3.4 Basis and Dimension

Def The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ form a basis for a vector space V iff
1. $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent
2. $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V

n in this case is the dimension of V and we write $\dim(V) = n$

Exp (standard basis)

1. The set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$
   $\Rightarrow \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$
2. The set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is the standard basis for $\mathbb{R}^{2\times 2}$
   where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
3. The set $\{1, x, x^2, \dots, x^{n-1}\}$ is the standard basis for P_n

Proof $\boxed{2}$
- If $c_1 E_{11} + c_2 E_{12} + c_3 E_{21} + c_4 E_{22} = 0$ then $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
  Hence, $E_{11}, E_{12}, E_{21}, E_{22}$ are linearly independent
- If A is any 2×2 matrix then

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} E_{11} + a_{12} E_{12} + a_{21} E_{21} + a_{22} E_{22}$$

  Hence, $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbb{R}^{2\times 2}$
- Thus, $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $\mathbb{R}^{2\times 2}$

Remark
- $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$

  Also $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$
  are bases for $\mathbb{R}^3$.
- Therefore, there are many bases we can choose for $\mathbb{R}^3$
- Any Basis for $\mathbb{R}^3$ must have exactly three elements.

Exp Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$. Find the basis for $N(A)$

- First we find $N(A)$: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right] R_2 - 2R_1$

$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right] -R_2 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$

Let $x_3 = \alpha$ and $x_4 = \beta$ $\Rightarrow x_2 = \beta - 2\alpha$

$\Rightarrow x_1 = \alpha - \beta$

$$N(A) = \left\{ \vec{x} \in \mathbb{R}^4 : x = \begin{pmatrix} \alpha - \beta \\ \beta - 2\alpha \\ \alpha \\ \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}$$

Note that $x = \begin{pmatrix} \alpha - \beta \\ \beta - 2\alpha \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

- Hence, any element of $N(A)$ can be written as a linear combination of these two vectos. Hence, they span $N(A)$

And these two vectors are linearly independent

- Hence, $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ form a basis for $N(A)$ $\Rightarrow \dim(N(A)) = 2$

Exp Are the vectors in what follows form a basis for $\mathbb{R}^2$

1. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ No because they are linearly dependent $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0$

2. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ Yes because they are linearly independent since $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \neq 0$ and they span $\mathbb{R}^2$ since if $\vec{v}$ is any vector in $\mathbb{R}^2$ then

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Leftrightarrow \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 3 & v_1 \\ 0 & 1 & 2v_2 - v_1 \end{array}\right] \Leftrightarrow \boxed{c_2 = 2v_2 - v_1}$ and $c_1 = v_1 - 3c_1$

$= v_1 - 6v_2 + 3v_1$

$\boxed{c_1 = 4v_1 - 6v_2}$

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* To find the dimension of V, we find linearly independent vectors that span V.

Th 4.1 If $\{\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_n\}$ is a spanning set for a vector space V, then any collection of m vectors in V, where $m > n$, is linearly dependent

Proof • Let $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m$ be a collection of m vectors in V where $m > n$.

• Since $\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_n$ span V $\Rightarrow$ $u_i = a_{i1}\vec{V}_1 + a_{i2}\vec{V}_2 + \cdots + a_{in}\vec{V}_n$

$$= \sum_{j=1}^{n} a_{ij}\vec{V}_j \quad , i = 1, 2, \ldots, m$$

• To show that $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m$ are linearly dependent :

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = c_1\sum_{j=1}^{n} a_{1j}\vec{V}_j + c_2\sum_{j=1}^{n} a_{2j}\vec{V}_j + \cdots + c_m\sum_{j=1}^{n} a_{mj}\vec{V}_j$$

$$= \sum_{i=1}^{m}\left(c_i\left[\sum_{j=1}^{n} a_{ij}\vec{V}_j\right]\right)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}c_i\right)\vec{V}_j = \vec{0} \text{ if } * \text{ holds}$$

• But $\overset{*}{\sum_{i=1}^{m} a_{ij}c_i = 0 \quad , j = 1, 2, \cdots, n}$ is a homogeneous system with more unknows m than equations n

$\Rightarrow$ Thus, by Th 2.1 the system must have nontrivial solution $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)^T$: $\bar{c}_1\vec{u}_1 + \bar{c}_2\vec{u}_2 + \cdots + \bar{c}_m\vec{u}_m = \vec{0} = \sum_{j=1}^{n} 0\vec{V}_j$

• Thus, $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m$ are linearly dependent.

