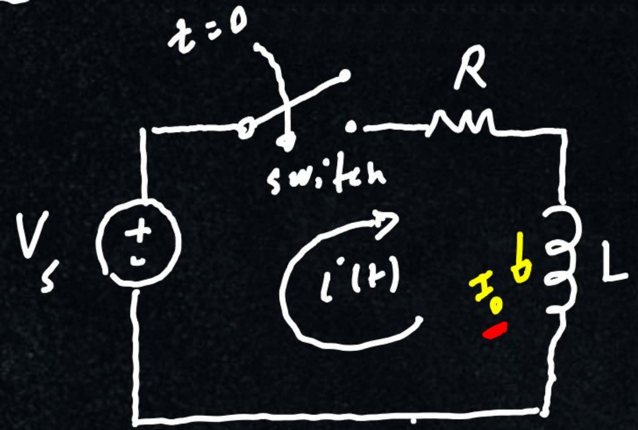
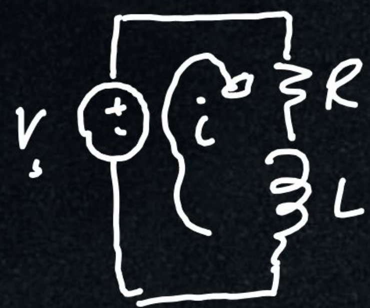
CH7: First Order RL & RC Circuits

(DC sources with RL & RC circuits)

① First order RL circuit



when $t \geq 0$



I_0: initial inductive current.

KVL in the loop

$$V_s = Ri + L\frac{di}{dt} \Rightarrow \frac{V_s - Ri}{L} = \frac{di}{dt}$$

$$\Rightarrow dt = \frac{L}{V_s - Ri}\, di$$

$$\int_0^t dt = \int_{I_0}^{i} \frac{-L}{R} \frac{1}{i - \frac{V_s}{R}}\, di$$

$$t\Big|_0^t = \frac{-L}{R} \ln\left|i - \frac{V_s}{R}\right|\Bigg|_{I_0}^{i}$$

$$t = \frac{-L}{R} \ln\left|\frac{i - V_s/R}{I_0 - V_s/R}\right|$$

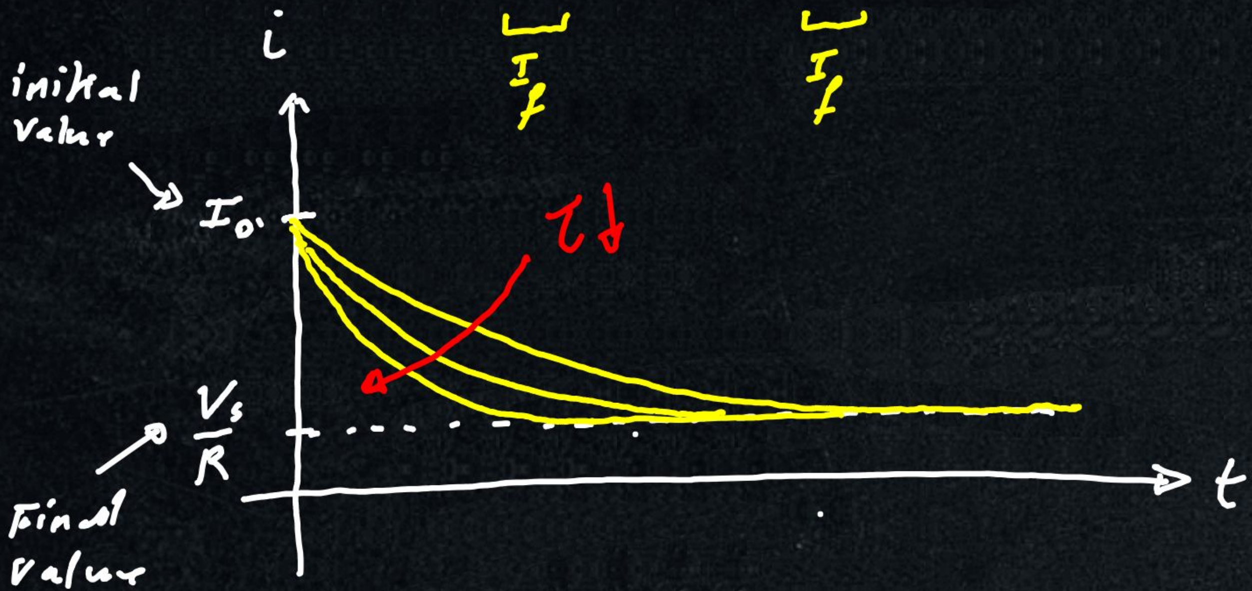
Solve for i

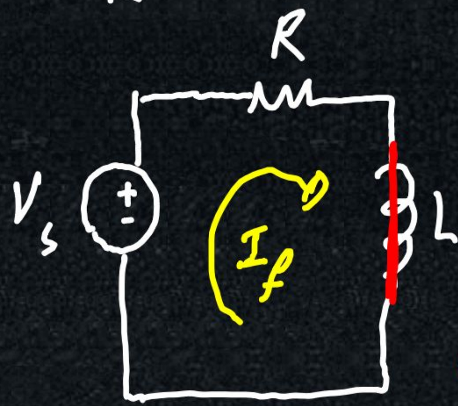
$$-\frac{R}{L}t = \ln\left|\frac{i - V_s/R}{I_o - V_s/R}\right|$$

$$i - \frac{V_s}{R} = \left(I_o - \frac{V_s}{R}\right)e^{-Rt/L}$$

$$i = \underbrace{\frac{V_s}{R}}_{I_f} + \left(I_o - \underbrace{\frac{V_s}{R}}_{I_f}\right)e^{-t/\tau} \quad ; \quad \tau = \frac{L}{R} \text{ time-constant}$$

$$\boxed{i = I_f + (I_o - I_f)\,e^{-t/\tau} \quad t \geq 0}$$

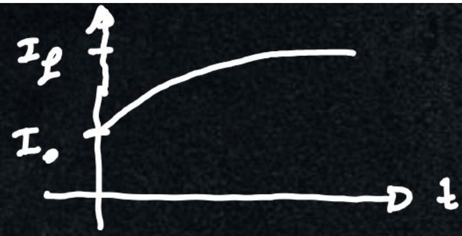




it appears as a short circuit at $t \to \infty$

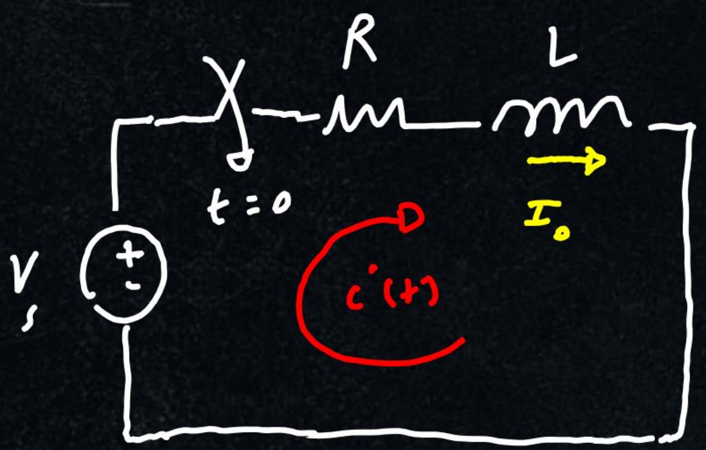
$$I_f = \frac{V_s}{R} : \text{Final Value}$$

When $V_s \neq 0$ or $I_f \neq 0 \Rightarrow$ step response



When $V_s = 0$ or $I_f = 0 \Rightarrow$ Natural response

First order RL circuit



$$V_s = Ri + L\frac{di}{dt}$$

1st order differential equation

$$i(t) = \frac{V_s}{R} + \left(I_o - \frac{V_s}{R}\right) e^{-t/\tau} \quad ; \quad \tau = L/R$$

