2.2 Properties of Determinants

Lemma 2.1: Let A be an $n \times n$ matrix. Then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} |A| & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases},$$

where A_{jk} is the cofactor of a_{jk} for $k = 1, \ldots, n$.

Proof

- If $i=j$, then $a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = |A|$ since this is just the cofactor expansion of $|A|$ along the i^{th} row of A.

- If $i \neq j$, take A^* to be the matrix obtained by replacing the j^{th} row of A by the i^{th} row of A:

$$A = \begin{array}{c} \\ \\ j^{th}\text{ row} \\ \\ i^{th}\text{ row} \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad A^* = \begin{array}{c} \\ \\ j^{th}\text{ row} \\ \\ i^{th}\text{ row} \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Since two rows of A^* are the same $\Rightarrow |A^*| = 0$

$\Rightarrow$ Using the cofactor expansion of $|A^*|$ along the j^{th} row:

$$0 = |A^*| = a_{i1}A^*_{j1} + a_{i2}A^*_{j2} + \cdots + a_{in}A^*_{jn}$$

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$$

$$= |A|$$

* What are the effects of the row operations I, II, III on the value of the determinant?

$\boxed{I}$ The effect of row operation I (two rows of A are interchanged)

- If A is 2×2 matrix given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and

  E is 2×2 elementary matrix $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$|EA| = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$$

$$= -\left[a_{11}a_{22} - a_{21}a_{12}\right]$$

$$= -|A|$$ That is $|EA| = |E|\,|A| = -|A|$

- For $n > 2$, Let E_{ij} be the elementary matrix that switches rows i and j of A. Then $|E_{ij}A| = -|A|$

Exp Let $n=3$. That is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

suppose that R_1 and R_3 have been interchanged $\Rightarrow$

$$|E_{13}A| = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$ expanding along the 2^{nd} row

$$= -a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} + a_{22}\begin{vmatrix} a_{31} & a_{33} \\ a_{11} & a_{13} \end{vmatrix} - a_{23}\begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix}$$

using the result of 2×2

$$= a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -|A|$$

- In general, if A is $n\times n$ matrix and E_{ij} is the $n\times n$ elementary matrix formed by interchanging the i^{th} and j^{th} rows of I, then

identity

$$\boxed{|E_{ij}A| = -|A|}$$

→ In particular $|E_{ij}| = |E_{ij}I| = -|I| = -1$

→ Thus, for any elementary matrix E of type I :

$$|EA| = -|A| = |E|\,|A|$$

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[2] The effect of row operation II (A row of A is multiplied by a nonzero constant)

- Let E denote the elementary matrix of type II formed from I (identity) by multiplying the i^{th} row by the nonzero constant α.

$$E = \begin{array}{c} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\ \alpha\ \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n} \end{array} \begin{array}{l} \\ \\ \\ i^{th}\text{ row} \\ \\ \\ \end{array} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$|EA| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \cdots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

expanding $|EA|$ along the i^{th} row

$$\boxed{|\alpha A| = \alpha^n |A|}$$

$$= \alpha a_{i1}A_{i1} + \alpha a_{i2}A_{i2} + \cdots + \alpha a_{in}A_{in}$$

$$= \alpha[a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}]$$

$$= \alpha|A|$$

Hence, $|E| = |EI| = \alpha|I| = \alpha$ and $|EA| = \alpha|A| = |E|\,|A|$

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3 The effect of row operation III (A multiple of one row is added to another row) 41

- Let E be the elementary matrix of type III obtained from I (identity) by adding c (i^{th} row) to the j^{th} row.

