

II

Bessel and Legendre Series

Fourier - Bessel Series

Df. the Fourier - Bessel series of f defined on $(0, b)$ is given by

$$(i) \quad f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x), \text{ where}$$

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx,$$

where α_i are defined by $J_n(\alpha b) = 0$.

$$(ii) \quad f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx,$$

where α_i are defined by

$$h J_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$$

$$(iii) \quad f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x)$$

$$c_1 = \frac{2}{b^2} \int_0^b x f(x) dx$$

$$c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_0(\alpha_i x) f(x) dx$$

where α_i are defined by $J_0'(\alpha_i b) = 0$

Convergence of a Fourier-Bessel Series

thm. Let f & f' be piecewise cont-
on $[0, b]$. Then for all $x \in (0, b)$, the

Fourier-Bessel series of f converges to

$f(x)$ at a point where f is continuous
and to $\frac{f(x^+) + f(x^-)}{2}$ at a point

where f is discontinuous.

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Ex. Expand $f(x) = x$, $0 < x < 3$ in a Fourier-Bessel series, using Bessel function of order n that satisfy the boundary condition $J_1(3\alpha) = 0$.

Sol. we use (i) in the last def'n.

with $b=3$, $n=1$

$$f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x), \text{ where}$$

$$c_i = \frac{2}{9 J_2^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x) \cdot x dx.$$

$$= \frac{2}{9 J_2^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x) dx.$$

Let $t = \alpha_i x \Rightarrow dt = \alpha_i dx$

or $dx = \frac{dt}{\alpha_i}$, $\begin{matrix} x=0 \Rightarrow t=0 \\ x=3 \Rightarrow t=3\alpha_i \end{matrix}$

and use $\frac{d}{dt} [t^2 J_2(t)] = t^2 J_1(t)$

$$= \frac{2}{9 J_2^2(3\alpha_i)} \int_0^{3\alpha_i} \left(\frac{t}{\alpha_i}\right)^2 J_1(t) \frac{dt}{\alpha_i}$$

$$= \frac{2}{9 \alpha_i^3 J_2^2(3\alpha_i)} \int_0^{3\alpha_i} t^2 J_1(t) dt$$

$$= \frac{2}{9 \alpha_i^3 J_2^2(3\alpha_i)} t^2 J_2(t) \Big|_0^{3\alpha_i}$$

$$= \frac{2}{9 \alpha_i^3 J_2^2(3\alpha_i)} [9 \alpha_i^2 J_2(3\alpha_i) - 0]$$

$$c_i = \frac{2}{\alpha_i J_2(3\alpha_i)}$$

$$\therefore f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x)$$

$$= 2 \sum_{i=1}^{\infty} \frac{J_1(\alpha_i x)}{\alpha_i J_2(3\alpha_i)}$$

Ex. If α_i in the last example are defined by $J_1(3\alpha) + \alpha J_1'(3\alpha) = 0$, then we multiply both sides by 3,

$$3 J_1(3\alpha) + 3\alpha J_1'(3\alpha) = 0$$

Now, we use (ii) in the def'n with $\boxed{h=3}$, $\boxed{b=3}$, $\boxed{n=1}$, thus

$$C_i = \frac{2\alpha_i^2}{(9\alpha_i^2 - 1^2 + 3^2) J_1^2(3\alpha_i)} \int_0^3 x J_1(\alpha_i x) \cdot x dx$$

$$= \frac{2\alpha_i^2}{(9\alpha_i^2 + 8) J_1^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x) dx$$

let $t = \alpha_i x$
 $dt = \alpha_i dx$

$$= \frac{2\cancel{\alpha_i^2}}{(9\alpha_i^2 + 8) J_1^2(3\alpha_i)} \int_0^{3\alpha_i} \frac{t^2}{\cancel{\alpha_i}} J_1(t) \frac{dt}{\alpha_i}$$

$$= \frac{2}{\alpha_i (9\alpha_i^2 + 8) J_1^2(3\alpha_i)} \int_0^{3\alpha_i} t^2 J_1(t) dt$$

$$= \frac{2}{\alpha_i (9\alpha_i^2 + 8) J_1^2(3\alpha_i)} t^2 J_2(t) \Big|_0^{3\alpha_i}$$

$$= \frac{2 (9\alpha_i^2 J_2(3\alpha_i))}{\alpha_i (9\alpha_i^2 + 8) J_1^2(3\alpha_i)}$$