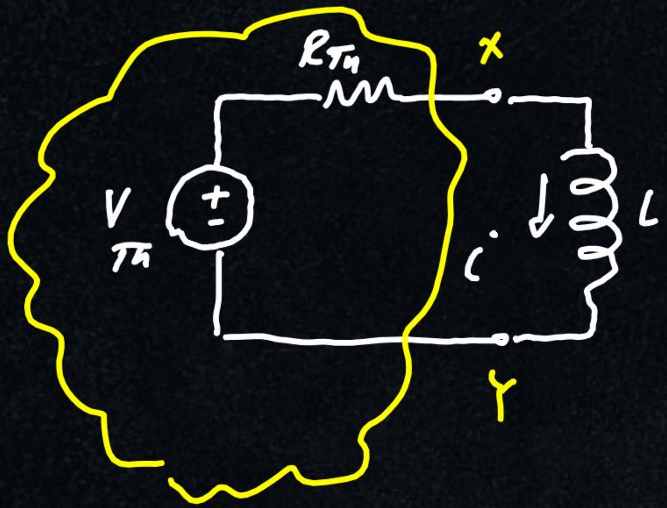
$$i(0) = I_o$$

$$i(\infty) = I_f = \frac{V_s}{R}$$

$$i(t) = I_f + (I_o - I_f) e^{-t/\tau}$$

Note: At $t \to \infty \Rightarrow I_f = \frac{V_s}{R}$ L is short circuit

$t \geq 0$

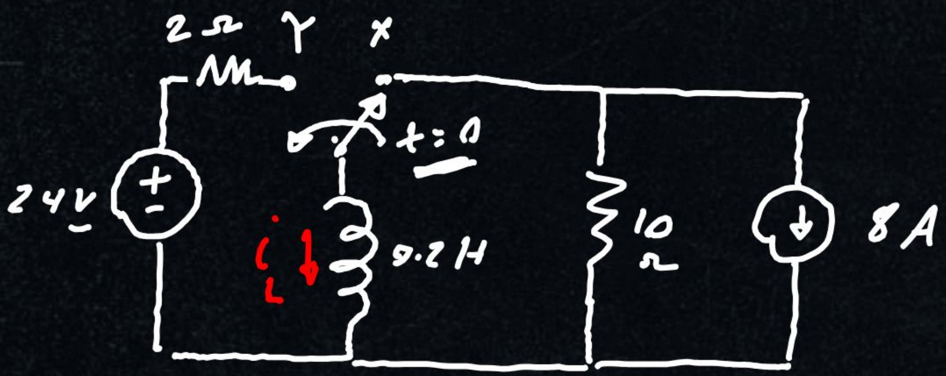


Thevenin Equivalent

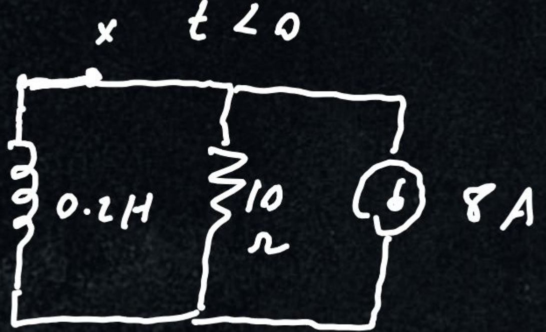
$$i = \frac{V_{Th}}{R_{Th}} + \left(I_0 - \frac{V_{Th}}{R_{Th}} \right) e^{-t/\tau} ;$$

$$\tau = \frac{L}{R_{Th}} .$$

EX :-



The switch is set in position x for a long time. At t = 0, the switch moves to position Y. Calculate $i(t)$ $t \geq 0$?



$t = 0^-$ switch at position x

$t = 0^+$ switch at position Y

L is a current continuous device

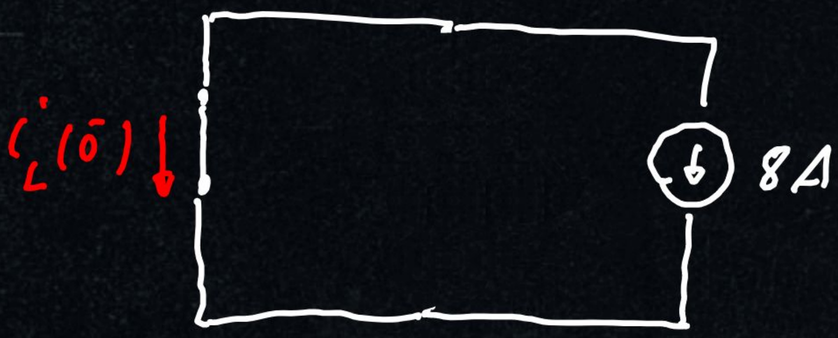
$$V_L = L \frac{d i_L}{dt}$$ continuous

$i_L(t_1^-) = i_L(t_1^+)$

$i_L(0^-) = i_L(0^+) = i_L(0) = I_0$

At $t = 0^-$, i_L reaches its steady state value ⇒ L appears as a short circuit

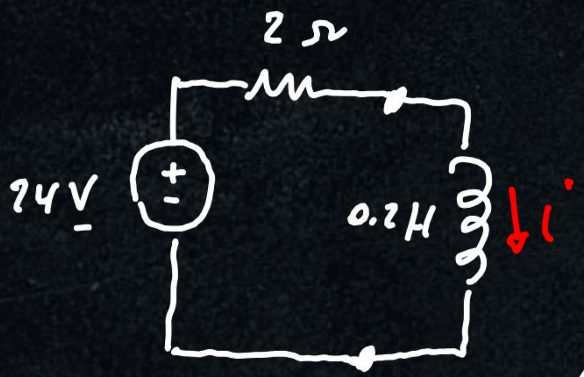
$\underline{t = 0^-}$



$i_L(0^-) = -8A$

$i_L(0^-) = i_L(0^+) = i_L(0) = I_o = -8A$.

$\underline{t \geq 0}$



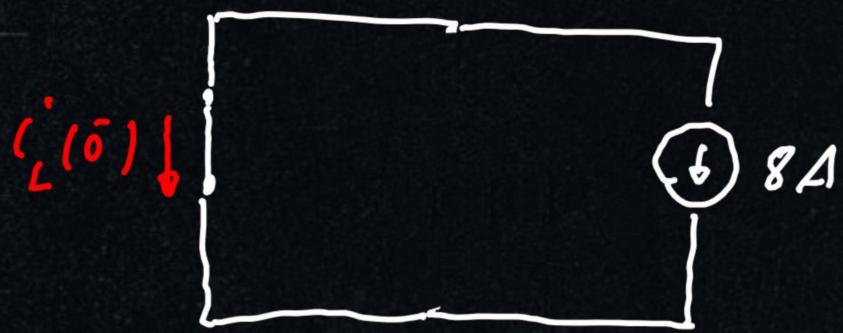
$V_{Th} = 24\,V$, $R_{Th} = 2\,\Omega$

$$i = \frac{V_{Th}}{R_{Th}} + \left(I_o - \frac{V_{Th}}{R_{Th}}\right) e^{-t/\tau} \quad ; \quad \tau = \frac{L}{R_{Th}} = \frac{0.2}{2}$$

$\tau = 0.1$ sec

$$i = 12 - 20\, e^{-10t} \;A, \quad t \geq 0$$

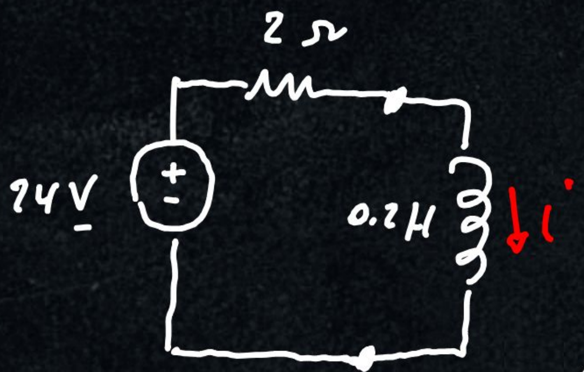
$\underline{t = 0^-}$



$i_L(0^-) = -8A$

$i_L(0^-) = i_L(0^+) = i_L(0) = I_0 = -8A$.

