Circuit Analysis

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Chapter 8

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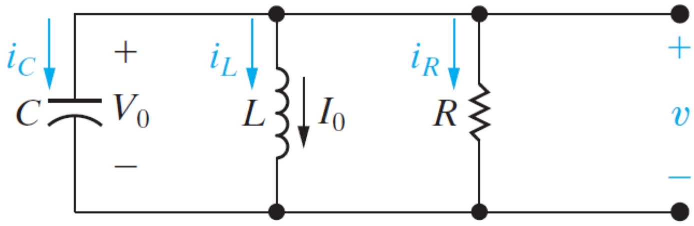
Chapter 8
Natural and Step Response of RLC Circuits

It's when we have resistor, inductor and capacitor (RLC) in the same circuit, either they are connected in series on parallel, and this is called second order differential equation

1 – parallel RLC circuits

We have two cases

1.1 – natural response



to find the DE apply KCL

$$i_R + i_L + i_C = 0$$

important to find the initial Conditions

$$\boxed{\frac{V(t)}{R} + \frac{1}{L}\int V(t)\,dt + C\frac{dV}{dt} = -i_L(0^-)}$$

resonant frequency

$$\omega = \frac{1}{\sqrt{LC}}$$

Neper frequency

$$\alpha = \frac{1}{2RC}$$

$$S_1 = -\alpha - \sqrt{\alpha^2 - \omega^2}$$

$$S_2 = -\alpha + \sqrt{\alpha^2 - \omega^2}$$

① if $\alpha^2 > \omega^2$, the Solutions are real distinct

$V(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$ ⟹ Over Damped

② if $\alpha^2 = \omega^2$, the Solutions are real Similar

$V(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$ ⟹ Critical Damped

③ if $\alpha^2 < \omega^2$, the Solutions are Complex

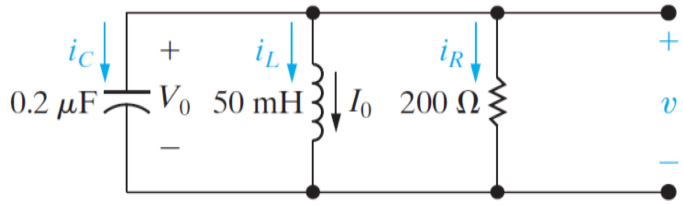
let $\omega_d = \sqrt{\omega^2 - \alpha^2}$

$V(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$ ⟹ Under Damped

Example 1

$V_C(0^+) = 12V$, $i_L(0^+) = 30 mH$

find $V(t)$?



Step 1 - Find s_1 and s_2

$\alpha = 1/2RC \Rightarrow \alpha = 1/2 * 200 * 0.2 * 10^{-6} \Rightarrow \alpha = 1.25 * 10^4$

$\omega = 1/\sqrt{LC} \Rightarrow \omega = 1/\sqrt{50 * 0.2 * 10^{-9}} \Rightarrow \omega = 10^4$

$S_1 = -\alpha - \sqrt{\alpha^2 - \omega^2} \Rightarrow S_1 = -1.25 * 10^4 - 7500 \Rightarrow S_1 = -20,000$

$S_2 = -\alpha + \sqrt{\alpha^2 - \omega^2} \Rightarrow S_2 = -1.25 * 10^4 + 7500 \Rightarrow S_2 = -5,000$

$V(t) = A_1 e^{-20000t} + A_2 e^{-5000t}$ ------ 1

Step 2 - find the initial Conditions

i have $V(0^+)$, so i have to find $V'(0^+)$ first derivative

$\boxed{\dfrac{dv}{dt} = \dfrac{i_C(0^+)}{C}}$, by applying KCl to find $i_C(0^+)$

$i_C(0^+) = -i_L - i_R \Rightarrow i_C = 30 * 10^{-3} - \dfrac{12}{200}$

$\Rightarrow i_C(0^+) = -90 mA$

$\dfrac{dv}{dt} = \dfrac{-90 * 10^{-3}}{0.2 * 10^{-6}} \Rightarrow \dfrac{dv}{dt} = -450KV$