- Note that $|E| = 1$ since E is triangular form with diagonal elements all 1.

$$|EA| = \begin{array}{c} \left| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{j1}+ca_{i1} & a_{j2}+ca_{i2} & \cdots & a_{jn}+ca_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right| \end{array} \begin{matrix} \\ \\ j^{th} \text{ row} \\ \\ \\ \\ \\ \end{matrix}$$

expanding $|EA|$ along the j^{th} row $\Rightarrow$

$$= (a_{j1} + c\,a_{i1})A_{j1} + (a_{j2} + c\,a_{i2})A_{j2} + \cdots + (a_{jn} + c\,a_{in})A_{jn}$$

$$= (a_{j1}A_{j1} + a_{j2}A_{j2} + \cdots a_{jn}A_{jn}) + c\underbrace{(a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots a_{in}A_{jn})}_{\text{This is zero by Lemma 2.1}}$$

$$= |A| + 0$$

$$= |A|$$

Hence, $|EA| = |A| = |E|\,|A|$

Summary

* If E is an elementary matrix, then

$\boxed{1}$ $|EA| = |E|\,|A|$ where

$$|E| = \begin{cases} -1 & \text{if } E \text{ is of type I} \\ \alpha & \text{if } E \text{ is of type II} \quad \text{"}\alpha \neq 0\text{"} \\ 1 & \text{if } E \text{ is of type III} \end{cases}$$

Similar results hold for column operations.

$\boxed{2}$ E^T is also an elementary matrix and

$$|AE| = |(AE)^T| = |E^T A^T| = |E^T|\,|A^T|$$
$$= |E|\,|A|$$

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* The effects of row "or column" operations on the value of the determinant can be summarized as follows

I. Interchanging two rows (or columns) of a matrix changes the sign of the determinant.

II. Multiplying a single row (or column) of a matrix by a scalar has the effect of multiplying the value of the determinant by that scalar.

III. Adding a multiple of one row (or column) to another does not change the value of the determinant.

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Note: If one row (or column) of a matrix is a multiple of another, then the determinant of the matrix $= 0$. "consequence of III $R_j = c R_i$"

Th An $n \times n$ matrix A is singular iff $|A| = 0$

Proof: The matrix A can be reduced to row echelon form with a finite number of row operations:

$$U = E_k \, E_{k-1} \cdots E_1 \, A$$

where U is in row echelon form and E_i's are all elementary matrices. Hence,

$$|U| = |E_k \, E_{k-1} \cdots E_1 \, A|$$
$$= |E_k| \, |E_{k-1}| \cdots |E_1| \; |A|$$

But $|E_i|$ are all nonzero. Hence, $|A| = 0$ iff $|U| = 0$.

$\Rightarrow$ • If A is singular, then U has a row consisting entirly of zeros.

$$\Rightarrow |U| = 0 \Rightarrow |A| = 0.$$

$\Leftarrow$ • If $|A| = 0$ "goes in the beging of the proof"

$\Rightarrow$ A can not be row equivalent to I

$\Rightarrow$ A must be singular

"Th 5.2"

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* From the proof above, we obtain a method for computing $|A|$:

- we reduce A to row echelon form: $U = E_k \, E_{k-1} \cdots E_1 \, A$
- If the last row of U is zero row, then A is singular and $|A| = 0$

  Otherwise, A is nonsingular and $|U| = |E_k \, E_{k-1} \cdots E_1 \, A|$

  $$1 = |E_k| \, |E_{k-1}| \cdots |E_1| \, |A|$$
  $$|A| = \left[|E_k| \, |E_{k-1}| \cdots |E_1| \right]^{-1}$$

- In fact, If A is nonsingular, we can reduce A to triangular form T using $\boxed{I}$ and $\boxed{II}$. Thus $T = E_m \, E_{m-1} \cdots E_1 \, A$

$|E| = -1$

$|E| = 1$

Hence, $|T| = \pm |A| \Leftrightarrow |A| = \pm t_{11} \, t_{22} \cdots t_{nn}$ diagonal entries of T. The sign will be positive if n is even and negative otherwise.

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Exp Find $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix}$ $\quad R_2 - \boxed{2} R_1$, $R_3 - \boxed{3} R_1$

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} \underset{R_3 \Leftrightarrow R_2}{=} (-1) \begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} = (-1)(2)(-6)(-5) = -60$$

Notes ① In evaluating the determinant of $n \times n$ matrix by cofactors, we need $n! - 1$ additions and

$$\sum_{K=1}^{n-1} \frac{n!}{K!} \text{ multiplications}$$

Exp • when $n = 2$ we need $2! - 1 = 2 - 1 = 1$ addition and

$\frac{2!}{1!} = 2$ multiplications

since $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$

• when $n = 3$ we need $3! - 1 = 6 - 1 = \underline{\underline{5}}$ additions and