$\underline{t \geq 0}$



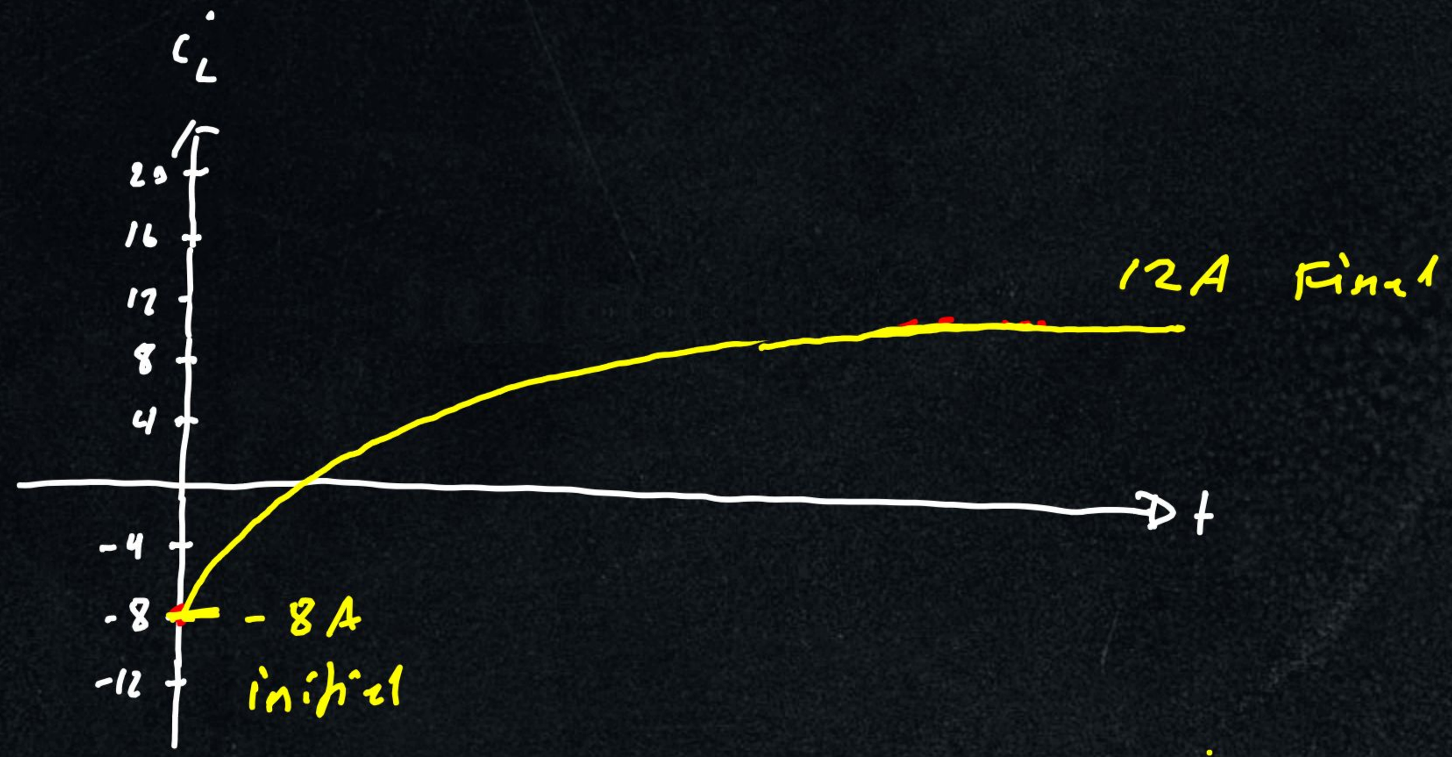
$V_{Th} = 24\,V$, $R_{Th} = 2\,\Omega$

$$i = \frac{V_{Th}}{R_{Th}} + \left(I_0 - \frac{V_{Th}}{R_{Th}}\right) e^{-t/\tau} \quad ; \quad \tau = \frac{L}{R_{Th}} = \frac{0.2}{2}$$

$\tau = 0.1$ sec

$$i = 12 - 20\,e^{-10t}\ A, \quad t \geq 0$$

$i_L = 12 - 20\, e^{-10t}$



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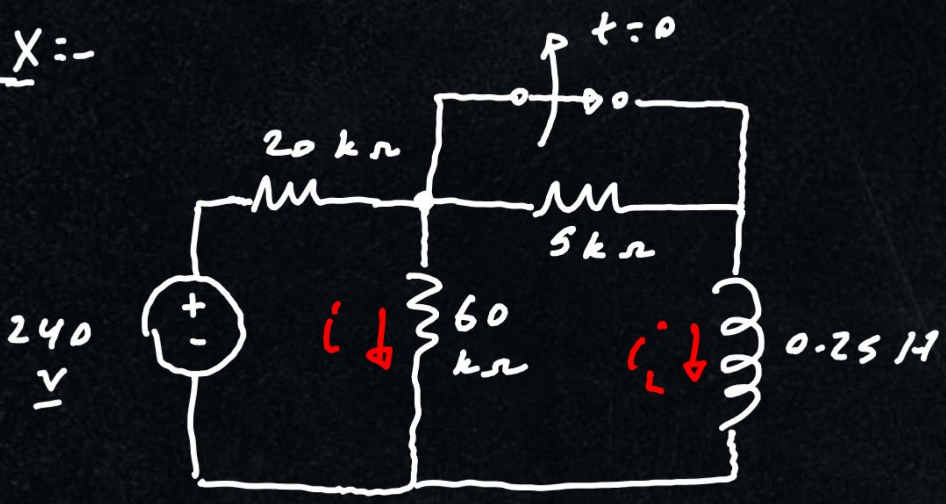
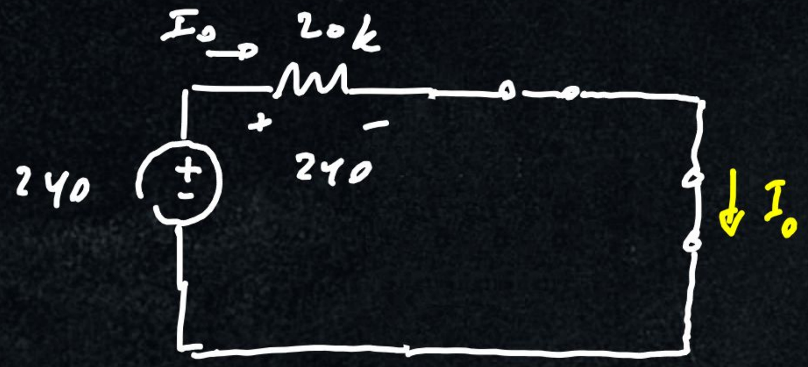
EX:-



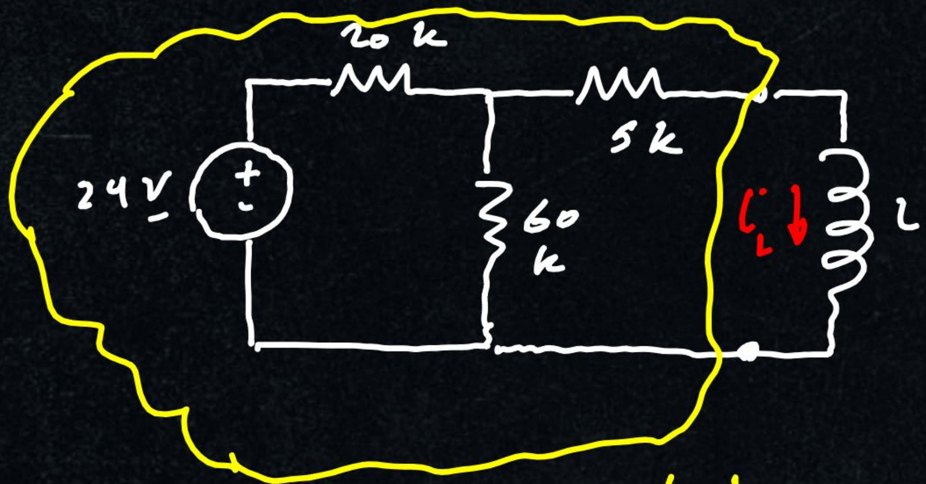
The switch has been closed for a long time. The switch opens at t = 0. Find $i_L(t)$ + $i(t)$

