ENEE3309

Communication Notes

Chapter 1

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Review of signals & systems

Real Scenario

process
Behavior

Mathematical analysis

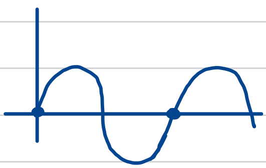
1D → Speech
2D → image
3D → video

Continuous & Discrete Signal

↓ $x(t)$

↓ $x[n]$

* Periodic & aperiodic



T_0

Fund. period

fund freq. : $f_0 = \frac{1}{T_0}$ (Hz)

$x(t) = A\cos(\omega_0 t)$

$\omega_0 = 2\pi f_0$ [rad/sec]

For periodic Signal : $x(t+T_0) = x(t)$

Remember:

- $\cos(\alpha \mp \beta) = \cos(\alpha)\cos(\beta) \pm \sin(\alpha)\sin(\beta)$
- $\sin(\alpha \mp \beta) = \sin(\alpha)\cos(\beta) \mp \cos(\alpha)\sin(\beta)$
- $\cos(\alpha)\cos(\beta) = ??$

To evaluate the expression of $\cos(\alpha)\cos(\beta) \Rightarrow$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad +$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

- $\sin(\alpha)\cos(\beta) = ?$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \quad \oplus$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$$
$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta)$$

- $\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$

- $\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$

⚠ Note: to check the periodicity

① if $x(t)$ is Single tone

(eg; $x(t) = A\cos(\underbrace{2\pi f_0}_{\omega_0} t)$ or $x(t) = A\sin(\underbrace{2\pi f_0}_{\omega_0} t)$) $\Rightarrow$ periodic Signal

$x(t) = 3\cos(\underbrace{20\pi}_{\omega_0} t)$, $x(t) = \sin(\underbrace{15}_{\omega_0} t)$) with $T_0 = \frac{2\pi}{\omega_0}$

$\omega_0 = 2\pi f_0 \Rightarrow f_0 = \frac{\omega_0}{2\pi} \Rightarrow T_0 = \frac{2\pi}{\omega_0}$

② If $x(t)$ is multiton (eg: $x(t)=A_1\cos(w_1t)+A_2\sin(w_0t)+\ldots$)
To check the periodity ① rational / a rational number $\frac{n_1}{n_2}$?!
② GCD

Ex: check if the following signal is periodic or aperiodic

$x(t) = 3\cos(\underbrace{20\pi}_{w_1} t + \frac{\pi}{4}) + 2\sin(\underbrace{60\pi}_{w_2} t)$

$2\pi = 2\pi f_1$ $\quad$ $w_2 = 2\pi f_2$
$10 = n_1 f_0$ $\quad$ $60\pi = 2\pi f_0$
$\quad$ $30 = n_2 f_0$

$\frac{10}{30} = \frac{n_1}{n_2} \Rightarrow$ Rational Number $\Rightarrow$ Periodic Signal

if $n_1 = 1 \Rightarrow 10 = (1) f_0 \longrightarrow f_0 = 10$ Hz

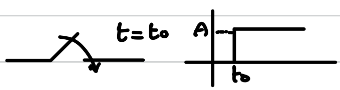
$\Rightarrow x(t) = 3\cos(2\pi(1)(10)t + \frac{\pi}{4}) + 2\sin(2\pi(3)(10)t)$

if we apply $x(t+T_0) = x(t)$ where $T_0 = \frac{1}{f_0} = \frac{1}{10}$

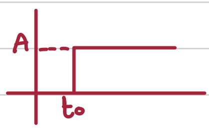
Periodic Signal أي
جمع sin + cos
نفس ال fundemental
frequency ثابتة
مضروبة ب integer
values مختلفة

Singularity Functions:

① Step function

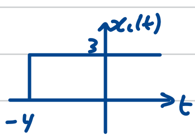


$AU(t+t_0) = \begin{cases} A & t > t_0 \\ 0 & \text{o.w} \end{cases}$

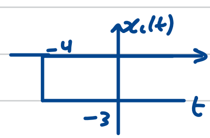


Sketch the following Signals:

① $x_1(t) = 3u(t+4)$



② $x_2(t) = -3u(t+4)$



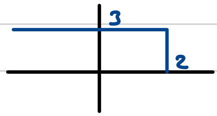
اشي مضروب بـ t دائماً باخذه عامل مشترك

③ $x_3(t) = 3u(-2t+4)$

$= 3u(-2(t-2))$

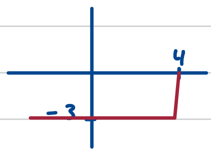
منوخذ الاشارة بعين الاعتبار

ما الهمية فيه لأنه العلاقة بالميل والميل u(t) = صفر



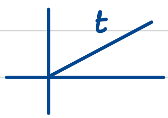
④ $x_4(t) = -3u(-2t+8)$

$= -3u(-2(t-4))$



② Ramp Function

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & \text{o.w} \end{cases}$$



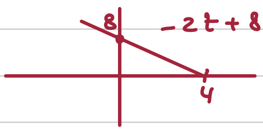
$r(2t) = 2r(t)$

Ex: Sketch the following signal

$x(t) = r(-2t+8)$

$= r(-2(t-4))$

اذا كان في سالب تقلب لتحت
والمعادلة بدخل علينا (-)