$*$ is like :

$$\begin{array}{l} j=1 \rightarrow \\ j=2 \rightarrow \\ \vdots \\ j=n \rightarrow \end{array} \underset{n\times m}{\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}} \underset{m\times 1}{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}} = \underset{n\times 1}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

Corollary 4.2 If both $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ are basis for a vector space V, then $n = m$

Proof: Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be bases for V.

$\Rightarrow$ $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent and span V

$\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ " " " " " ".

$\Rightarrow$ By Th 4.1 since

$\vec{v}_1, \dots, \vec{v}_n$ span V and $\vec{u}_1, \dots, \vec{u}_m$ are linearly indep. $\Rightarrow m \leq n$

$\vec{u}_1, \dots, \vec{u}_m$ span V and $\vec{v}_1, \dots, \vec{v}_n$ " " " $\Rightarrow m \geq n$

Def . Let V be a vector space.

- If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for V, then V has **dimension** n. And we write $\dim(V) = n$.
- The subspace $\{\vec{0}\}$ of V has dimension 0.
- V is called **finite dimensional** if there is a finite set of vectors that spans V. Otherwise,

V is called **infinite dimensional**.

Exp Consider the vectors $x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $x_3 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$

what is the dimension of span (x_1, x_2, x_3)?

- Note that x_1 and x_2 are linearly independent vectors in $\mathbb{R}^2$,

and $\dim(\mathbb{R}^2) = 2$. Hence, x_1 and x_2 span $\mathbb{R}^2$

$\Rightarrow$ span $(x_1, x_2, x_3) =$ span (x_1, x_2)

$\Rightarrow$ dimension is 2

- We could choose x_1 and x_3 or x_2 and x_3 in the same manner.

since x_1, x_2, x_3 are linearly dependent. That is

$$\left(-\frac{33}{2}\alpha\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \left(\frac{13}{2}\alpha\right)\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \alpha\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Th 4.3 Let V be a vector space with $\dim(V) = n > 0$.
Then: $\boxed{1}$ any set of n linearly independent vectors spans V
$\boxed{2}$ any n vectors span V are linearly independent.

Proof $\boxed{1}$ • Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be linearly independent. Let $\vec{v} \in V$.

• Since $\dim(V) = n \Rightarrow V$ has basis of n vectors that span V

• By Th 4.1 $\Rightarrow \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n, \vec{v}$ are linearly dependent

$\Rightarrow$ there exists $c_1, c_2, \dots, c_n, c_{n+1}$ not all zero s.t

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n + c_{n+1} \vec{v} = \vec{0}$$

Note that $c_{n+1} \neq 0$. Since if $c_{n+1} = 0$, then $\vec{v}_1, \dots, \vec{v}_n$ will become linearly dependent.

$$\Rightarrow \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n \quad , \text{where } \alpha_i = \frac{c_i}{c_{n+1}} \ \forall i$$

• Hence, $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V since $\vec{v}$ was arbitrary.

$\boxed{2}$ • Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V. (Proof by Contradiction)

• If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent, then one of the v_i's say $\vec{v}_n$ can be written as a linear combination of the others.

• $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ still span V

• If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ are linearly dependent, then we eliminate another vector and the new set will still span V

• We continuoue eliminating vectors until we arrive a linearly independent spanning set $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ with $k < n$

• $\times$ since $\dim(V) = n$

• Hence, $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ must be linearly independent.

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Exp show that $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

- since $\dim(\mathbb{R}^3) = 3$
- It's enough to show either $\boxed{1}$ or $\boxed{2}$ from Th 4.3
- We prove $\boxed{1}$: $\begin{vmatrix}1 & -2 & 1\\2 & 1 & 0\\3 & 0 & 1\end{vmatrix} = \begin{vmatrix}1 & -2 & 1\\0 & 5 & -2\\0 & 6 & -2\end{vmatrix} = -10+12 = 2$

Hence, the vectors are linearly independent

By Th 4.3 $\Rightarrow$ they span $\mathbb{R}^3$

$\Rightarrow$ They are basis

Th 4.4 Let V be a vector space s.t $\dim(V) = n > 0$. Then

$\boxed{1}$ No set of fewer than n vectors can span V

$\boxed{2}$ Any subset of fewer than n linearly independent vectors can be extended to form a basis for V

$\boxed{3}$ Any spanning set containing more than n vectors can be pared down to form a basis for V.