$t = 0^-$ L is a short circuit



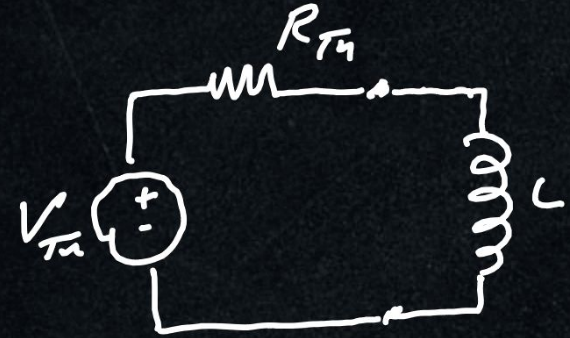
$$I_o = \frac{240}{20} = 12 \text{ mA}.$$

$t \geq 0$

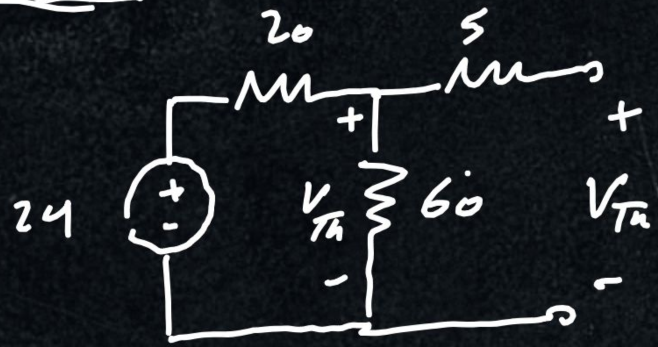


Thevenin Equivalent circuit

$$i_L = \frac{V_{Th}}{R_{Th}} + \left(I_0 - \frac{V_{Th}}{R_{Th}}\right) e^{-t/\tau} \quad t \geq 0 .$$

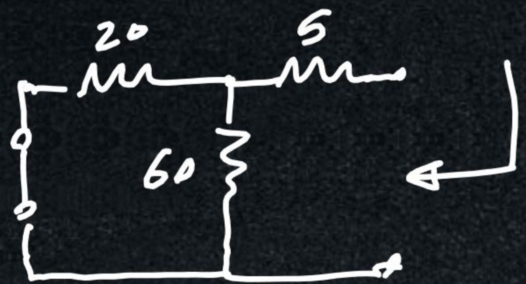


$V_{Th} = V_{oc}$



$$V_{Th} = \frac{60}{60 + 20}(24) = 18\ V$$

R_{Th} "Method II"



$R_{Th} = (20 \,//\, 60) + 5$

$R_{Th} = 20\ k\Omega$

$$\tau = \frac{L}{R_{Th}} = \frac{0.25}{20{,}000} = 12.5\ \mu sec$$

$$i_L = \frac{V_{Th}}{R_{Th}} + \left(I_o - \frac{V_{Th}}{R_{Th}} \right) e^{-t/\tau}$$

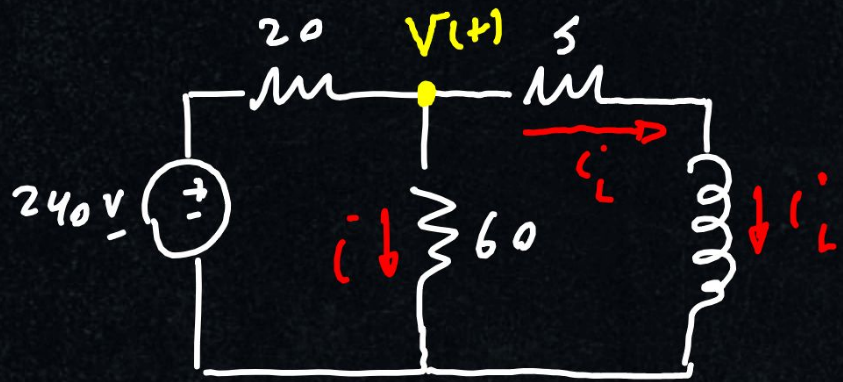
$V_{Th} = 180\,V$

$R_{Th} = 20\,k\Omega$ $\qquad \tau = 12.5\ \mu sec$

$I_o = 12\ mA$

$$i_L = 90 + (12 - 90)\, e^{-80{,}000t} \quad mA, \quad t \geq 0$$

$t \geq 0$



$$\frac{V-240}{20} + \frac{V}{60} + i_L = 0$$

$$3V - 3(240) + V + 60\, i_L = 0$$

$$4V = 720 - 60\, i_L$$

$$V = 180 - 15\, i_L$$

$$i = \frac{V}{60} = 3 - 0.25\, i_L$$

$$\boxed{i = 3 - 0.25\, i_L}$$

$$i_L = \frac{V_{Th}}{R_{Th}} + \left(I_o - \frac{V_{Th}}{R_{Th}}\right) e^{-t/\tau}$$

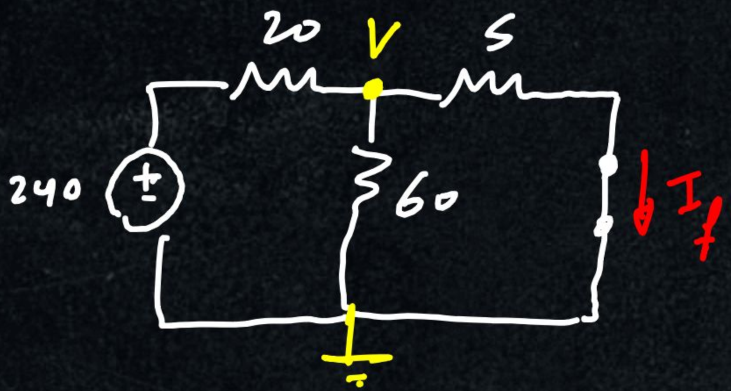
180

20

$$i_L = I_f + (I_o - I_f)\, e^{-t/\tau}$$

To find I_f, draw the circuit at $t \to \infty$ with L is short circuit

$t \to \infty$ (L is short circuit)



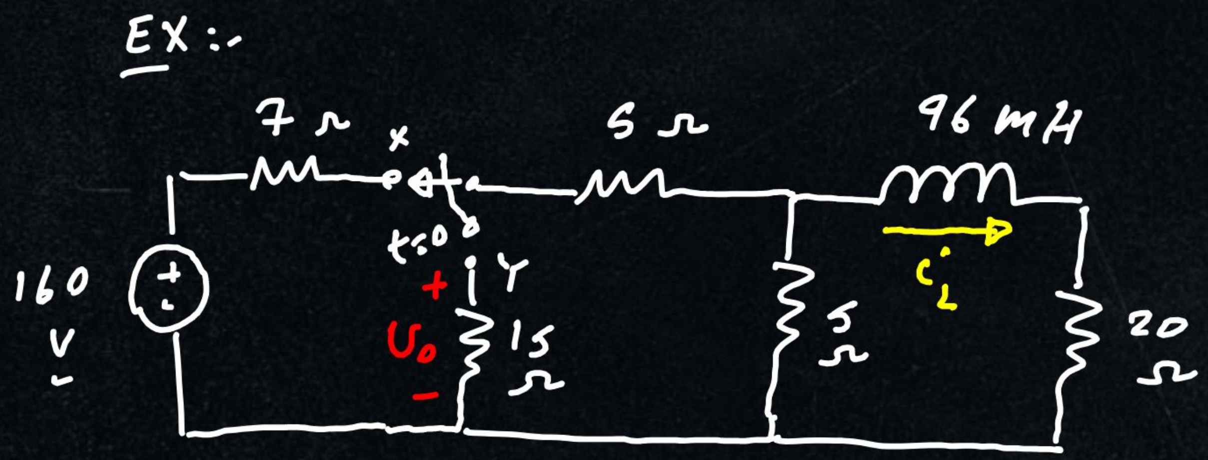
$$\frac{V-240}{20} + \frac{V}{60} + \frac{V}{5} = 0$$

