discontinuous at every Integer.

Recall: $\lfloor 6.2 \rfloor = 6$, $\lfloor 4 \rfloor = 4$, $\lfloor -6.7 \rfloor = -7$.

$$\lim_{t \to \frac{\pi}{2}} \vec{r}(t) = 0\vec{i} + 1\vec{j} + \lfloor \tfrac{\pi}{2} \rfloor \vec{k} =$$
$$= \vec{j} + \vec{k}.$$

Derivatives

Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. be a position vector the difference between the particle's position at t & $t+\Delta t$ is

$$\Delta \vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\frac{\Delta \vec{r}}{\Delta t} = \left[f(t+\Delta t)\vec{i} + g(t+\Delta t)\vec{j} + h(t+\Delta t)\vec{k} \right] - \left[f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \right]$$
$$= \left[\frac{f(t+\Delta t) - f(t)}{\Delta t} \right]\vec{i} + \left[\frac{g(t+\Delta t) - g(t)}{\Delta t} \right]\vec{j} + \left[\frac{h(t+\Delta t) - h(t)}{\Delta t} \right]\vec{k}.$$

Now $$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \left[\frac{df}{dt} \right]\vec{i} + \left[\frac{dg}{dt} \right]\vec{j} + \left[\frac{dh}{dt} \right]\vec{k}.$$

$$\Rightarrow \vec{V}(t) = r'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}.$$

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Remarks: $\boxed{1}$ The position vector $\vec{r}(t)$ is called smooth if $\frac{dr}{dt}$ is continuous and never $\vec{0}$.

That is, if f, g, and h have continuous derivatives that are not all zero.

$\boxed{2}$ The Velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt}$

($\vec{v}$ is tangent on the Curve)

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$\boxed{3}$ Speed is the magnitued of the Velocity $S = |\vec{v}(t)|$.

$\boxed{4}$ Direction of the motion is a unit vector given by:

$$\text{Direction} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

Example: $\vec{v}_1 = (\cos t)\vec{i} + (\sin t)\vec{j}$.

$$S = |\vec{v}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

so $\vec{v}_1$ is a Unit vector.

Example: Find Direction of particle moves along a curve with position vector $\vec{r}(t) = 3t\vec{i} + 4t\vec{j}$.

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$$\vec{v}(t) = 3\vec{i} + 4\vec{j}.$$

$$|\vec{v}(t)| = \sqrt{9+16} = \sqrt{25} = 5.$$

$$\therefore \text{Direction} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

Note: $|\text{Direction}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1.$

Note: Velocity = (speed) . (Direction)

$$= |\vec{v}| . \frac{\vec{v}}{|\vec{v}|}.$$

Differentiation Rules:

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t.

$\vec{C}$ a constant vector, c any scalar and f any differentiable scalar function.

1. Constant function Rule : $\frac{d}{dt}\vec{C} = \vec{0}$.

2. Scalar multiple Rule : $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$.

3. Sum Rule / Difference : $\frac{d}{dt}[\vec{u}(t) \pm \vec{v}(t)] = \vec{u}'(t) \pm \vec{v}'(t)$.

4. Scalar multiple Rule: $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

5. Dot product Rule : $\frac{d}{dt}[\vec{u}(t) . \vec{v}(t)] = \vec{u}'(t) . \vec{v}(t) + \vec{u}(t) . \vec{v}'(t)$

   $\frac{du}{dt} \cdot v(t) + u(t) . \frac{dv}{dt}$

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6. Cross product Rule : $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

7. Chain Rule : $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\,\vec{u}'(f(t))$

Example: $\vec{w}_1 = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{w}_2 = -\vec{i} - 7\vec{k}$.

$\vec{w}_1 \cdot \vec{w}_2 = -2 + 7 = +5$. $\Rightarrow (w_1 . w_2)' = 0$

① $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

Recall: ② $\vec{w}_1 \cdot \vec{w}_2 = |\vec{w}_1||\vec{w}_2| \cos\theta$.