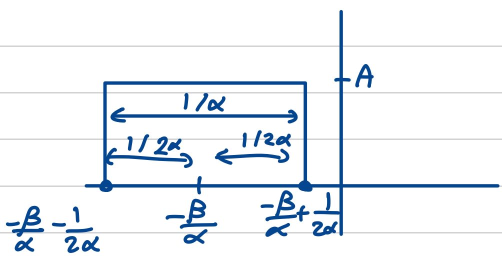
③ Pulse Function $\Pi(\ )$

Sketch $x(t) = A\Pi(\alpha t+\beta)$

assume $A, \alpha, \beta > 0$

Ans: $x(t) = A\Pi\left(\alpha\left(t+\frac{\beta}{\alpha}\right)\right)$

Amp. scale center



Ex: Sketch the following signals

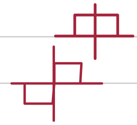
$x_1(t) = 3\Pi(-2t+6)$

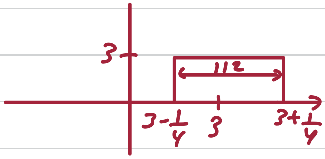
$= 3\Pi(-2(t-3))$

$= 3\Pi(2(t-3))$

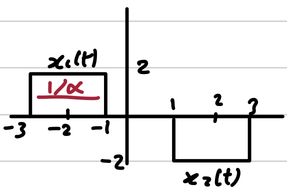
Remember: $x(t) = x(-t)$ even

$x(t) = -x(-t)$ odd

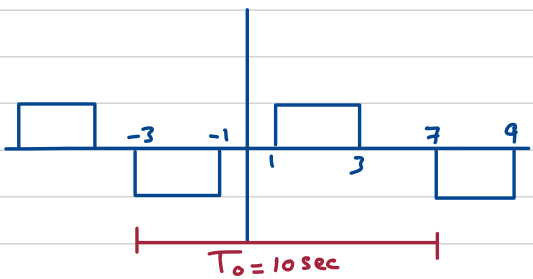




Ex: Express the following signal



at $f = 10$



a) in terms of pulse function

$x(t) = x_1(t) + x_2(t)$

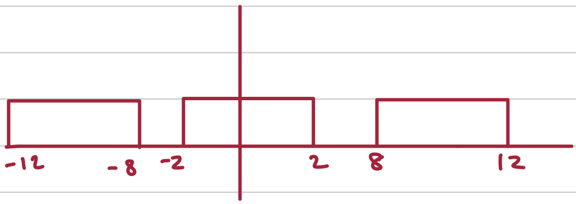
where $x_1(t) = 2\Pi\left(\frac{1}{2}(t+2)\right)$

$x_2(t) = -2\Pi\left(\frac{1}{2}(t-2)\right)$

$x(t) = \sum_{n=-\infty}^{\infty} x_1(t-10n)$

where $x_1(t) = 2\Pi\left(\frac{t+2}{2}\right) - 2\Pi\left(\frac{t-2}{2}\right)$

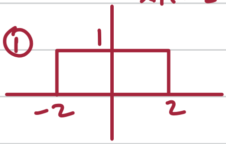
$x(t) = \sum_{-\infty}^{\infty} 2\Pi\left(\frac{t-10n+2}{2}\right) - 2\Pi\left(\frac{t-10n-2}{2}\right)$



To evaluate $x(t)$

$x_1(t) = \Pi\left(\frac{t}{4}\right)$

①



① Evaluate the fundamental period T_0

$T_0 = 10$ sec

② Evaluate the fund. freq.

$f_0 = \frac{1}{T_0} = \frac{1}{10}$ Hz

$$x(t) = \sum_{-\infty}^{\infty} x_1(t-10n) \quad ②$$

$$= \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-10n}{4}\right)$$

③ Express $x(t)$ in terms of pulse function

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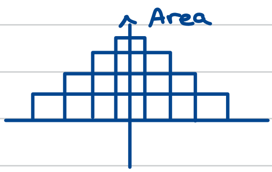
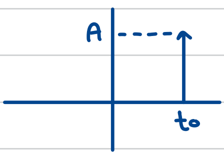
④ Impulse function $\delta(t)$



$$A\,\delta(t-t_0) = \begin{cases} A & t = t_0 \\ 0 & , \text{ o.w}, \; t_0 > 0 \end{cases}$$

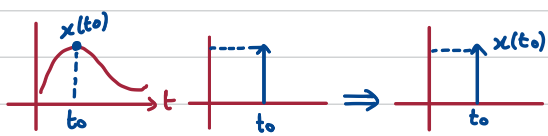


if $A = 1 \Rightarrow$ unit impulse function

Impulse function properties

① $\delta(-t) = \delta(t)$ even

② $\delta(at) = \frac{1}{|a|}\delta(t)$



③ $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$ [Sampling theorem]

Example: Evaluate:

$$(2t^2+5)\delta(-2t+8) = (2t^2+5)\delta(-2(t-4))$$
$$= (2t^2+5)\delta(2(t-4))$$
$$= (2t^2+5)\cdot\frac{1}{2}\delta(t-4)$$
$$= 37\cdot\frac{1}{2}\delta(t-4)$$