$$3V - 720 + V + 12V = 0$$

$$16V = 720 \Rightarrow V = 45\,\underline{V}$$

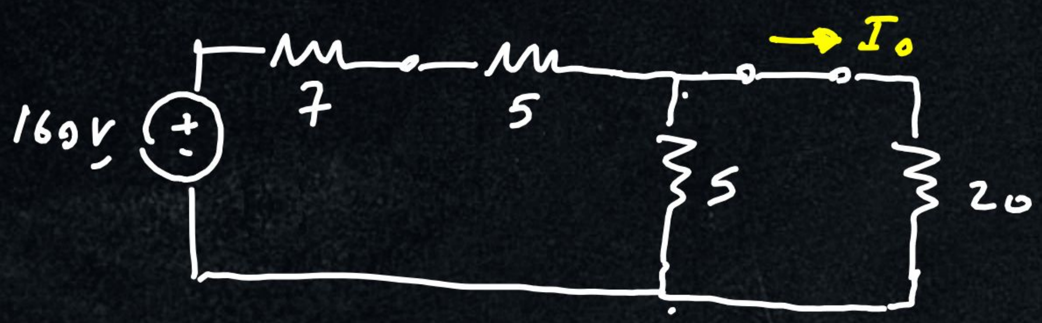
$$I_f = \frac{45}{5} = 9\,mA$$

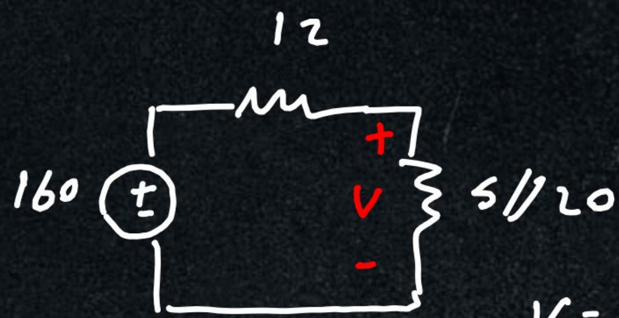
EX :-



The switch has been in position X for a long time. At $t=0$, the switch moves to position Y. Find $v_o(t)$ $t \geq 0$.

$t = 0^-$ (L is s/c)





$$V = \frac{4}{16} 160 = 40 V$$

$$I_o = \frac{40}{20} = 2 A$$

$\underline{t \geq 0}$

$V_{Th} = 0$

"Natural Response"

$i_L = I_o e^{-t/\tau}$, $t \geq 0$, A

$I_o = 2A$

$\tau = L/R_{Th}$;

R_{Th} "Method II"

$R_{Th} = [(5+15) \parallel 5] + 20 = 24\,\Omega$

$\tau = 96/24 = 4$ msec

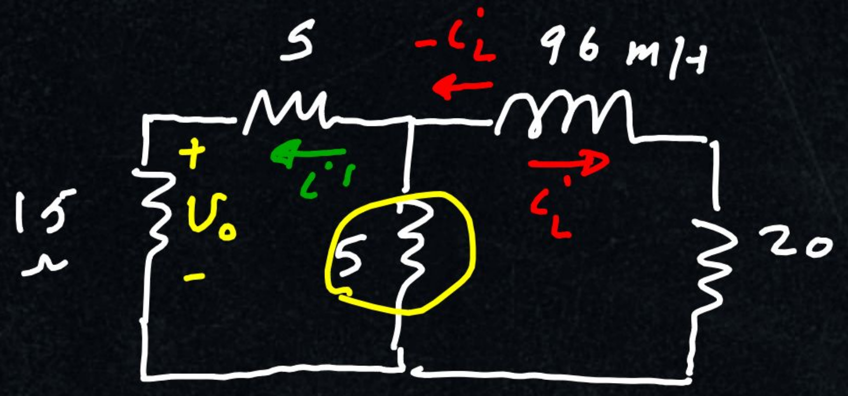
$i_L(t) = 2e^{-250t}$ A , $t \geq 0$

$i' = \frac{5}{25}(-i_L)$

$i' = -0.4e^{-250t}$

$V_o = 15i' = -6e^{-250t}$ $\underline{V}$

$t \geq 0$



$V_{Th} = 0$

"Natural Response"

$i_L = I_o e^{-t/\tau}, \quad t \geq 0, \quad A$

$I_o = 2A$

$\tau = L/R_{Th}$;

R_{Th} "Method II"

$R_{Th} = [(5+15) \parallel 5] + 20 = 24\ \Omega$

$\tau = 96/24 = 4$ msec

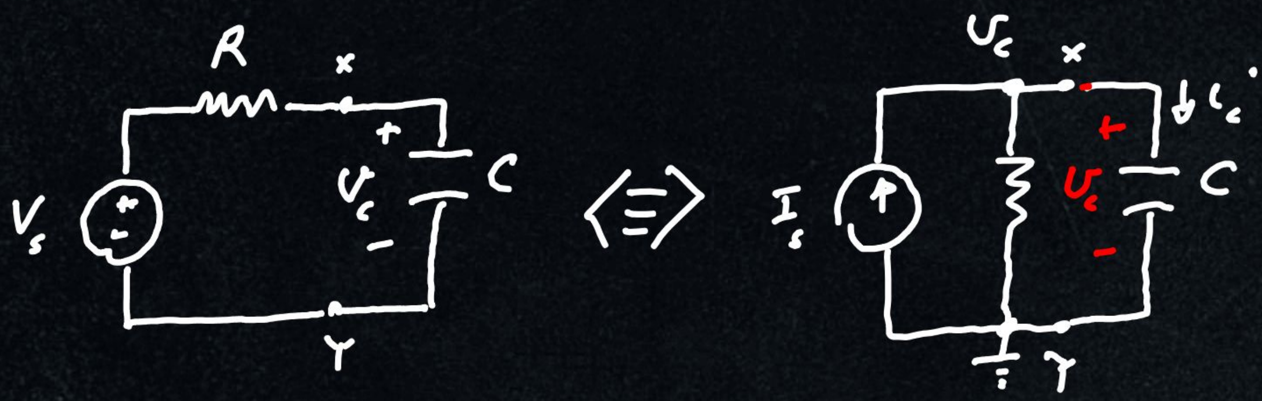
$i_L(t) = 2e^{-250t}\ A, \quad t \geq 0$

$i' = \frac{5}{25}(-i_L)$

$i' = -0.4e^{-250t}$

$V_o = 15i' = -6e^{-250t}\ \underline{V}$

Natural + step Responses of RC circuit



$$i_c = C\frac{dV_c}{dt}$$

KCL at V_c :- $-I_s + \frac{V_c}{R} + C\frac{dV_c}{dt} = 0$

$$I_s = \frac{V_c}{R} + C\frac{dV_c}{dt}$$

1st order diff. equation (Linear)

$$\frac{1}{C}\left(I_s - \frac{V_c}{R}\right) = \frac{dV_c}{dt} \Rightarrow -\frac{1}{RC}V_c + \frac{1}{C}I_s = \frac{dV_c}{dt} \Rightarrow \frac{-1}{RC}\left[V_c - RI_s\right] = \frac{dV_c}{dt}$$

$$-\frac{1}{RC}\left[V_c - RI_s\right] = \frac{dV_c}{dt} \Rightarrow \int_0^t -\frac{1}{RC}\,dt = \int_{V_0}^{V_c} \frac{dV_c}{V_c - RI_s}$$

where V_0 is the initial capacitive voltage

$$-\frac{1}{RC}t = \ln|V_c - RI_s| \Big|_{V_0}^{V_c} = \ln\left|\frac{V_c - RI_s}{V_0 - RI_s}\right|$$

$$(V_c - RI_s) = (V_0 - RI_s)e^{-t/RC}$$

$$\boxed{V_c = RI_s + (V_0 - RI_s)e^{-t/\tau} \; ; \quad \tau = RC} \quad t \geq 0$$

when $I_s \neq 0 \Rightarrow$ step response $\rightarrow V_c = RI_s + (V_0 - RI_s)e^{-t/\tau}$