④ $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$

Note: ③ If $\vec{u}_1 \cdot \vec{u}_2 = 0 \Rightarrow \vec{u}_1 \perp \vec{u}_2$

Example: $\vec{w}_1 = t\vec{i} + 3t\vec{j}$, $\vec{w}_2 = 3\vec{i} + 7t\vec{j}$.

$\vec{w}_1 \cdot \vec{w}_2 = 3t + 21t^2$.

$(\vec{w}_1 \cdot \vec{w}_2)' = 3 + 42t$.

Using Rule $\boxed{5}$: $(w_1 . w_2)' = (\underbrace{\vec{i} + 3\vec{j}}_{\vec{w}_1'}) . (\underbrace{3\vec{i} + 7t\vec{j}}_{\vec{w}_2}) + (\overbrace{t\vec{i} + 3t\vec{j}}^{\vec{w}_1}) \cdot (\underbrace{7\vec{j}}_{\vec{w}_2'})$

$= 3 + 21t + 21t = 3 + 42t$

Example: $\vec{w}_1 \times \vec{w}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 0 & -7 \end{vmatrix} = -21\vec{i} + 15\vec{j} + 3\vec{k}$.

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Recall • $\vec{w}_1 \times \vec{w}_2 = -\vec{w}_2 \times \vec{w}_1$

• while $\vec{w}_1 \cdot \vec{w}_2 = \vec{w}_2 \cdot \vec{w}_1$.

• $\vec{w}_1 \times \vec{w}_2 = (|\vec{w}_1||\vec{w}_2| \sin\theta) \vec{n}$

where $\vec{n}$: normal unit vector (RHR)
Right hand Rule.

$|\vec{w}_1 \times \vec{w}_2| = |\vec{w}_1||\vec{w}_2| \sin\theta$.

Corollary:

Remark: Assume $\vec{r}(t)$ is differentiable vector function with Constant Length. then : $\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$.

Proof: $|\vec{r}(t)| = C$

$$|\vec{r}(t)|^2 = C^2$$

but $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = C^2$.

Now $\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \vec{r}(t) \cdot \frac{d\vec{r}}{dt} + \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$

$$\Rightarrow 2\,\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$$

$$\Rightarrow \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0.$$

Result: $\vec{r}(t) \perp \frac{d\vec{r}}{dt}$ iff $\vec{r}(t)$ has Constant Length.

Example: $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$.

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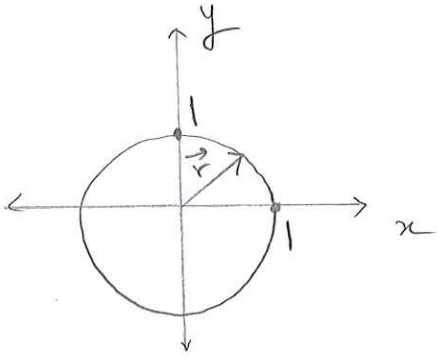
$|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$ (Constant length)

Cartesian equation : $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

paremetric equations

$$x^2 + y^2 = 1$$



$$r^2 = x^2 + y^2 = 1$$

$$\tan\theta = \frac{y}{x}$$

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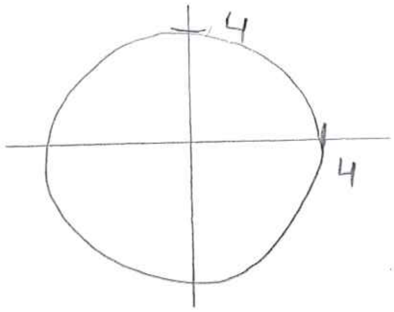
Example: Given a position vector of a particle moves along the path $\vec{r}(t) = \left(4\cos\frac{t}{2}\right)\vec{i} + \left(4\sin\frac{t}{2}\right)\vec{j}$

