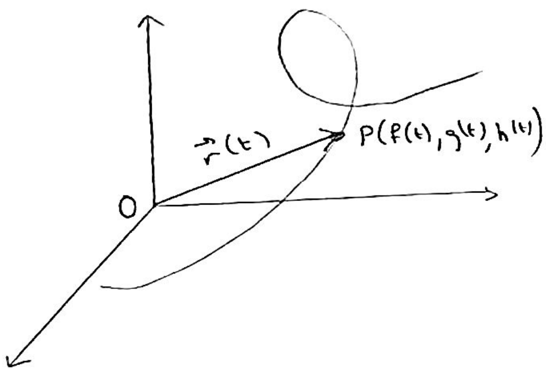
Chapter 13

Vector_ Valued Functions and Motion in Space.

13.1 Curves in Space and Their Tangents:

Assume a particle moves in space along a Curve C during a time Interval I



- The position of the particle at any time is determined by the position vector:

$$\vec{r}(t) := f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

- The parametrization of the Curve C is:

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

- $f(t)$, $g(t)$, $h(t)$ are called Component functions.

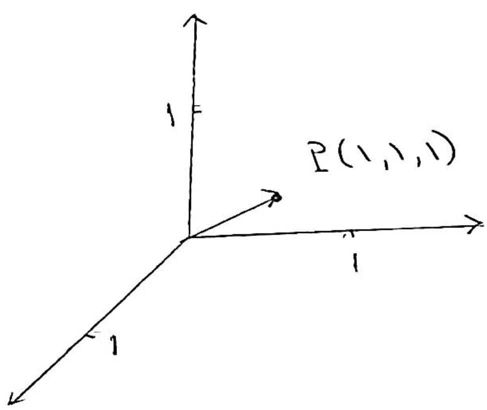
Parametrization:

1. $x = t$, $y = t$, $z = t$

   $0 \le t \le 1$.

   Initial point at $t=0$

   Terminal pt at $t=1$



2. $x = t-1$, $y = t-1$, $z = t-1$

   $1 \le t \le 2$.

3. Example: $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

   $x^2 + y^2 = 1$ circle.

Note: The domain of the position vector $\vec{r}(t)$ is the Intersection of the domains of $f(t)$, $g(t)$ & $h(t)$.

Note: $\vec{r}(t)$ is called a vector-valued function; that assigns a vector in space to each element in the domain D.

Def: The vector $\vec{C} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$, where c_1, c_2, c_3 are constants is called Constant vector.

Limits and Continuity:

Def: Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D, and $\vec{L}$ a vector, then

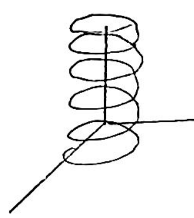
$$\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} f(t)\vec{i} + \lim_{t \to t_0} g(t)\vec{j} + \lim_{t \to t_0} h(t)\vec{k} = \vec{L}$$

Example: Let $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + 2t\,\vec{k}$.

Find $\lim_{t \to \frac{\pi}{2}} \vec{r}(t)$.

$$\lim_{t \to \frac{\pi}{2}} \vec{r}(t) = \lim_{t \to \frac{\pi}{2}} (\cos t)\vec{i} + \lim_{t \to \frac{\pi}{2}} (\sin t)\vec{j} + \lim_{t \to \frac{\pi}{2}} 2t\,\vec{k}$$

$$= 0\vec{i} + 1\vec{j} + \pi\vec{k} = \vec{j} + \pi\vec{k}.$$



Example: $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + \lfloor t \rfloor \vec{k}$ ← greatest Integer function. or "floor fun."

$\vec{r}$ is discontinuous at every Integer.

Recall: $\lfloor 6.2 \rfloor = 6$, $\lfloor 4 \rfloor = 4$, $\lfloor -6.7 \rfloor = -7$.

$$\lim_{t \to \pi/2} \vec{r}(t) = 0\vec{i} + 1\vec{j} + \lfloor \tfrac{\pi}{2} \rfloor \vec{k} =$$
$$= \vec{j} + \vec{k}.$$

Derivatives

Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a position vector the difference between the particle's position at t & $t+\Delta t$ is

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\Delta\vec{r} = \left[f(t+\Delta t)\vec{i} + g(t+\Delta t)\vec{j} + h(t+\Delta t)\vec{k}\right] - \left[f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}\right]$$
$$= \left[f(t+\Delta t) - f(t)\right]\vec{i} + \left[g(t+\Delta t) - g(t)\right]\vec{j} + \left[h(t+\Delta t) - h(t)\right]\vec{k}$$