④ $$\int_{t_1}^{t_2} x(t)\delta(t-t_0)\,dt = \begin{cases} x(t_0) & t_1 \le t \le t_2 \\ 0 & \text{o.w} \end{cases}$$ Sifting

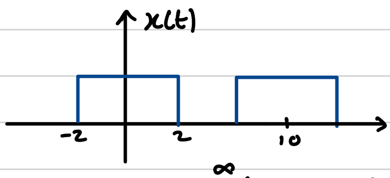
Evaluate :

$$\int_{-5}^{5}(2t^2+5)\delta(-2t+8) = \frac{37}{2}$$

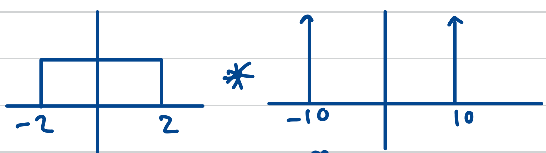
⑤ $$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(t-\lambda)\delta(\lambda-t_0)\,d\lambda = x(t-t_0)$$ Convolution

أي periodic signal بقدر أعملها

Ex:

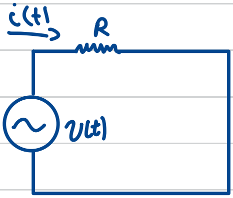


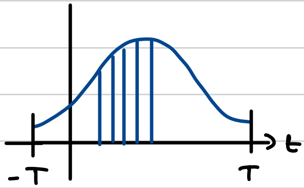
$$\Pi\left(\frac{t}{4}\right) * \sum_{n=-\infty}^{\infty}\left(\delta(t-10n)\right)$$



$$x(t) = \Pi\left(\frac{t}{4}\right) * \sum_{-\infty}^{\infty}\delta(t-10n)$$
$$= \sum_{n=-\infty}^{\infty}\Pi\left(\frac{t-10n}{4}\right)$$

Power signal & Energy signal





$$P(t) = v(t)\, i(t)$$
$$= i^2(t)(R) \longrightarrow$$ ثابتة

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} i^2(t)$$

$$E = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)\,dt$$

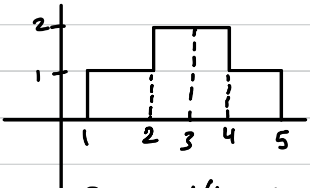
In general $$E = \lim_{T\to\infty} \int_{-T}^{T} (x(t))^2\,dt$$

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt$$

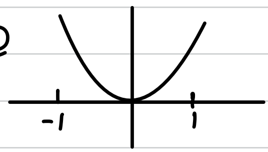
Note: ① $0 < P < \infty$ & $E = \infty \longrightarrow$ Power Signal (Ex: Periodic Signal)

② $0 < E < \infty$ & $P = 0 \longrightarrow$ Energy Signal (Ex: Time limited or bounded signal)

For the following Signals

① 

Bounded / time limited
↳ a periodic

② 

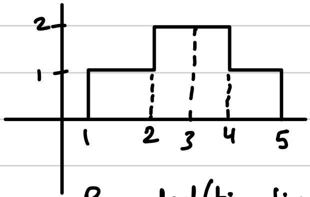
aperiodic → periodic
Conv. + train of delta

③ $x(t) = 3\sin(30\pi t)$
④ $x(t) = 3\cos(30\pi t)$

Check Energy / power
for each signal

For the following Signals Check Energy/power for each signal:

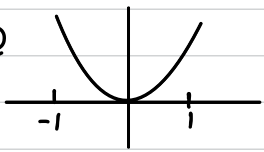
①



Bounded/time limited

↳ aperiodic

②



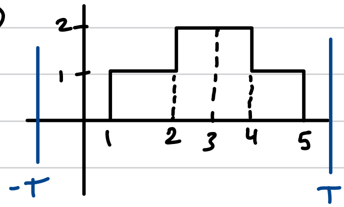
aperiodic → periodic

Conv. + train of delta

③ $x(t) = 3\sin(30\pi t)$

④ $x(t) = 3\cos(30\pi t)$

①



$$E = \lim_{T\to\infty} \int_{-T}^{T} (x(t))^2 dt$$

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \left[\int_{-T}^{1} (0)^2 dt + \int_{1}^{2} (1)^2 dt + \int_{2}^{4} (2)^2 dt + \int_{4}^{5} (1)^2 dt + \int_{5}^{T} (0)^2 dt \right]$$

$$= \lim_{T\to\infty} \left(0 + 1 + (4)(2) + (1) + 0 \right)$$

Answer $< \infty$

$\therefore$ Energy Signal

$$P_{avg} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \frac{1}{\infty} /// = 0$$

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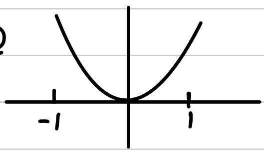
②