1. Find a Cartesian equation in x-y plane, Draw it.



$x = 4\cos\frac{t}{2}$, $y = 4\sin\frac{t}{2}$

$r^2 = x^2 + y^2 = 16\cos^2\frac{t}{2} + 16\sin^2\frac{t}{2} = 16$

2. Find Velocity and acceleration at $t = \pi, \frac{3\pi}{2}$.

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(-2\sin\frac{t}{2}\right)\vec{i} + \left(2\cos\frac{t}{2}\right)\vec{j}$

$\vec{a}(t) = \frac{d\vec{v}}{dt} = \left(-\cos\frac{t}{2}\right)\vec{i} - \left(\sin\frac{t}{2}\right)\vec{j}$.

135

270

$\vec{v}(\pi) = -2\vec{i}$. , $\vec{v}\left(\frac{3\pi}{2}\right) = -\sqrt{2}\,\vec{i} - \sqrt{2}\,\vec{j}$

$\vec{a}(\pi) = -\vec{j}$, $\vec{a}\left(\frac{3\pi}{2}\right) = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$.

3. Find the speed & the Direction of the motion at $t = 2\pi$.

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$S = |\vec{v}(t)| = \sqrt{2^2\sin^2\frac{t}{2} + 2^2\cos^2\frac{t}{2}} = \sqrt{4} = 2$, $\forall t$.

$D|_{2\pi} = \frac{\vec{v}(2\pi)}{|\vec{v}(2\pi)|} = \frac{-2\vec{j}}{2}$

(13.1) Q(2) $\vec{r}(t) = \frac{t}{t+1}\vec{i} + \frac{1}{t}\vec{j}$, $t = -\frac{1}{2}$

$x = \frac{t}{t+1}$ and $y = \frac{1}{t}$.

$\Rightarrow \quad x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y} \Longleftrightarrow y = \frac{1}{x} - 1$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{1}{(t+1)^2}\vec{i} - \frac{1}{t^2}\vec{j}$

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{-2}{(t+1)^3}\vec{i} + \frac{2}{t^3}\vec{j}$

$\vec{v}(-\frac{1}{2}) = 4\vec{i} - 4\vec{j}$ & $\vec{a} = -16\vec{i} - 16\vec{j}$.

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for Q20

Recall: The tangent line to a smooth curve $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\frac{d\vec{r}}{dt}(t_0) = \vec{v}(t_0)$

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Recall: A parametrization of the line through a point $a = (f(t_0), g(t_0), h(t_0))$ & parallel to $\vec{v} = \frac{dr}{dt}(t_0)$ is $I = a + t\vec{v}$

13.1 17 $\vec{r}(t) = (\ln(t^2+1))\vec{i} + (\tan^{-1}t)\vec{j} + \sqrt{t^2+1}\,\vec{k}$.

Find the angle between $\vec{v}$ & $\vec{a}$ when $t = 0$.

$$\vec{v}(t) = \frac{2t}{t^2+1}\vec{i} + \frac{1}{t^2+1}\vec{j} + \frac{t}{\sqrt{t^2+1}}\vec{k}$$

$$\vec{a}(t) = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\vec{i} - \left[\frac{2t}{(t^2+1)^2}\right]\vec{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\vec{k}$$

$\vec{v}(0) = \vec{j}$ & $\vec{a}(0) = 2\vec{i} + \vec{k}$.

$|v||\vec{a}|\cos\theta = \vec{v}(0)\vec{a}(0)$

$\vec{v}(0) \cdot \vec{a}(0) = 0 \implies \cos\theta = 0 \implies \boxed{\theta = \frac{\pi}{2}}$

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20 Find parametric equations for the line that is tangent to the given curve at $t = t_0$.

$\vec{r}(t) = t^2\vec{i} + (2t-1)\vec{j} + t^3\vec{k}$, $t_0 = 2$.

To find the parametric equations for the line, we need

1) a point (x_0, y_0, z_0) on the line

2) vector that is // to the line $[\vec{u} = \langle a, b, c \rangle]$

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• We know that $v(t_0)$ // line. tangent line.