Now $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \left[\frac{df}{dt}\right]\vec{i} + \left[\frac{dg}{dt}\right]\vec{j} + \left[\frac{dh}{dt}\right]\vec{k}$.

$$\Rightarrow \quad \underline{\underline{\vec{V}(t)}} = r'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}.$$

Remarks: $\boxed{1}$ The position vector $\vec{r}(t)$ is called <u>smooth</u> if $\frac{dr}{dt}$ is continuous and never $\vec{0}$. that is, if f, g, and h have continuous derivatives that are not all zero.

$\boxed{2}$ The Velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt}$

($\vec{v}$ is tangent on the Curve)

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$\boxed{3}$ Speed is the magnitued of the Velocity $S = |\vec{v}(t)|$.

$\boxed{4}$ Direction of the motion is a unit vector given by:

$$\text{Direction} = \frac{\vec{v}(t)}{|\vec{v}(t)|}$$

<u>Example:</u> $\vec{v}_1 = (\cos t)\vec{i} + (\sin t)\vec{j}$.

$$S = |\vec{v}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

So $\vec{v}_1$ is a Unit vector.

<u>Example:</u> Find Direction of particle moves along a curve with position vector $\vec{r}(t) = 3t\vec{i} + 4t\vec{j}$.

$$\vec{v}(t) = 3\vec{i} + 4\vec{j}.$$

$$|\vec{v}(t)| = \sqrt{9+16} = \sqrt{25} = 5.$$

$$\therefore \text{Direction} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

<u>Note:</u> $|\text{Direction}| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1.$

Note: Velocity = (speed) . (Direction)

$$= |\vec{v}| \cdot \frac{\vec{v}}{|\vec{v}|}.$$

Differentiation Rules:

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t. $\vec{C}$ a constant vector, c any scalar and f any differentiable scalar function.

1. Constant function Rule: $\frac{d}{dt}\vec{C} = \vec{0}$.

2. Scalar multiple Rule: $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$.

3. Sum Rule / Difference: $\frac{d}{dt}[\vec{u}(t) \pm \vec{v}(t)] = \vec{u}'(t) \pm \vec{v}'(t)$.

4. Scalar multiple Rule: $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

5. Dot product Rule: $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

   $\frac{du}{dt} \cdot v(t) + u(t) \cdot \frac{dv}{dt}$

6. Cross product Rule: $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

7. Chain Rule: $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\,\vec{u}'(f(t))$

<u>Example:</u> $\vec{w}_1 = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{w}_2 = -\vec{i} - 7\vec{k}$

$\vec{w}_1 \cdot \vec{w}_2 = -2 + 7 = +5$. $\Rightarrow (w_1 \cdot w_2)' = 0$

<u>Recall:</u> $\vec{w}_1 \cdot \vec{w}_2 = |\vec{w}_1||\vec{w}_2| \cos\theta$.

<u>Example:</u> $\vec{w}_1 = t\vec{i} + 3t\vec{j}$, $\vec{w}_2 = 3\vec{i} + 7t\vec{j}$.

$\vec{w}_1 \cdot \vec{w}_2 = 3t + 21t^2$.

$(\vec{w}_1 \cdot \vec{w}_2)' = 3 + 42t$.

Using Rule $\boxed{5}$: $(w_1 \cdot w_2)' = (\vec{i} + 3\vec{j}) \cdot (3\vec{i} + 7t\vec{j}) + (t\vec{i} + 3t\vec{j}) \cdot (7\vec{j})$

$= 3 + 21t + 21t = 3 + 42t$

<u>Example:</u> $\vec{w}_1 \times \vec{w}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 0 & -7 \end{vmatrix} = -21\vec{i} + 15\vec{j} + 3\vec{k}$.

<u>Recall</u>
- $\vec{w}_1 \times \vec{w}_2 = -\vec{w}_2 \times \vec{w}_1$
- while $\vec{w}_1 \cdot \vec{w}_2 = \vec{w}_2 \cdot \vec{w}_1$.
- $\vec{w}_1 \times \vec{w}_2 = (|\vec{w}_1||\vec{w}_2| \sin\theta)\,\vec{n}$

where $\vec{n}$: normal unit vector (RHR)

Remark: Assume $\vec{r}(t)$ is differentiable vector function with constant length. then: $\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$.

proof: $|\vec{r}(t)| = C$

$|\vec{r}(t)|^2 = C^2$

but $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = C^2$.