Not energy or power Signal

③ $A\sin(w_0 t)$ (In general)

$$E = \lim_{T\to\infty} \int_{-T}^{T} A^2 \sin^2(w_0 t)\,dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} \left[\frac{A^2}{2} - \frac{A^2}{2}\cos(2w_0 t) \right] dt$$

$$= \lim_{T\to\infty} \left[\frac{A^2}{2}\cdot 2T - \frac{A^2}{2}\cdot \frac{T}{2(2\pi)} \cancelto{0}{\left[\sin\left(2\cdot\frac{2\pi}{T}\cdot T\right) - \sin\left(2\cdot\frac{2\pi}{T}\cdot -T\right) \right]} \right]$$

$$= \lim_{T\to\infty} \frac{A^2}{2}\cdot 2T = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{2\cancel{T}}\cdot\frac{A^2}{2}\cdot 2\cancel{T} = \frac{A^2}{2}$$

In General, if $x(t)$ is periodic Signal $\Rightarrow P_{avg} = \frac{1}{T}\int_{T} |x(t)|^2\,dt$

* اذا كان عنا Square wave وبدنا نوجدلها ال Power منحولها لـ Sin/cos ومنوجد لكل وحدة
ال Power $\frac{A^2}{2}$

Fourier series

- Trigonometric FS

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + \sum_{n=1}^{\infty} b_n \sin(n w_0 t)$$

where:

a_0, a_n, b_n are trigonometric coefficients FS.

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \qquad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n w_0 t)\,dt \qquad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n w_0 t)\,dt$$

Remember:

- if $x(t)$ even $\Rightarrow a_n \checkmark,\ b_n = 0$
- if $x(t)$ odd $\Rightarrow a_n = 0,\ b_n \checkmark$

- Complex exponential FS.

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j n w_0 t}$$

where X_n: Complex exp. Coefficient FS.

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j n w_0} \, dt$$

Remember:

if Trig. coefficient FS are Known

$$X_n = \begin{cases} \frac{1}{2}(a_n - j b_n) & n>0 \\ \frac{1}{2}(a_{-n} + j b_{-n}) & n<0 \\ a_0 & n=0 \end{cases} = \begin{cases} \sqrt{\left(\frac{a_n}{2}\right)^2 + \left(\frac{-b_n}{2}\right)^2} \ \measuredangle -\tan^{-1}\left(\frac{b_n}{a_n}\right) & n>0 \\ \sqrt{\left(\frac{a_{-n}}{2}\right)^2 + \left(\frac{b_{-n}}{2}\right)^2} \ \measuredangle \tan^{-1}\left(\frac{b_{-n}}{a_{-n}}\right) & n<0 \\ a_0 \ \measuredangle 0 & n=0 \end{cases}$$

- $|X_n| = |X_{-n}|$ even and $\measuredangle \theta X_n = -\measuredangle \theta X_{-n}$ odd

$a_n = a_{-n}$ and $b_n = b_{-n}$

* If X_n Known $\Rightarrow$

$a_n = 2\,\mathrm{Re}\{X_n\}$

$b_n = -2\,\mathrm{Im}\{X_n\}$

Ex: Consider the following Signal:

$$x(t) = 3 + 4\sin(50\pi t) - 6\cos(30\pi t)$$

a- Evaluate the trig. coefficient FS.
b- Evaluate the complex Exp coefficient FS.
c- Plot the line spectra amp. and phase shift.
d- Evaluate the total power.

Ans: $x(t) = 3 + 4\sin(\underline{50}\pi t) - 6\cos(30\pi t)$

$\omega_1 = 50\pi \qquad \omega_2 = 30\pi \qquad \frac{5}{3} = \frac{n_1}{n_2}$ when $n_1 = 5 \Rightarrow f_0 = 5$

$f_1 = 25 \qquad f_2 = 15$

$f_0 = n_1 f_0 \qquad f_0 = n_2 f_0$

$$x(t) = 3 + 4\sin(2\pi(5)(5)t) - 6\cos(2\pi(3)(5)t)$$

(a) $\Rightarrow a_0 = 3$

$$a_n = \begin{cases} -6 & n=3 \\ 0 & \text{o.w} \end{cases} \qquad b_n = \begin{cases} 4 & n=5 \\ 0 & \text{o.w} \end{cases} \qquad a_0 = 3$$

(b)

$$X_n = \begin{cases} \frac{1}{2}(-6-j0) & n=3 \\ \frac{1}{2}(-6+j0) & n=-3 \\ \frac{1}{2}(-j4) & n=5 \\ \frac{1}{2}(j4) & n=-5 \\ 3 & n=0 \\ 0 & \text{o.w} \end{cases}$$

(c)

$$X_n = \begin{cases} 3\angle\pi & n=3 \\ 3\angle-\pi & n=-3 \\ 2\angle-\pi/2 & n=5 \\ 2\angle\pi/2 & n=-5 \\ 3\angle 0 & n=0 \\ 0 & \text{o.w} \end{cases}$$

$-3 = 3\angle\pi$
$3\angle-\pi$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$
$3e^{j\pi} = 3[\cos(\pi) + j\sin(\pi)]$
$= -3$
$3e^{-j\pi} = 3[\cos(-\pi) + j\sin(\pi)]$
$= -3$

Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

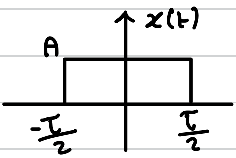
$$x(t) = \int_{-\infty}^{\infty} X(t) e^{j2\pi f t} df$$