$\implies v(t) = 2t\vec{i} + 2\vec{j} + 3t^2\vec{k}$

$\therefore \vec{v}(2) = 4\vec{i} + 2\vec{j} + 12\vec{k} \implies \vec{u} = \langle 4, 2, 12 \rangle$

& $\vec{r}(2) = 4\vec{i} + 3\vec{j} + 8\vec{k} \implies (4, 3, 8)$

$$\implies \begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned} \implies \begin{aligned} x &= 4 + 4t \\ y &= 3 + 2t \\ z &= 8 + 12t \end{aligned}$$

13.2 Integrals of Vector functions

Def: If $\vec{R}(t)$ is any antiderivative of $\vec{r}(t)$, then the indefinite Integral of $\vec{r}$ is:

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

Def: If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a,b]$, then so is $\vec{r}$, and the definite Integral of $\vec{r}$ is:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Example: Let $\vec{r}(t) = (\cos t)\vec{i} + 3\vec{j} - 2t\vec{k}$.

Find (1) $\int \vec{r}(t)\,dt$

$$\int \vec{r}(t)\,dt = (\sin t + c_1)\vec{i} + (3t + c_2)\vec{j} - \left(\frac{2t^2}{2} + c_3\right)\vec{k}$$

$$= \sin t\,\vec{i} + 3t\,\vec{j} - t^2\,\vec{k} + \underbrace{c_1\vec{i} + c_2\vec{j} + c_3\vec{k}}_{\vec{C}}$$

(2) $$\int_0^{\pi} \vec{r}(t)\,dt = \sin t\Big|_0^{\pi}\vec{i} + 3t\Big|_0^{\pi}\vec{j} - t^2\Big|_0^{\pi}\vec{k}$$

$$= 0\vec{i} + 3\pi\vec{j} - \pi^2\vec{k} = 3\pi\vec{j} - \pi^2\vec{k}.$$

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<u>Example:</u> Let $\vec{a}(t) = (-3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$ be the acceleration of a particle. Assume that at time $t=0$, the particle departed from the point $(3,0,0)$ with velocity $\vec{v}(0) = 3\vec{j}$.

Find the position of the particle at time $t=\pi$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{k} + \vec{C}$$

$$\vec{v}(0) = 3\vec{j} + \vec{C} = 3\vec{j} \Rightarrow \boxed{\vec{C} = 0}$$

$$\therefore \vec{v}(t) = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{k}.$$

Now $\vec{r}(t) = \int \vec{v}(t)\,dt = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k} + \vec{C_2}$

$\vec{r}(0) = 3\vec{i} + \vec{C_2} = 3\vec{i} + 0\vec{j} + 0\vec{k}$ (from $(3,0,0)$)

$$\therefore \vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2 k$$

$$\Rightarrow \vec{r}(\pi) = -3\vec{i} + \pi^2\vec{k}.$$

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13.2 Q10 Evaluate $\int_0^{\frac{\pi}{4}} \left[\sec t\,\vec{i} + \tan^2 t\,\vec{j} - t\sin t\,\vec{k}\right] dt$

$$= \int_0^{\frac{\pi}{4}} \left[\sec t\,\vec{i} + (\sec^2 t - 1)\vec{j} - \underbrace{(t\sin t)}_{\text{by parts}}\vec{k}\right] dt$$

$$= \ln(\sec t + \tan t)\Big]_0^{\frac{\pi}{4}}\vec{i} + \Big[\tan t - t\Big]_0^{\frac{\pi}{4}} + \Big[t\cos t - \sin t\Big]_0^{\frac{\pi}{4}}\vec{k}$$

$$= \ln(1+\sqrt{2})\vec{i} + \left(1 - \frac{\pi}{4}\right)\vec{j} + \left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\vec{k}$$