$\frac{3!}{1!} + \frac{3!}{2!} = 6 + 3 = 9$ multiplications

since $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

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② In evaluating the determinant of $n \times n$ matrix by Elimination (as in exp above) we need $\frac{n(n-1)(2n-1)}{6}$ additions and

$\frac{(n-1)(n^2+n+3)}{3}$ multiplications

Exp • when $n = 2$: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $R_2 - \frac{a_{21}}{a_{11}} R_1 \Leftrightarrow a_{11} R_2 - a_{21} R_1$ 3 multiplication, 1 addition

$\begin{vmatrix} a_{11} & a_{12} \\ 0 & a'_{22} \end{vmatrix} = a_{11} a'_{22}$

• when $n = 3$: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\begin{matrix} R_2 - \frac{a_{21}}{a_{11}} R_1 \\ R_3 - \frac{a_{31}}{a_{11}} R_1 \end{matrix}$ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{vmatrix}$ $R_3 - \frac{a'_{32}}{a'_{22}} R_2$ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{vmatrix} = a_{11} a'_{22} a''_{33}$

10 multiplications

5 additions $= 10/2$

3 We have seen for any elementary matrix E we have $|EA| = |E|\,|A| = |AE|$ $*$

Th If A an B are $n\times n$ matrices, then $|AB| = |A|\,|B|$

Proof
- If B is singular $\Rightarrow$ AB is singular $\Rightarrow |AB| = 0 = |A|\,|B|$
- If B is nonsingular $\Rightarrow$ B can be written as product of elementary matrices $\Rightarrow B = E_k E_{k-1} \cdots E_1$

$$\Rightarrow AB = A E_k E_{k-1} \cdots E_1$$
$$\Rightarrow |AB| = |A|\,|E_k| \cdots |E_1|$$
$$= |A|\,|E_k \cdots E_1| \quad \text{by } *$$
$$= |A|\,|B|$$

Th Let A, B be $n\times n$ matrices. Show that AB is nonsingular iff A and B are both nonsingular

Proof AB is nonsingular iff $|AB| \neq 0$

$|A|\,|B| \neq 0 \iff |A| \neq 0$ and $|B| \neq 0$
$\iff A$ is nonsingular and B is nonsingular.

Exp Let A, B be $n\times n$ matrices and AB is singular what you can say about A and B

Proof AB is singular $\Rightarrow |AB| = 0 \iff |A|\,|B| = 0 \iff$ $|A| = 0$ or $|B| = 0 \iff A$ is singular or B is singular

Exp If A is non singular then $|A^{-1}| = \frac{1}{|A|}$

Proof $A^{-1}A = I \iff |A^{-1}A| = |I| \iff |A^{-1}|\,|A| = 1$
$\iff |A^{-1}| = \frac{1}{|A|}$

Exp Find $|A|$ if

1) $A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix} \Rightarrow |A| = 0$ since A has equal rows $R_1 = R_3$

2) $A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow |A| = 0$ since A has a zero row

3) $A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} \Rightarrow |A| = 0$ since $R_3 = R_1 + R_2$ "linear combination"

4) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{bmatrix} \Rightarrow |A| = (2)(2)(6) = 24$

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5) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 5 & 6 \end{bmatrix}$ if $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 16$, $|A| = -16$ since $R_1 \Leftrightarrow R_2$

6) $A = \begin{bmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{bmatrix}$, $|A| = 16$ since $C_3 = 2C_1 + C_2 + C_3$

7) $A = \begin{vmatrix} \boxed{4} & 1 & 2 \\ \boxed{2} & 2 & 0 \\ \boxed{6} & 5 & 6 \end{vmatrix}$, $|A| = 2 \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 32$

8) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{matrix} \\ r_2 - r_1 \\ r_3 - r_2 \end{matrix} \Leftrightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \Rightarrow |A| = 0$

9) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1$

10) $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & a+3 & b+4 \\ 2 & c+3 & d+4 \end{bmatrix} \begin{matrix} \\ r_2 - r_1 \\ r_3 - r_1 \end{matrix} \Rightarrow A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \Rightarrow |A| = 2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= 2(ad - bc).$