Now $\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \vec{r}(t) \cdot \frac{d\vec{r}}{dt} + \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$

$\Rightarrow \; 2\,\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$

$\Rightarrow \; \vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$.

Result: $\vec{r}(t) \perp \frac{d\vec{r}}{dt}$ iff $\vec{r}(t)$ has constant length.

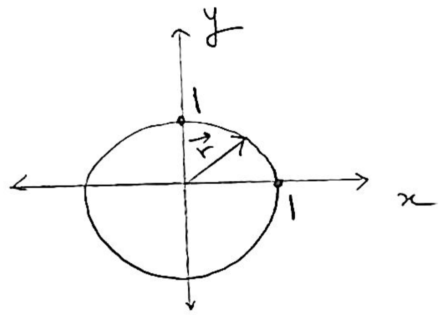
Example: $\vec{r}(t) = (\cos t)\,\vec{i} + (\sin t)\,\vec{j}$.

$|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$

Cartesian equation: $x = \cos t$

$y = \sin t$

$x^2 + y^2 = 1$

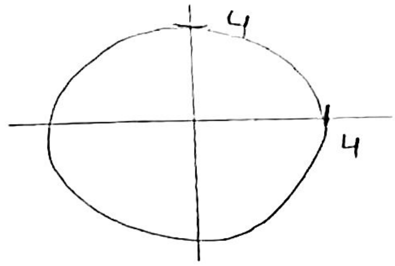


<u>Example:</u> Given a position vector of a particle moves along the path $\vec{r}(t) = (4\cos\frac{t}{2})\vec{i} + (4\sin\frac{t}{2})\vec{j}$

1. Find a Cartesian equation in x-y plane. Draw it. p 629

$x = 4\cos\frac{t}{2}$, $y = 4\sin\frac{t}{2}$

$x^2+y^2 = 16\cos^2\frac{t}{2} + 16\sin^2\frac{t}{2} = 16$



2. Find Velocity and acceleration at $t = \pi, \frac{3\pi}{2}$.

$\vec{v}(t) = \frac{d\vec{r}}{dt} = (-2\sin\frac{t}{2})\vec{i} + (2\cos\frac{t}{2})\vec{j}$

$\vec{a}(t) = \frac{d\vec{v}}{dt} = (-\cos\frac{t}{2})\vec{i} - (\sin\frac{t}{2})\vec{j}$.

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$\vec{v}(\pi) = -2\vec{i}$. , $\vec{v}(\frac{3\pi}{2}) = -\sqrt{2}\,\vec{i} - \sqrt{2}\,\vec{j}$ 270

$\vec{a}(\pi) = -\vec{j}$, $\vec{a}(\frac{3\pi}{2}) = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$.

3. Find the speed & the Direction of the motion at $t = 2\pi$.

$S = |\vec{v}(t)| = \sqrt{4^2\sin^2\frac{t}{2} + 4^2\cos^2\frac{t}{2}} = \cancel{\sqrt{4}} = \cancel{2}$ $\sqrt{32}$, $\forall t$.

$D|_{2\pi} = \frac{\vec{v}(2\pi)}{|\vec{v}(2\pi)|} = \frac{-2\vec{j}}{\sqrt{32}}$

13.2 Integrals of Vector functions

Def: If $\vec{R}(t)$ is any anti derivative of $\vec{r}(t)$, then the indefinite Integral of $\vec{r}$ is:

$$\int r(t)\,dt = \vec{R}(t) + \vec{C}$$

Def: If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a,b]$, then so is $\vec{r}$, and the definite Integral of $\vec{r}$ is:

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Example: Let $\vec{r}(t) = (\cos t)\vec{i} + 3\vec{j} - 2t\vec{k}$.