Q15 Solve the following IVP:

$$\frac{d^2\vec{r}}{dt^2} = -32\,\vec{k} \quad , \vec{r}(0) = 100\vec{k} \text{ and } \frac{d\vec{r}}{dt}\Big|_{t=0} = 8\vec{i} + 8\vec{j}.$$

$$\frac{d\vec{r}}{dt} = \int -32\,\vec{k}\,dt = -32t\,\vec{k} + \vec{C_1}.$$

$$\frac{d\vec{r}}{dt}\Big|_{t=0} = 8\vec{i} + 8\vec{j} = -32(0)\vec{k} + \vec{C_1} \Rightarrow \vec{C_1} = 8\vec{i} + 8\vec{j}$$

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$$\Rightarrow \frac{d\vec{r}}{dt} = 8\vec{i} + 8\vec{j} - 32t\,\vec{k}.$$

Therefore $\vec{r}(t) = \int \left(8\vec{i} + 8\vec{j} - 32t\,\vec{k}\right) dt$

$$\vec{r}(t) = 8t\,\vec{i} + 8t\,\vec{j} - 16t^2\,\vec{k} + \vec{C_2}$$

Using $\vec{r}(0) = 100\vec{k} \Rightarrow \vec{C_2} = 100\vec{k}$

$$\Rightarrow \vec{r}(t) = 8t\,\vec{i} + 8t\,\vec{j} + (100 - 16t^2)\vec{k}.$$

13.2 Q18 A particle traveling in a straight line is located at the point $\overset{P}{(1,-1,2)}$ and has speed 2 at $t=0$. The particle moves toward the point $\overset{S}{(3,0,3)}$ with Constant acceleration $2\vec{i}+\vec{j}+\vec{k}$. Find $\vec{r}(t)$?

sol:

$$\vec{V}(t) = \int (2\vec{i}+\vec{j}+\vec{k})\,dt = 2t\vec{i}+t\vec{j}+t\vec{k}+\vec{C}_1 \quad \text{Eq (1)}$$

We know that

$$\vec{V}(t)\Big|_{t=0} = \begin{pmatrix}\text{Unit}\\ \text{Direction}\\ \text{vector}\end{pmatrix}(\text{speed})$$

$$\Rightarrow \vec{V}(0) = \frac{\vec{PS}}{|\vec{PS}|}\cdot(2) = \frac{(3-1)\vec{i}+(0-(-1))\vec{j}+(3-2)\vec{k}}{|\vec{PS}|}(2).$$

$$\vec{V}(0) = \frac{1}{\sqrt{6}}(4\vec{i}+2\vec{j}+2\vec{k}) = \frac{4}{\sqrt{6}}\vec{i}+\frac{2}{\sqrt{6}}\vec{j}+\frac{2}{\sqrt{6}}\vec{k}$$

substitute $t=0$ in Eq.(1) $\Rightarrow \vec{C}_1 = \frac{4}{\sqrt{6}}\vec{i}+\frac{2}{\sqrt{6}}\vec{j}+\frac{2}{\sqrt{6}}\vec{k}$

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$\therefore$ Eq(1): $\vec{V}(t) = \left(2t+\frac{4}{\sqrt{6}}\right)\vec{i}+\left(t+\frac{2}{\sqrt{6}}\right)\vec{j}+\left(t+\frac{2}{\sqrt{6}}\right)\vec{k}$

Now $\vec{r}(t) = \int \vec{V}(t)\,dt = \left(t^2+\frac{4t}{\sqrt{6}}\right)\vec{i}+\left(\frac{t^2}{2}+\frac{2t}{\sqrt{6}}\right)\vec{j}+\left(\frac{t^2}{2}+\frac{2t}{\sqrt{6}}\right)\vec{k}+\vec{D}$

But $\vec{r}(0) = (1)\vec{i}+(-1)\vec{j}+(2)\vec{k} = \vec{D}$

$$\Rightarrow \vec{r}(t) = \left(t^2+\frac{4}{\sqrt{6}}t+1\right)\vec{i}+\left(\frac{t^2}{2}+\frac{2}{\sqrt{6}}t-1\right)\vec{j}+\left(\frac{t^2}{2}+\frac{2t}{\sqrt{6}}+2\right)\vec{k}.$$