Step 3 - Find the first derivative of eq. 1 then find A_1, A_2

$V'(t) = -20000 A_1 e^{-20000t} - 5000 A_2 e^{-5000t}$ --- 2

apply the initial Conditions to get 2 equations

$$\left.\begin{aligned} 12 &= A_1 + A_2 \\ -450K &= -20000 A_1 - 5000 A_2 \end{aligned}\right\} \quad \text{Solve} \quad \begin{aligned} A_1 &= -14 \\ A_2 &= 26 \end{aligned}$$

hence :- $V(t) = 26 e^{-20000t} - 14 e^{-5000}$

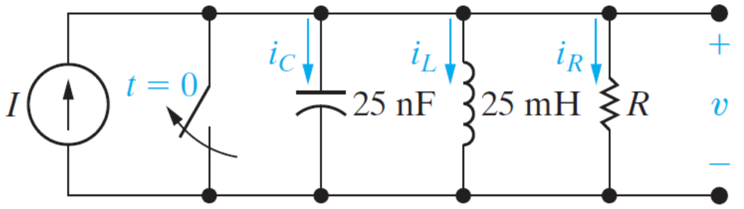
1.2 – step response

$X(t) = X_n(t) + X_f(t)$

- $X_n(t)$: natural response.
- $X_f(t)$: step response

* $i_f(t)$ = current source , $V_f(t)$ = voltage source

Example 2



The initial stored energy = 0

$R = 625\,\Omega$, $I = 24\,mA$

Find $i_L(t)$

Step 1 – Solve normal as natural

$\alpha = 1/2RC \Rightarrow \alpha = 1/3.125 * 10^{-5} \Rightarrow \alpha = 32 * 10^{3}$

$\omega = 1/\sqrt{LC} \Rightarrow \omega = 1/2.5 * 10^{-5} \Rightarrow \omega = 40 * 10^{3}$

Since $\omega > \alpha$ we apply $\Rightarrow \omega_d = \sqrt{\omega^2 - \alpha^2}$

$\omega_d = \sqrt{1.6 * 10^9 - 1.024 * 10^9} \Rightarrow \omega_d = 24 * 10^3$

$$i_n(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$
$$= e^{-32000t}(B_1 \cos 24000t + B_2 \sin 24000t)$$

$$\therefore\ i(t) = 24 + e^{-32000t}(B_1 \cos 24000t + B_2 \sin 24000t)$$

* Note : initial Conditions are found for the final equation

Since the initial stored energy = 0 $\Rightarrow i(0^+) = V(0^+) = 0$

$$V(0^+) = L\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{V(0^+)}{L} = \frac{0}{L} \Rightarrow \frac{di}{dt} = 0$$

Step 2 – get the first derevative of $i(t)$ then solve when $t=0$

$24 + B_1 = 0 \Rightarrow B_1 = -24$ ------ (1)

$-32000 B_1 + 24000 B_2 = 0$ ------ (2)

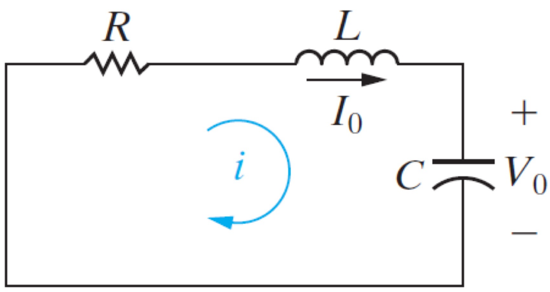
$$\left\{ \begin{array}{l} B_1 = -24 \\ B_2 = -32 \end{array} \right.$$

$$\boxed{i(t) = 24 + e^{-32000t}(-24 \cos 24000t - 32 \sin 24000t)}$$

2 – Series RLC circuits

We have two cases

2.1 – step response



to find the DE apply KVL

$V_R + V_L + V_C = 0$

important to find the initial Conditions

$$\boxed{iR + \frac{1}{C}\int_0^t i(t)\,dt + L\frac{di}{dt} = -V_c(0^-)}$$

Same as parallel

resonant frequency

$$\omega = \frac{1}{\sqrt{LC}}$$

Neper frequency

$$\alpha = \frac{R}{2L}$$

Same as parallel

$$S_1 = -\alpha - \sqrt{\alpha^2 - \omega^2}$$

$$S_2 = -\alpha + \sqrt{\alpha^2 - \omega^2}$$

Same as parallel

1. if $\alpha^2 > \omega^2$, the Solutions are real distinct

$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \Rightarrow$ Over Damped

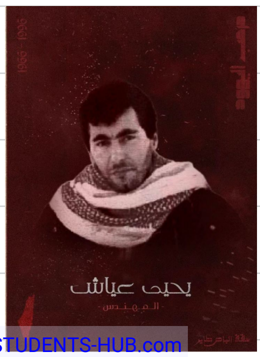
2. if $\alpha^2 = \omega^2$, the Solutions are real Similar

$V(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t} \Rightarrow$ Critical Damped

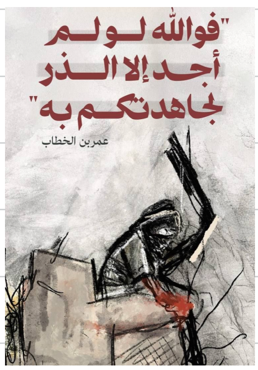
3. if $\alpha^2 < \omega^2$, the Solutions are Complex

let $\omega_d = \sqrt{\omega^2 - \alpha^2}$

$V(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t) \Rightarrow$ Under Damped