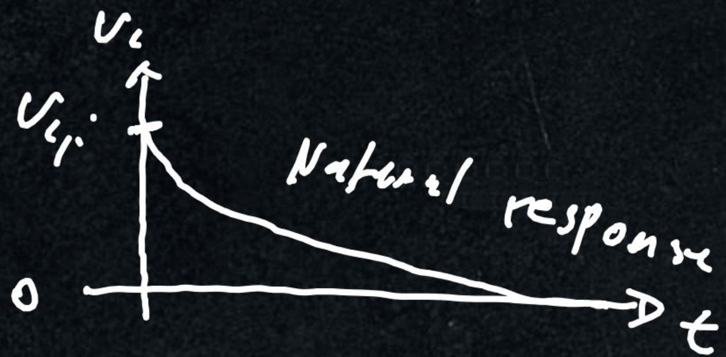
when $I_s = 0 \Rightarrow$ Natural response $\rightarrow V_c = V_0 e^{-t/\tau}$

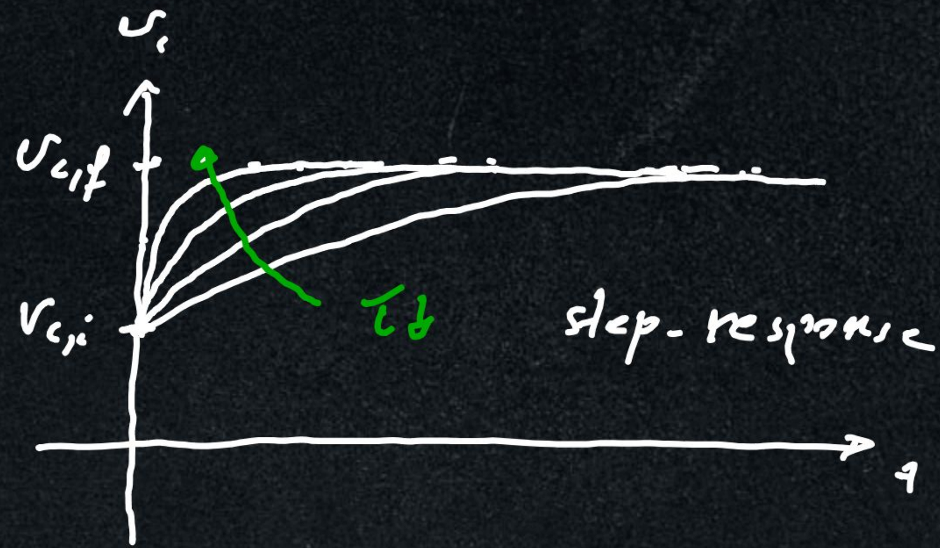
$$V_c = RI_s + (V_0 - RI_s)\,e^{-t/\tau} \; ; \; t \geq 0$$

$$\underbrace{V_c(0)}_{\text{initial}} = V_0 = V_c(0^-) = V_c(0^+) = V_c(0)$$

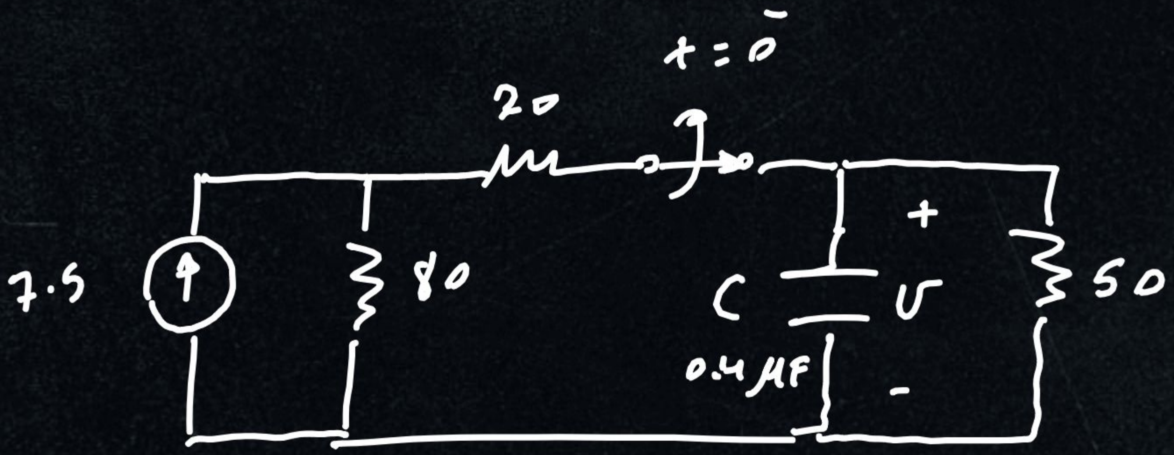
$$\underbrace{V_c(\infty)}_{\text{Final}} = RI_s \Rightarrow \text{The capacitor appears as an open circuit}$$

$$V_c = \underbrace{V_{c,f}}_{\text{Final}} + (\underbrace{V_{c,i}}_{\text{initial}} - V_{c,f})\,e^{-t/\tau}$$

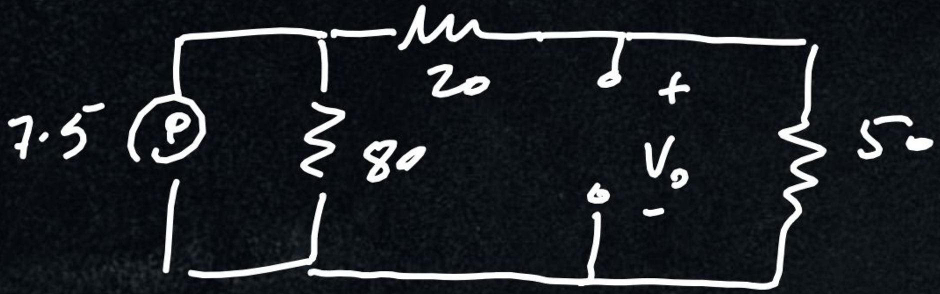




Find $v(t)$ $t > 0$



$t = 0^-$



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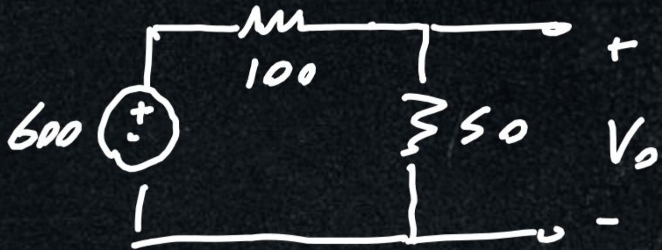
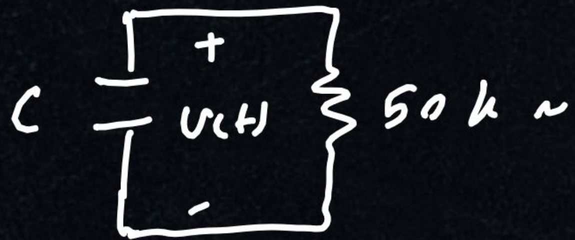