13.3 Arc Length in Space.

Def: The length of a smooth curve:

$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$, $a \le t \le b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

Remark: $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\vec{i} + \dfrac{dy}{dt}\vec{j} + \dfrac{dz}{dt}\vec{k}$

$$\Rightarrow \quad |\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Hence
$$L = \int_a^b |\vec{v}(t)|\, dt$$

Example: Given $\vec{r}(t) = (2+t)\vec{i} - (1+t)\vec{j} + \sqrt{2}\,t\,\vec{k}$ for $0 \le t \le 3$. Find the arc length for the curve parametrized by the position vector $\vec{r}(t)$.

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$\vec{v}(t) = \vec{i} - \vec{j} + \sqrt{2}\,\vec{k}$

$|\vec{v}(t)| = \sqrt{1+1+(\sqrt{2})^2} = \sqrt{4} = 2.$

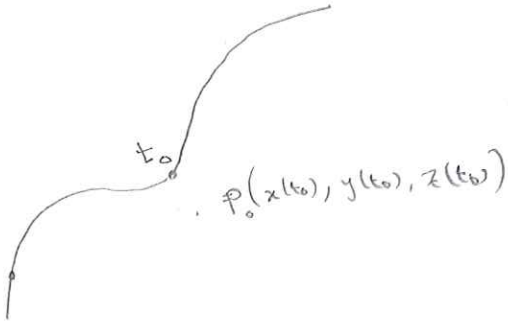
$\therefore\; L = \int_0^3 2\, dt = 6 - 0 = 6.$

Unit Tangent Vector: $\dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{1}{2}\vec{i} - \dfrac{1}{2}\vec{j} + \dfrac{\sqrt{2}}{2}\vec{k}.$

Def: Arc length parameter.

If we choose a basepoint $P(t_0)$ on a smooth curve C then the arc length parameter from the based point $P(t_0)$ is defined by:

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau \quad \ldots \circledast$$



Remark: Note that $L = s(b) - s(a)$.

proof: $s(b) - s(a) = \int_{t_0}^{b} |\vec{v}(t)|\,dt - \int_{t_0}^{a} |v(t)|\,dt$

$$= -\int_{b}^{t_0} |\vec{v}(t)|\,dt - \int_{t_0}^{a} |\vec{v}(t)|\,dt =$$

$$= -\left[\int_{b}^{a} |v(t)|\,dt\right] = \int_{a}^{b} |v(t)|\,dt = L.$$

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Speed on a smooth curve: $\boxed{\dfrac{ds}{dt} = |v(t)|}$ FTOC from (*)

Unit Tangent vector: $T = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$. V.I

Note: $\boxed{\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \vec{v} \cdot \dfrac{1}{|\vec{v}|} = \dfrac{d\vec{r}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{d\vec{r}}{ds}}$

Example: Find the arc Length parameter along the Curve

$$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$$

Consider the base point $P(1,0,0)$ and $\frac{\pi}{2} \le t \le \pi$.

Sol: we need to find t_0 such that:

$$\cos t_0 + t_0 \sin t_0 = 1 \quad \& \quad \sin t_0 - t_0 \cos t_0 = 0$$

clearly $t_0 = 0$.

$$\therefore \vec{v}(t) = \frac{d\vec{r}}{dt} = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\vec{i} + (\cancel{\cos t} + t\sin t - \cancel{\cos t})\vec{j}$$

$$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$$

$$|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = |t| = t$$

$$\therefore S(t) = \int_0^t \tau \, d\tau = \frac{t^2}{2}.$$

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2] Find the arc length L

$$L = S(\pi) - S(\tfrac{\pi}{2}) = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}.$$

OR: $$L = \int_{\frac{\pi}{2}}^{\pi} t \, dt = \frac{3\pi^2}{8}$$

13.3 (16) Find the point on the curve

$$\vec{r}(t) = (12 \sin t)\vec{i} - (12 \cos t)\vec{j} + 5t\vec{k}.$$

at a distance 13π units along the curve from the point $(0,-12,0)$ in the direction opposite to the direction of increasing arc length.