Find (1) $\int \vec{r}(t)\,dt$

$$\int \vec{r}(t)\,dt = (\sin t + c_1)\vec{i} + (3t + c_2)\vec{j} - \left(\frac{2t^2}{2} + c_3\right)\vec{k}$$

$$= \sin t\,\vec{i} + 3t\,\vec{j} - t^2\vec{k} + \underbrace{c_1\vec{i} + c_2\vec{j} + c_3\vec{k}}_{\vec{C}}$$

(2) $$\int_0^{\pi} \vec{r}(t)\,dt = \sin t\Big|_0^{\pi}\vec{i} + 3t\Big|_0^{\pi}\vec{j} - t^2\Big|_0^{\pi}\vec{k}$$

$$= 0\vec{i} + 3\pi\vec{j} - \pi^2\vec{k} = 3\pi\vec{j} - \pi^2\vec{k}.$$

Example: Let $\vec{a}(t) = (-3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$ be the acceleration of a particle. Assume that at time $t=0$, the particle departed from the point $(3,0,0)$ with velocity $\vec{v}(0) = 3\vec{j}$.

Find the position of the particle at time $t=\pi$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{k} + \vec{C}$$

$$\vec{v}(0) = 3\vec{j} + \vec{C} = 3\vec{j} \implies \boxed{\vec{C} = 0}$$

$$\therefore \vec{v}(t) = (-3\sin t)\vec{i} + (3\cos t)\vec{j} + 2t\vec{k}.$$

Now $\vec{r}(t) = \int \vec{v}(t)\,dt = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k} + \vec{C_2}$

$\vec{r}(0) = 3\vec{i} + \vec{C_2} = 3\vec{i} + 0\vec{j} + 0\vec{k}$ (from $(3,0,0)$)

$$\therefore \vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2 k$$

$$\Rightarrow \vec{r}(\pi) = -3\vec{i} + \pi^2\vec{k}.$$

13.3 Arc length in Space

Def: The length of a smooth curve:

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Remark: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$$\Rightarrow |\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Hence $L = \int_a^b |\vec{v}(t)|\, dt$

Example: Given $\vec{r}(t) = (2+t)\vec{i} - (1+t)\vec{j} + \sqrt{2}\, t\,\vec{k}$ for $0 \leq t \leq 3$. Find the arc length for the curve parametrized by the position vector $\vec{r}(t)$.

$\vec{v}(t) = \vec{i} - \vec{j} + \sqrt{2}\,\vec{k}$

$|\vec{v}(t)| = \sqrt{1+1+(\sqrt{2})^2} = \sqrt{4} = 2.$

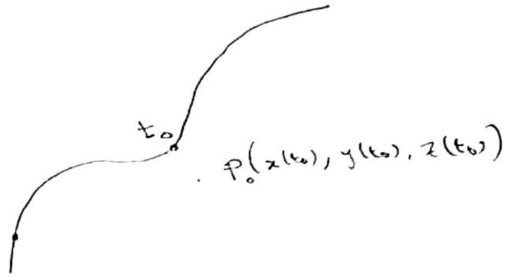
$\therefore L = \int_0^3 2\, dt = 6 - 0 = 6.$

Unit Tangent Vector: $\frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}.$

Def: Arc length parameter.

If we choose a base point $P(t_0)$ on a smooth curve C then the arc length parameter from the based point $P(t_0)$ is defined by:

$$S(t) = \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau \quad \dots \circledast$$



Remark: Note that $L = S(b) - S(a)$.

proof: $S(b) - S(a) = \int_{t_0}^{b} |\vec{v}(t)|\,dt - \int_{t_0}^{a} |v(t)|\,dt$

$$= -\int_{b}^{t_0} |\vec{v}(t)|\,dt - \int_{t_0}^{a} |\vec{v}(t)|\,dt =$$

$$= -\left[\int_{b}^{a} |v(t)|\,dt\right] = \int_{a}^{b} |v(t)|\,dt = L.$$

Speed on a smooth curve: $\boxed{\dfrac{ds}{dt} = |v(t)|}$ FTOF from $(\ast)$

Unit Tangent vector: $T = \dfrac{\vec{v}(t)}{|\vec{v}(t)|}$.

V.I

$$\boxed{\text{Note: } \vec{T} = \frac{\vec{v}}{|\vec{v}|} = \vec{v} \cdot \frac{1}{|\vec{v}|} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{d\vec{r}}{ds}}$$

<u>Example:</u> Find the arc Length parameter along the Curve

$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$

Consider the base point $P(1,0,0)$ and $\frac{\pi}{2} \le t \le \pi$.

Sol: we need to find t_0 such that:

$\cos t_0 + t_0 \sin t_0 = 1$ & $\sin t_0 - t_0 \cos t_0 = 0$

clearly $t_0 = 0$.