Note: $|X(f)| = |X(-f)|$ Even

$\measuredangle \theta X(f) = - \measuredangle \theta X(f)$ odd

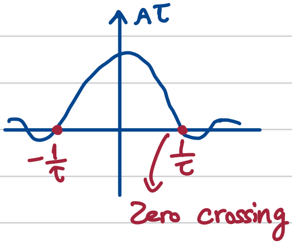
- if $x(t)$ even $\longrightarrow X(f)$ real
- if $x(t)$ odd $\longrightarrow X(f)$ imaginary

Ex: Evaluate the FT of the following signal



$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi f t} dt$$

$= A\tau \,\text{sinc}(\tau f)$



Theorem of FT

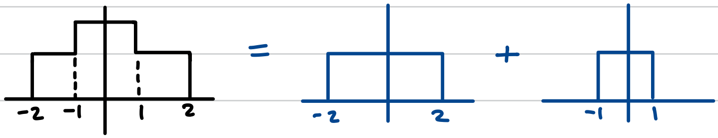
① Linearity

$$\mathcal{F}[x_1(t) + x_2(t)] = X_1(f) + X_2(f)$$

② Scaling

$$\mathcal{F}[x(at)] = \frac{1}{a} X\left(\frac{f}{a}\right)$$

Ex: Evaluate the Fourier transform of the following signals:



Ans: $x(t) = x_1(t) + x_2(t)$

$= \Pi\left(\frac{t}{4}\right) + \Pi\left(\frac{t}{2}\right)$

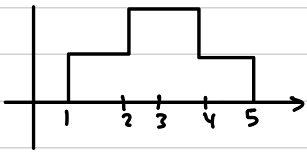
$\Rightarrow X(f) = 4\text{sinc}(4f) + 2\text{sinc}(2f)$

---

③ Time-delay

$$\mathcal{F}[x(t-t_0)] = X(f)e^{-j2\pi f t_0}$$

Ex: Evaluate the FT of the following signal:



Ans: $x(t) = x_1(t) + x_2(t)$

$= \Pi\left(\frac{t-3}{4}\right) + \Pi\left(\frac{t-3}{2}\right)$

$X(f) = 4\text{sinc}(4f)e^{-j2\pi f(3)} + 2\text{sinc}(2f)e^{-j2\pi f(3)}$

Ex: Evaluate FT of $\Pi(-3t+6)$

Ans: $x(t) = \Pi(-3(t-2)) = \Pi(3(t-2))$

$X(f) = \frac{1}{3}\text{sinc}\left(\frac{1}{3}f\right)e^{-j2\pi f(2)}$

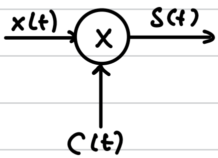
④ Frequency transition

$$\mathcal{F}[x(t)e^{j2\pi f_o t}] = X(f-f_o)$$

⑤ Modulation Theorem

$$\mathcal{F}[x(t)\cos(2\pi f_o t)] = \frac{1}{2}X(f+f_o) + \frac{1}{2}X(f-f_o)$$

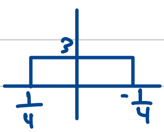
Ex: Consider the following System, Evaluate & plot the FT of x(t) and c(t)

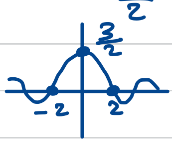


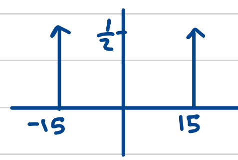
where $x(t) = 3\pi(3t)$

$c(t) = \cos(30\pi t)$

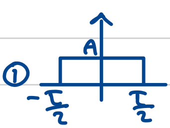
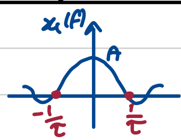
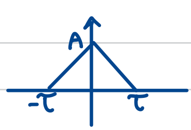
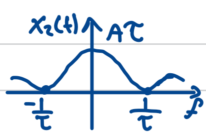
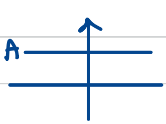
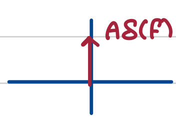
$X(t) = 3\pi(2t)$ $\quad X(f) = \frac{3}{2}\text{sinc}\left(\frac{f}{2}\right)$







Remember:

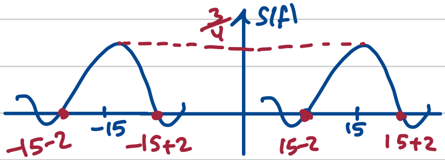
	Time-domain	Freq.-domain
①	 $x_1(t) = A\pi\left(\frac{t}{\tau}\right)$	 $x_1(f) = A\tau\,\text{sinc}(\tau f)$
②	 $x_2(t) = A\chi\left(\frac{t}{\tau}\right)$	 $x_2(f) = A\tau\,\text{sinc}^2(\tau f)$
③	 $x_3(t) = A$	 $x_3(t) = A\,\delta(f)$

b- Evaluate & plot the Fourier transform of $S(t)$

Ans. $S(t) = C(t)x(t)$

$= 3\Pi(2t)\cos(30\pi t)$

$= 3.\frac{1}{2}.\frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}(f-15)\right) + 3.\frac{1}{2}.\frac{1}{2}\operatorname{sin}\left(\frac{1}{2}(f+15)\right)$