Proof $\boxed{2}$ • Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ be linearly independent where $k < n$.

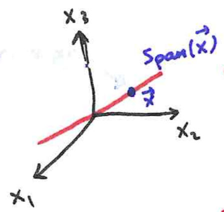
- by $\boxed{1}$ $\exists\ \vec{v}_{k+1} \in V$ s.t $\vec{v}_{k+1} \notin \text{span}(\vec{v}_1, \dots, \vec{v}_k) \Longrightarrow$ $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}$ are linearly independent
- If $k+1 < n$, then $\exists\ \vec{v}_{k+2} \in V$ s.t $\vec{v}_{k+2} \notin \text{span}(\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1})$ $\Rightarrow \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \vec{v}_{k+2}$ are linearly independent.
- We do extension simillarly until $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is obtained. by Th 4.3 this set spans V. Hence, it forms basis.

$\boxed{3}$. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a spannig set for V where $m > n$.

- by Th 4.1 $\Rightarrow \vec{v}_1, \dots, \vec{v}_m$ are linearly dependent.
- One of these vectors say $\vec{v}_m$ can be written as a linear combination of the others. Hence, we eliminate $\vec{v}_m$ and remains $m-1$ vectors that still span V.
- If $m-1 > n$, we eliminate another vector, until arriving a spanning set containing n vectors. by Th 4.3 they are LI. Hens they are basis.

Remarks

* If $\vec{x}$ is a nonzero vector in $\mathbb{R}^3$, then $\vec{x}$ spans one-dimensional subspace of $\mathbb{R}^3$. That is,



➡ $\text{span}(\vec{x}) = \{\alpha \vec{x} : \alpha \text{ is a scalar}\}$

➡ A vector $(a,b,c)^T \in \text{span}(\vec{x})$ iff the point (a,b,c) is on the line determined by $(0,0,0)$ and (x_1, x_2, x_3).

➡ Thus, one-dimensional subspace of $\mathbb{R}^3$ can be represented geometrically by a line through the origin.

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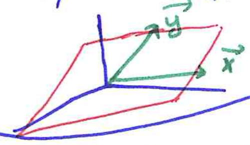
* If $\vec{x}$ and $\vec{y}$ are linearly independent in $\mathbb{R}^3$, then

$$\text{Span}(\vec{x}, \vec{y}) = \{\alpha \vec{x} + \beta \vec{y} : \alpha \text{ and } \beta \text{ are scalars}\}$$

is a two-dimensional subspace of $\mathbb{R}^3$

➡ A vector $(a,b,c)^T \in \text{span}(\vec{x}, \vec{y})$ iff (a,b,c) lies on the plane determined by $(0,0,0)$, (x_1, x_2, x_3) and (y_1, y_2, y_3).

➡ Thus, two-dimensional subspace of $\mathbb{R}^3$ is a plane through the origin.



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* If $\vec{x}, \vec{y}, \vec{z}$ are linearly independent in $\mathbb{R}^3$, then they form a basis for $\mathbb{R}^3$ and $\text{Span}(\vec{x}, \vec{y}, \vec{z}) = \mathbb{R}^3$. Hence, any point $(a,b,c)^T \in \text{Span}(\vec{x}, \vec{y}, \vec{z})$

Exp Let P be the vector space of all polynomials. Then $\dim(P) = \infty$

Proof. If P were finite dimensional, say $\dim(P) = n$, then any set of $n+1$ vectors would be linearly dependent.

• Take $1, x, x^2, \ldots, x^n$. But $1, x, x^2, \ldots, x^n$ are linearly independent

That is $W(1, x, x^2, \ldots, x^n) > 0$ ✗

• Hence, P can not be of dimension $n \Rightarrow \dim(P) = \infty$.

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* Similliarly one can show that $C[a,b]$ is infinite dimensional.