$$V_o = \frac{50}{150}(600) = 200\ \underline{V}$$

$t \geq 0$



Natural response

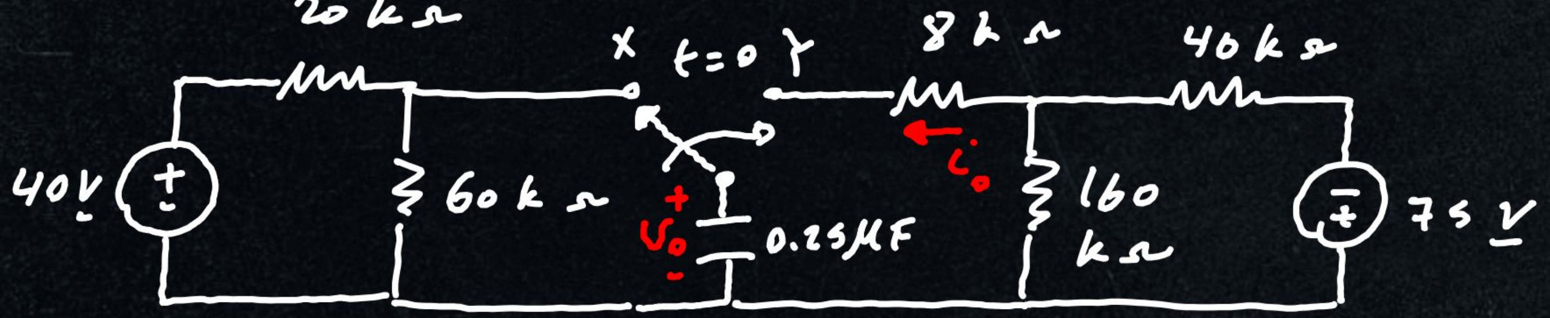
$I_s = 0$

$$v(t) = V_0 e^{-t/\tau}, \quad t \geq 0$$

$$\tau = RC = 50\times10^{3} \times 0.4\times10^{-6} = 20 \text{ msec}$$

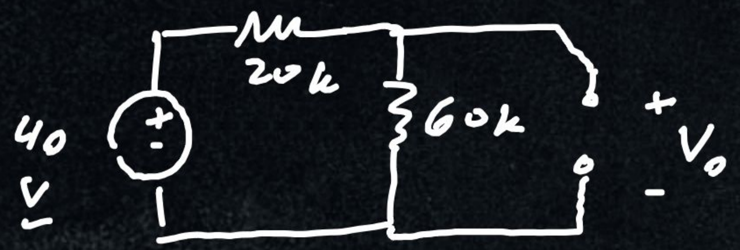
$$v(t) = 200 e^{-50t} \quad t \geq 0, \ \underline{V}$$

Ex:-



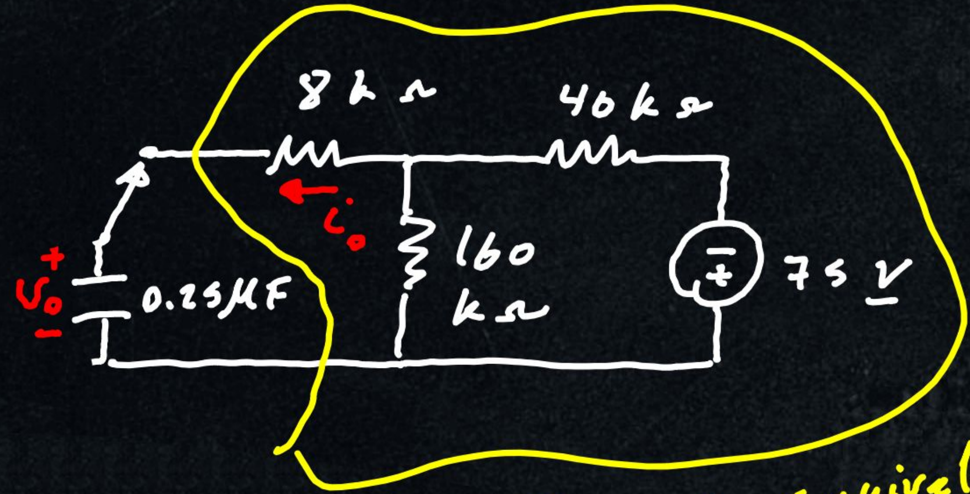
The switch has been in position x for a long time. At t=0, the switch moves to position Y. Find $V_o(t)$ & $i_o(t)$
$t \geq 0$ $t \geq 0^+$.

$t = 0^-$ (C is o/c)



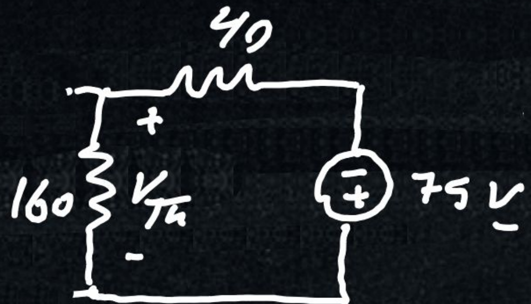
$$V_o = \frac{60}{60+20}(40) = 30\ V$$

$t \geq 0$



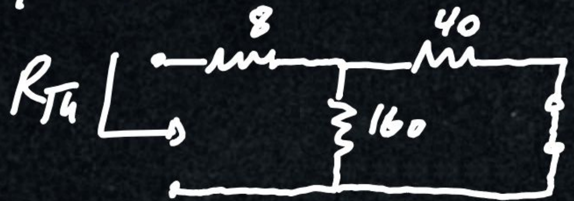
Thevenin Equivalent circuit

V_{Th} ??



$$V_{Th} = \frac{160}{160+40}(-75) = -60\ V$$

R_{Th} ?? "Method II"



$$R_{Th} = (40 \parallel 160) + 8 = 40\ k\Omega$$

$$\tau = R_{Th} C = 40 \times 0.25 = 10\ \mu sec$$

$V_{Th} = -60\ \underline{V}$

$\tau = 10\ msec$

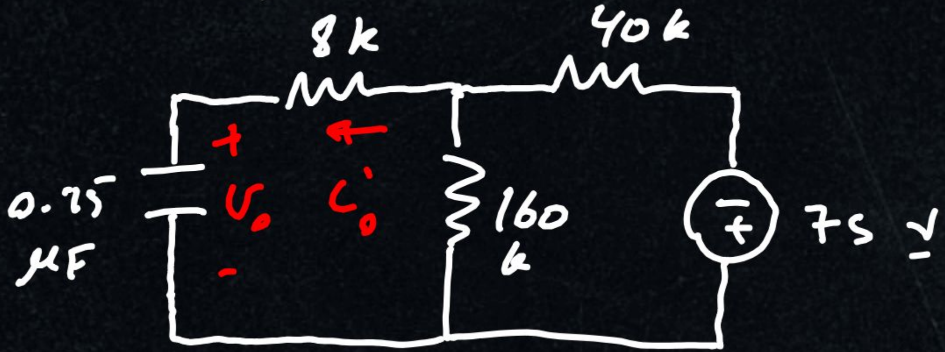
$R_{Th} = 40\ k\Omega$

$$V_o = \underbrace{R_{Th} I_N}_{V_{Th}} + (V_o - \underbrace{R_{Th} I_N}_{V_{Th}})\, e^{-t/\tau}$$