Sol:

Let $P(t_0)$ denote the point. Then:

$$\vec{v}(t) = (12 \cos t)\vec{i} + (12 \sin t)\vec{j} + 5\vec{k}.$$

It starts from the point $(0,-12,0)$ in $\vec{r}(t)$ $\Rightarrow t=0$, then:

$$-13\pi = \int_0^{t_0} \sqrt{(12\cos t)^2 + (12 \sin t)^2 + 5^2}\, dt. \quad \text{(Arc length)}$$

opposite

$$= \int_0^{t_0} \sqrt{169}\, dt = \int_0^{t_0} 13\, dt = 13 \Big|_0^{t_0} = 13t_0$$

$$\Rightarrow \boxed{t_0 = -\pi}$$

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even or odd ??

Then: $P(-\pi) = (12 \sin(-\pi))\vec{i} - 12\cos(-\pi)\vec{j} + 5(-\pi)\vec{k}$

$$= -12\vec{j} - 5\pi\vec{k}.$$

13.3 (Q 15) Find the length of the curve

$$\vec{r}(t) = (\sqrt{2}\,t)\vec{i} + (\sqrt{2}\,t)\vec{j} + (1-t^2)\vec{k}$$

from $\underbrace{(0,0,1)}_{t=0}$ to $\underbrace{(\sqrt{2},\sqrt{2}}_{t=1},0)$

$$\left.\begin{array}{l} \sqrt{2}\,t = \sqrt{2} \\ \& \ \sqrt{2}\,t = \sqrt{2} \\ \& \ 1-t^2 = 0 \end{array}\right\} \Rightarrow t=1$$

Sol: $L = \int_a^b |\vec{v}(t)|\,dt$

$$\vec{v}(t) = \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j} - 2t\,\vec{k}$$

$$|\vec{v}(t)| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (-2t)^2} = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$$

$$\therefore L = \int_0^1 2\sqrt{1+t^2}\,dt.$$

Now (Calc. 1 section 8.3) Trigonometric substitution:

Let $t = \tan\theta \Rightarrow dt = \sec^2\theta\,d\theta.$

when $t = 0 \Rightarrow \theta = 0$

$t = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\Rightarrow L = \int_0^{\pi/4} 2\sqrt{1+\tan^2\theta}\,(\sec^2\theta)\,d\theta = \int_0^{\pi/4} 2\sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta$$

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$$\therefore \; L = \int_0^{\frac{\pi}{4}} 2 \sec^3\theta \, d\theta.$$

Using Integration by parts:

Let $u = \sec\theta$, $dv = \sec^2\theta \, d\theta$

$du = \sec\theta \tan\theta \, d\theta$, $v = \tan\theta$

$$\therefore \; * \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \left[\sec\theta \tan\theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta \tan^2\theta \, d\theta \right]$$

$$= \sec\theta \tan\theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1) \sec\theta \, d\theta$$

$$= \sec\theta \tan\theta \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta + \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

to the first part *

$$\Rightarrow \frac{2}{2} \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \frac{\sec\theta \tan\theta}{2} \Big|_0^{\frac{\pi}{4}} + \frac{\ln(\sec\theta + \tan\theta)}{2} \Big|_0^{\frac{\pi}{4}}$$

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Therefore : $L = \cancel{2} \left(\frac{\sec\theta \tan\theta}{\cancel{2}} \right) \Big|_0^{\frac{\pi}{4}} + \cancel{2} \frac{\ln(\sec\theta + \tan\theta)}{\cancel{2}} \Big|_0^{\frac{\pi}{4}}$

$$\therefore L = \sqrt{2} + \ln(\sqrt{2} + 1).$$