Method: $\mathcal{F}[s(t)] = \mathcal{F}[x(t)c(t)]$

$= \mathcal{F}[x(t)] * \mathcal{F}[c(t)]$

$= \frac{3}{2}\operatorname{sinc}(\frac{f}{2}) * \left[\frac{1}{2}\delta(f-15) + \frac{1}{2}\delta(f+15)\right]$

$= \frac{3}{2}.\frac{1}{2}\operatorname{sinc}(\frac{f}{2}) * \delta(f-15) + \frac{3}{2}.\frac{1}{2}\operatorname{sinc}(\frac{f}{2})\delta(f+15)$

$= \frac{3}{4}\operatorname{sinc}\left(\frac{f-15}{2}\right) + \frac{3}{4}\operatorname{sinc}\left(\frac{f+15}{2}\right)$

---

6 Convolution Theorem

$\mathcal{F}[x_1(t) * x_2(t)] = X_1(f)X_2(f)$

Ex: Evaluate the FT of :

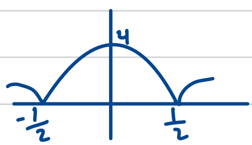
$y(t) = x_1(t) * x_2(t)$

$x_1(t) = \Pi(\frac{t}{2})$ and $x_2(t) = (\frac{t}{2})$

Ans: $\mathcal{F}[y(t)] = \mathcal{F}[x_1(t)].\mathcal{F}[x_2(t)]$

$= 2\operatorname{sinc}(2f)\,2\operatorname{sinc}(2f)$

$= 4\operatorname{sinc}^2(2f)$



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⑦ Duality theorem:

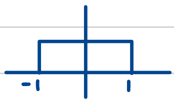
$$X(t) \longrightarrow x(f)$$
$$x(t) \longrightarrow X(f)$$

e.g. $\Pi(t) \longrightarrow \text{sinc}(f)$

$\text{sinc}(t) \longrightarrow \Pi(-f)$

Ex: Evaluate & plot the FT of $x(t) = 2\,\text{sinc}(2t)$

Ans: $X(f) = 2 \cdot \frac{1}{2} \Pi\left(-\frac{f}{2}\right) = \Pi\left(\frac{f}{2}\right)$



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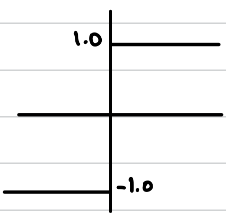
⑧ Differentiation & Integration

$$\mathcal{F}\left[\frac{d^n x(t)}{dt^n}\right] = (j2\pi f)^n X(f)$$

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\lambda)\,d\lambda\right] = \frac{1}{j2\pi f} X(f) + c\,\delta(f)$$

Ex: Evaluate the FT of the signum function, which is defined as:



Ans: $\text{Sgn}(t) = 2u(t) - 1$

$$\mathcal{F}\left[\frac{d\,\text{sgn}(t)}{dt}\right] = \mathcal{F}[2\delta(t)]$$

$$j2\pi f\,\mathcal{F}[\text{Sgn}(t)] = 2$$

$$\mathcal{F}[\text{sgn}(t)] = \frac{1}{j\pi f}$$

Ex: Evaluate the FT of the following signal: $\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda}$

$= x(t) * \frac{1}{\pi t}$

Ex: Evaluate the Hilbert transform of $x(t) = \cos(20\pi t)$

Ans: Hilbert Transform can expressed as: phase shift يعمل
للاشارة بمقدار 90°

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\lambda)}{t-\lambda}\, d\lambda = \frac{1}{\pi t} * x(t)$$

$$-\hat{x}(f) = \mathcal{F}\left[\frac{j.1}{j.\pi t}\right]\mathcal{F}[x(t)]$$

since $\mathcal{F}[\mathrm{sgn}(t)] = \frac{1}{j\pi f}$ and $\mathcal{F}\left[\frac{1}{j\pi t}\right] = -\mathrm{sgn}(f)$

$$\Rightarrow \mathcal{F}\left[\frac{1}{\pi t}\right] = -j\,\mathrm{sgn}(f)$$

$$\Rightarrow \hat{x}(f) = -j\,\mathrm{sgn}(f) . X(f)$$

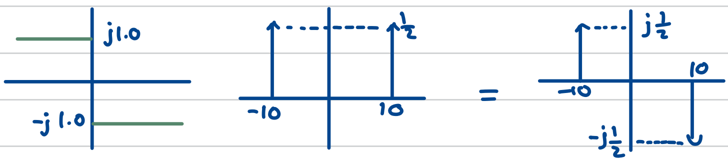
$x(t) = \cos(20\pi t)$

In our example:

$\hat{x}(f) = -j\,\mathrm{sgn}(f) X(f)$

where $X(f) = \frac{1}{2}\delta(f-10) + \delta(f+10)$

$$\Rightarrow \hat{x}(f) = -j\,\mathrm{sgn}(f)\left[\frac{1}{2}\delta(f-10) + \frac{1}{2}\delta(f+10)\right]$$



$$\hat{x}(f) = j\frac{1}{2}\delta(f+10) - j\frac{1}{2}\delta(f-10)$$

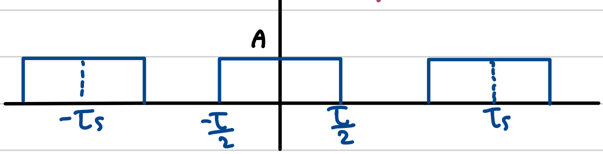
$$\hat{x}(f) = j\frac{1}{2} e^{-j2\pi(10)t} - j\frac{1}{2} e^{j2\pi(10)t}$$

$$= \frac{e^{j2\pi(10)t} - e^{-j2\pi(10)t}}{j2} = \sin(20\pi t)$$

Fourier transform for periodic signal

Ex: Consider the following periodic signal

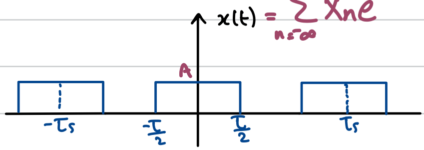
$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_s t} \xrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_s)$$



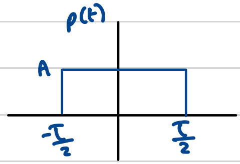
Calculate FT of x(t)

Ans:

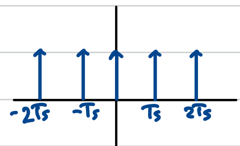
$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_s t}$$



$=$



$*$



$$\mathcal{F}[x(t)] = \mathcal{F}[p(t)]\,\mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] \qquad \sum_{m=-\infty}^{\infty} \delta(t - mT_s) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_s t}$$

To Evaluate $\mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right]$

$$\Rightarrow \mathcal{F}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s)\right] = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_s t}\right] = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s)$$

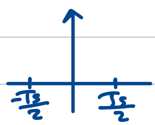
Remember: Evaluate FT for the following:

① $X_1(t) = 3 \xrightarrow{\mathcal{F}} X_1(f) = 3\,\delta(f)$

② $X_2(t) = 3e^{j20\pi t} \xrightarrow{\mathcal{F}} X_2(f) = 3\,\delta(f - 10)$

$$\Rightarrow X(f) = p(t) \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s)$$

To Evaluate C_n:



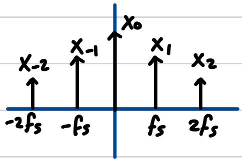
$$C_n = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-j2\pi n f_s t}\, dt = f_s$$

$$\Rightarrow X(f) = P(f) \sum_{n=-\infty}^{\infty} f_s \delta(f - nf_s) = \sum_{n=-\infty}^{\infty} f_s P(f) \delta(f - nf_s)$$

$$= \sum_{n=-\infty}^{\infty} f_s P(nf_s) \delta(f - nf_s)$$

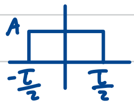
$$\Rightarrow X_n = f_s P(nf_s)$$

Plot $X(f)$



In Our Example: ⓐ $X(f) = \sum_{n=-\infty}^{\infty} f_s P(nf_s)\,\delta(f-nf_0)$

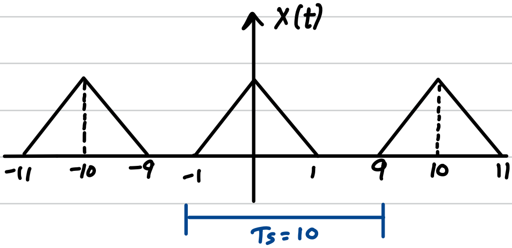
where $p(t) = A\,\Pi\left(\frac{t}{\tau}\right)$



$\xrightarrow{\mathcal{F}} P(f) = A\tau\,\mathrm{sinc}(\tau f)$

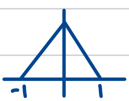
$\Rightarrow X(f) = \sum_{n=-\infty}^{\infty} f_s A\tau \sin(\tau n f_s)\,\delta(f-nf_s)$

Ex: Consider the following periodic signal:



ⓐ Evaluate the FT of $x(t)$

$X(f) = \sum f_s P(nf_s)\,\delta(f-nf_s)$ where $f_s = \frac{1}{10}$



$p(t) = \Lambda(t) \longrightarrow P(f) = \mathrm{sinc}^2(f)$

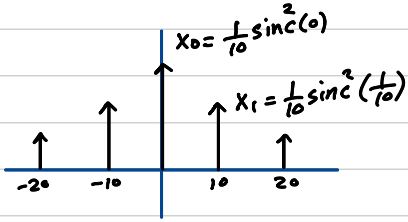
$P\left(\frac{n}{10}\right) = \mathrm{sinc}^2\left(\frac{n}{10}\right)$

$\Rightarrow X(f) = \sum_{n=-\infty}^{\infty} \frac{1}{10}\,\mathrm{sinc}^2\left(\frac{n}{10}\right)\delta\left(f - \frac{n}{10}\right)$