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$$= \frac{18 \alpha_i J_2(3\alpha_i)}{(9\alpha_i^2 + 8) J_1^2(3\alpha_i)}$$

$$\therefore f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x)$$

$$= 18 \sum_{i=1}^{\infty} \frac{\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2 + 8) J_1^2(3\alpha_i)} J_1(\alpha_i x).$$

Fourier-Legendre Series.

Df. the Fourier-Legendre series of a function f defined on $(-1, 1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \text{ where}$$

$$c_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx,$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left((x^2 - 1)^n \right), n=0, 1, 2, \dots$$

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• Convergence of a Fourier-Legendre Series.

Thm Let f & f' be piece-wise cont. on $[-1, 1]$. Then, $\forall x \in (-1, 1)$, the Fourier-Legendre series of f converges to $f(x)$ at a point where f is continuous and to the average $\frac{f(x^+) + f(x^-)}{2}$ at a point where f is discontinuous.

ex. Find the first three non-zero terms in the Fourier-Legendre expansion of $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$.

Sol. we have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \\ = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \dots$$

(8)

where

$$c_0 = \frac{1}{2} \int_{-1}^1 f(x) p_0(x) dx = \frac{1}{2} \int_0^1 x dx = \boxed{\frac{1}{4}}.$$

$$c_1 = \frac{3}{2} \int_{-1}^1 f(x) p_1(x) dx = \frac{3}{2} \int_0^1 x \cdot x dx = \boxed{\frac{1}{2}}.$$

$$c_2 = \frac{5}{2} \int_{-1}^1 f(x) p_2(x) dx$$
$$= \frac{5}{2} \int_0^1 x \cdot \frac{1}{2} (3x^2 - 1) dx$$

$$= \frac{5}{4} \int_0^1 (3x^3 - x) dx = \frac{5}{4} \left[\frac{3x^4}{4} - \frac{x^2}{2} \right]_0^1$$
$$= \boxed{\frac{5}{16}}$$

that is $f(x) \approx c_0 p_0(x) + c_1 p_1(x) + c_2 p_2(x) + \dots$

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$$= \frac{1}{4} p_0(x) + \frac{1}{2} p_1(x) + \frac{5}{16} p_2(x) + \dots$$

$$= \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot x + \frac{5}{16} \cdot \frac{1}{2} (3x^2 - 1) + \dots$$

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Alternative Form of Series

If we let $x = \cos \theta$, then $x = 1 \Rightarrow \theta = 0$
 $x = -1 \Rightarrow \theta = \pi$.

since $dx = -\sin \theta d\theta$, then Df page 6

becomes:

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos \theta), \text{ where}$$

$$c_n = \frac{2n+1}{2} \int_0^{\pi} F(\theta) P_n(\cos \theta) \sin \theta d\theta,$$

where $F(\theta) = f(\cos \theta)$.

$$P_0(\cos \theta) = 1, \quad P_1(\cos \theta) = \cos \theta, \quad P_2(x) = \frac{1}{2}(3\cos^2 \theta - 1),$$

H.w. Write the first three nonzero terms of the Fourier Legendre Series of

$$f(x) = |x|, \quad -1 < x < 1.$$

(simplify your final answer as a polynomial of degree 4)

$$\text{Ans. } f(x) = -\frac{105}{128}x^4 + \frac{105}{64}x^2 + \frac{15}{128} + \dots$$