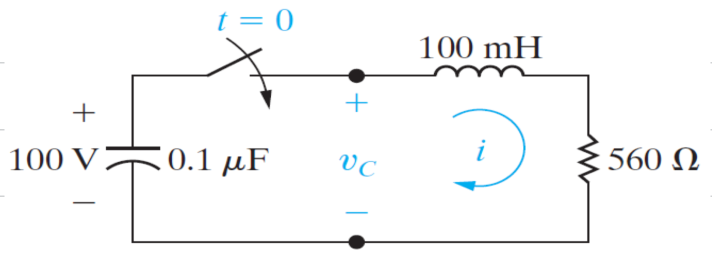
وَمَن يَتَوَكَّل عَلَى اللهِ فَهُوَ حَسْبُه



Example 1

$V_c(0^-) = 100\,V$

Find $i(t)$



$\alpha = R/2L \Rightarrow \alpha = 560 / 2 * 100 * 10^{-3} \Rightarrow \alpha = 2800$

$\omega = 1/\sqrt{LC} \Rightarrow \omega = 1/\sqrt{10^{-8}} \Rightarrow \omega = 10000$

$\omega_d = \sqrt{\omega^2 - \alpha^2} \Rightarrow \omega_d = \sqrt{92.16 * 10^6} \Rightarrow \omega_d = 9600$

$$i(t) = e^{-2800t}\left(\beta_1 \cos 9600t + \beta_2 \sin 9600t\right)$$

initial Conditions :-

the Circuit was opened $\Rightarrow i(0^-) = 0$

$$V_c = L\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{V_c(0^-)}{L} = \frac{100}{100*10^{-3}} \Rightarrow \frac{di}{dt} = 1000\,A$$

differentiate $i(t)$:-

$$\underbrace{\left(\left(e^{-2800t} * -9600\,\beta_1 \sin 9600t\right) + \left(\beta_1 \cos 9600 * -2800\,e^{-2800t}\right)\right)}_{\text{First Part}}$$

$$+\underbrace{\left(\left(e^{-2800t} * 9600\,\beta_2 \cos 9600t\right) + \left(\beta_2 \sin 9600 * -2800\,e^{-2800t}\right)\right)}_{\text{Second Part}}$$

Solve when $t = 0$:-

from the first equation $\Rightarrow \beta_1 = 0$

from the second equation :

$$(1 * 0) + (\beta_1 * -2800) + (1 * 9600\,\beta_2) + (0 * 1) = 1000$$

$$-2800\,\beta_1 + 9600\,\beta_2 = 1000$$

$$-2800 * 0 + 9600\,\beta_2 = 1000$$

$$9600\,\beta_2 = 1000 \Rightarrow \beta_2 = \frac{1000}{9600} \Rightarrow \beta_2 = 0.104$$

$$i(t) = e^{-2800t} * 0.104 \sin 9600t$$

2.2 – step response

$X(t) = X_n(t) + X_f(t)$

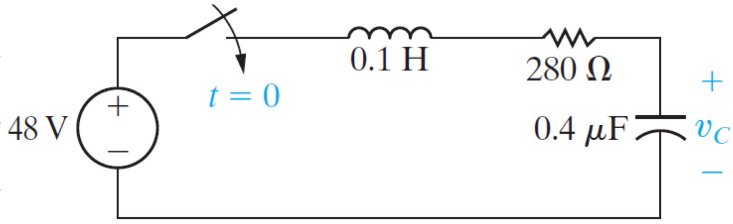
- $X_n(t)$: natural response
- $X_f(t)$: step response

* $i_f(t)$ = current source , $V_f(t)$ = voltage source

Example 2

No energy is stored at C, L

find $V(t)$



$\alpha = R/2L \Rightarrow \alpha = 280/2*0.1 \Rightarrow \alpha = 1400$

$\omega = 1/\sqrt{LC} \Rightarrow \omega = 1/\sqrt{4*10^{-8}} \Rightarrow \omega = 5000$

$\omega_d = \sqrt{\omega^2 - \alpha^2} \Rightarrow \omega_d = \sqrt{23*10^6} \Rightarrow \omega_d = 4800$

$V_n(t) = e^{-1400t}(\beta_1 \cos 4800t + \beta_2 \sin 4800t)$

$V(0^-) = 0$ since the initial stored energy $= 0$

$i(0^-) = 0 = C\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = 0$

After getting the equations $\Rightarrow \beta_1 = -48$

$\beta_2 = -14$

$V(t) = V_f(t) + V_n(t)$

$$V(t) = 48 + e^{-1400t}(-48 \cos 4800t - 14 \sin 4800t)$$