ⓑ Evaluate the Complex Exp. FS

$X_n = f_s P(nf_s) = \frac{1}{10}\,\mathrm{sinc}^2\left(\frac{n}{10}\right)$

ⓒ Plot the spectrum of $X(f)$

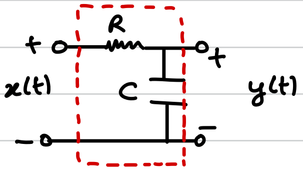


Filters:

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$LTI \Rightarrow y(t) = x(t) * h(t)$$

let



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

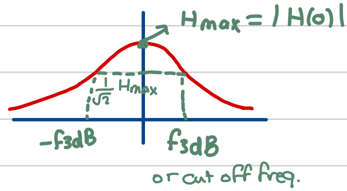
$$\mathcal{F}\left[RC\frac{dy(t)}{dt} + y(t) = x(t)\right]$$

$$\Rightarrow RC\, j2\pi f\;\; Y(f) + Y(f) = X(f)$$

Remember:

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f)$$

LTI



$$\Rightarrow Y(f) = X(f)\,H(f)$$

$$H(f) = \frac{Y(f)}{X(f)}$$

$$[j2\pi f RC + 1]\,Y(f) = X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + j2\pi f RC} = \frac{1\angle 0}{\sqrt{(1)^2 + (2\pi f RC)^2}\angle \tan^{-1}(2\pi f RC)}$$

$$\Rightarrow |H(f)| = \frac{1}{\sqrt{1 + (2\pi f RC)^2}}$$

$$\angle \theta_{H(f)} = -\tan^{-1}(2\pi f RC)$$

Remember:

$$|H(f)|^2 = \frac{1}{2}$$

$$|H(f)| = \frac{1}{\sqrt{2}}$$

To convert from Linear $\longrightarrow$ dB

$$X \longrightarrow 10\log(x)$$

$$P = X^2 \longrightarrow 10\log^2(x)$$

$$20\log(x)$$

$$\searrow \frac{1}{\sqrt{2}}$$

To Evaluate f_{3dB}

$$\frac{1}{\sqrt{2}} H_{max} = |H(f_{3dB})|$$

$$\frac{1}{\sqrt{2}} \cdot (1) = \frac{1}{\sqrt{(1)^2 + (2\pi f_{3dB} RC)^2}}$$

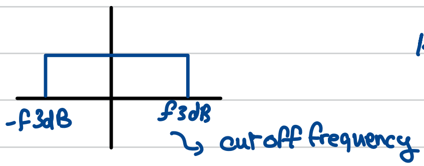
$$2 = 1 + (2\pi f_{3dB} RC)^2$$

$$\Rightarrow f_{3dB} = \frac{1}{2\pi RC}$$

if $f_{3dB} = 2KHz = \dfrac{1}{2\pi RC}$, and if $R = 2K\Omega$

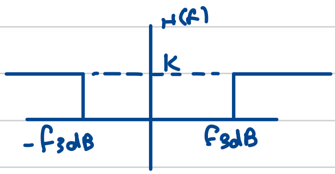
$\Rightarrow C = ?$

① Low Pass Filter (L.P.F)

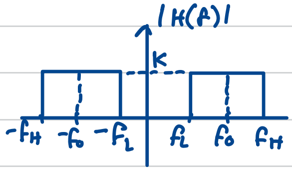


$K \leftarrow$ gain $= \dfrac{\text{Amp. of Output}}{\text{Amp. of input}}$

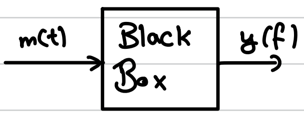
② High Pass filter



③ Band Pass Filter



Ex: Consider the following System



Assume

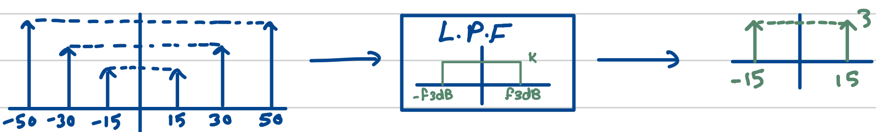
$m(t) = 2\cos(30\pi t) + 4\cos(60\pi t) + 10\cos(100\pi t)$

ⓐ Specify the CKt inside black box to obtain $y(t) = 6\cos(30\pi t)$

Ans:

$M(f) = \delta(f-15) + \delta(f+15) + 2\delta(f-30) + 2\delta(f+30) + 5\delta(f-50) + 5\delta(f+50)$

$Y(f) = 3\delta(f-15) + 3\delta(f+15)$

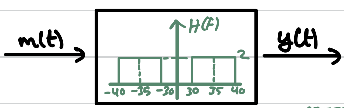


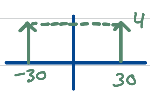
$15 \leq f_{3dB} < 30$

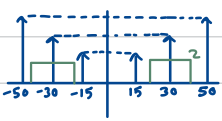
$$\boxed{K = \frac{\text{Amp of output}}{\text{Amp. of input}}} = \frac{3}{1}$$

Types of Filters:

ⓑ Specify the output of the System if black Box contain B.P.F as shown below







$Y(f) = 4\delta(f+30) + 4\delta(f-30)$

$y(t) = 8\cos(60\pi t)$