$$V_o = -60 + (30 - -60)\, e^{-100t}$$

$$V_o = -60 + 90\, e^{-100t}\ \underline{V},\quad t \geq 0$$

$t \geq 0$



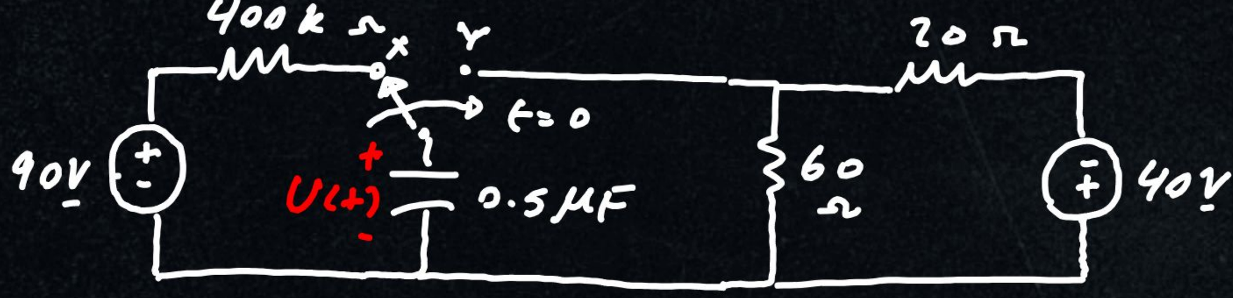
$$i_o = C\frac{dv_o}{dt}$$

$$v_o = -60 + 90e^{-100t} \quad t \geq 0$$

$$i_o = (0.25\times10^{-6})(-9000e^{-100t})$$

$$i_o = -2.25e^{-100t} \text{ mA}, \quad t \geq 0^+$$

EX :-



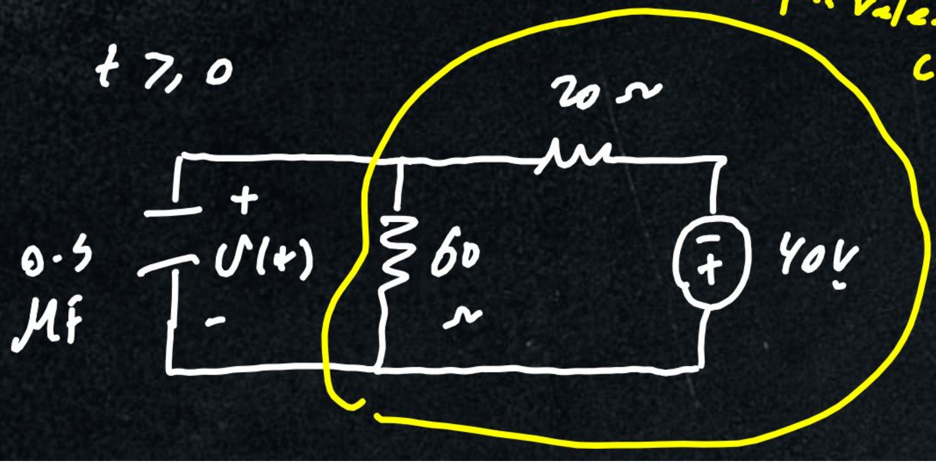
The switch has been in position X for a long time. At t = 0, the switch moves to position Y. Calculate $v(t)$ $t \geq 0$?

$t = 0^-$ (C is o/c)

$V_0 = 90\ V$

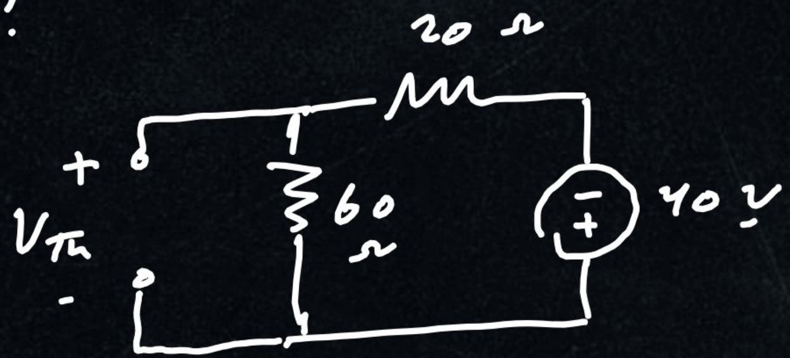
$t \geq 0$

Thevenin Equivalent circuit



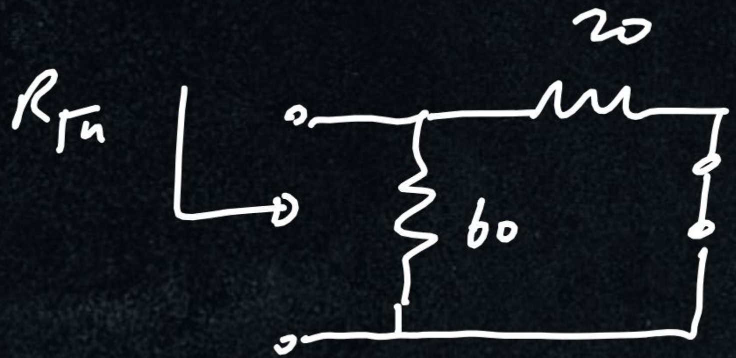
step response.

V_{Th} ??



$$V_{Th} = \frac{60}{60+20}(-40) = -30\ V$$

R_{Th} "Method II"



$$R_{Th} = 20 // 60 = 15\ \Omega$$

$$\tau = R_{Th} C = 15(0.5 \times 10^{-6}) = 7.5\ \mu sec$$

$$v(t) = V_{Th} + (V_0 - V_{Th})\, e^{-t/\tau}$$

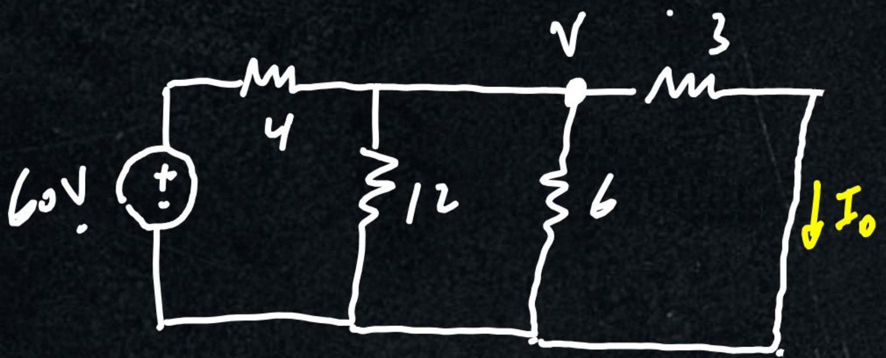
$$v(t) = -30 + (90 - -30)\, e^{-1.33 \times 10^{5} t}\ V, \quad t \geq 0$$

EX:-

The two switches have been closed for a long time. Find $i_L(t)$ $t \geq 0$?

$$i_L(t) = \begin{cases} i_{L_1} & 0 \leq t \leq 35 \\ i_{L_2} & t \geq 35 \end{cases}$$

$t = 0^-$ (L is s/c)



$$\frac{V}{6} + \frac{V}{12} + \frac{V}{3} + \frac{V-60}{4} = 0$$

$$2V + V + 4V + 3V = 180$$

$$V = 18 \text{ V}$$

$$I_o = \frac{18}{3} = 6 \text{ A}$$

$$i_L(0^-) = i_L(0^+) = i_L(0) = I_o = 6 \text{ A}$$

$0 \le t \le 3\,s \qquad i_{L_1}\ ??$

$i_{L_1} = I_0\, e^{-t/\tau_1}$

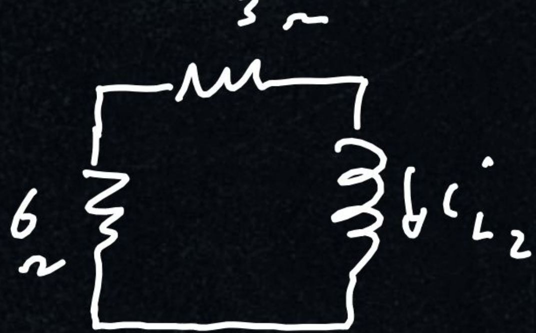
$\tau_1 = L/R_{Th_1}$

$R_{Th_1} = (3+6) \,//\, 18 = 6\,\Omega$

$\tau_1 = \dfrac{150}{6} = 25$ msec.

$i_{L_1} = 6\, e^{-40t}$ A

$t \geq 35$ msec $\quad i_{L_2}$



Natural response

$$i_L = \begin{cases} i_{L_1} & 0 \leq t \leq 35 \\ i_{L_2} & t \geq 35 \end{cases}$$

$$i_{L_2} = \underline{I_0'}\, e^{-\frac{(t-35m)}{\tau_2}}$$

$$I_0' = i_{L_2}(35ms) = i_{L_1}(35ms)$$

$$I_0' = 6e^{-40\times 0.035}$$

$$= 1.48A.$$

$$\tau_2 = \frac{L}{R_{Th_2}}, \quad R_{Th_2} = 3+6 = 9\,\Omega$$

$$\tau_2 = \frac{150}{9} = 16.667 \text{ msec}$$

$$i_{L_2} = 1.48\, e^{-60(t-0.035)} \quad t \geq 35\text{ ms}$$

$$i_L = \begin{cases} 6e^{-40t} & 0 \le t \le 35\text{ ms} \\ 1.48e^{-60(t-0.035)} & t \ge 35\text{ ms} \end{cases}$$