$\therefore \vec{v}(t) = \frac{d\vec{r}}{dt} = (\cancel{-\sin t} + t\cos t + \cancel{\sin t})\vec{i} + (\cancel{\cos t} + t\sin t - \cancel{\cos t})\vec{j}$

$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$

$|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = |t| = t$

$\therefore s(t) = \int_0^t \tau \, d\tau = \frac{t^2}{2}$.

$\boxed{2}$ Find the arc length L

$L = S(\pi) - S(\frac{\pi}{2}) = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{3\pi^2}{8}$.

OR: $L = \int_{\frac{\pi}{2}}^{\pi} t \, dt = \frac{3\pi^2}{8}$

15 Find the length of the Curve

$$\vec{r}(t) = (\sqrt{2}\, t)\vec{i} + (\sqrt{2}\, t)\vec{j} + (1-t^2)\vec{k}$$

from $\overset{t=0}{(0,0,1)}$ to $\overset{t=1}{(\sqrt{2}, \sqrt{2}, 0)}$ $\quad \sqrt{2}t = \sqrt{2}$, $t=1$

$$L = \int_a^b |\vec{v}(t)|\, dt$$

$$v(t) = \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j} + 2t\,\vec{k}$$

$$|\vec{v}(t)| = \sqrt{(2)+(2)+4t^2} = 2\sqrt{1+t^2}$$

$$\therefore L = \int_0^1 2\sqrt{1+t^2}\, dt$$

Let $t = \tan\theta \Rightarrow 1+t^2 = 1+\tan^2\theta = \sec^2\theta$

$dt = \sec^2\theta\, d\theta$

$$\therefore L = \int 2\sqrt{\sec^2\theta}\,.\,\sec^2\theta\, d\theta \qquad \theta \text{ positive}$$

$$= \int 2\sec^3\theta\, d\theta$$

$(\frac{2}{3})(13)$

We Integrate by parts.

$u = \sec\theta \quad , \quad dv = \sec^2\theta \, d\theta$

$du = \sec\theta \tan\theta \, d\theta \quad\quad v = \tan\theta$

$$\Rightarrow \int \sec^3\theta \, d\theta = \sec\theta \tan\theta - \int \tan\theta (\sec\theta \tan\theta) \, d\theta$$

$$= \sec\theta \tan\theta - \int (\sec^2\theta - 1) \sec\theta \, d\theta$$

$$= \sec\theta \tan\theta - \int (\sec^3\theta - \sec\theta) \, d\theta$$

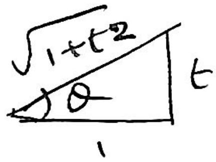
$$\therefore \int \sec^3\theta \, d\theta = \sec\theta \tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

$$\Rightarrow 2 \int \sec^3\theta \, d\theta = \sec\theta \tan\theta + \int \sec\theta \, d\theta$$

$$\therefore \int \sec^3\theta \, d\theta = \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln |\sec\theta + \tan\theta|$$

$\sec\theta = \sqrt{1+t^2}$

$\tan\theta = t$



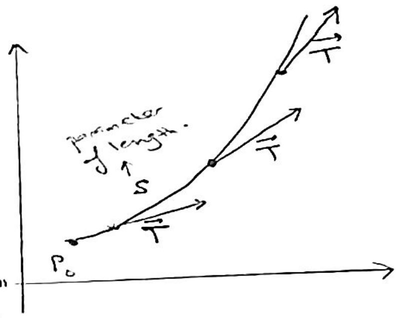
$\left(\frac{3}{3}\right)^{(13)}$

13.4 Curvature and Normal Vectors of a Curve.

انحناء

Def: If $\vec{T}$ is the unit tangent vector of a smooth curve, the curvature function of the curve is

$$\kappa := \left| \frac{d\vec{T}}{ds} \right|$$



"The rate at which $\vec{T}$ turns per unit of length along the curve is called Curvature."

Formula for Calculating Curvature.

If $\vec{r}(t)$ is a smooth curve, then the curvature is

$$K = \frac{1}{|\vec{v}(t)|} \left| \frac{d\vec{T}}{dt} \right|.$$

$$\frac{ds}{dt} = |v(t)|.$$

when $\vec{T} = \dfrac{v(t)}{|\vec{v}(t)|}$.

proof:

$$K = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \right|$$

$$= \left| \frac{d\vec{T}}{dt} \right| \cdot \left| \frac{dt}{ds} \right| = \left| \frac{d\vec{T}}{dt} \right| \cdot \frac{1}{|ds/dt|}$$

$$= \frac{1}{|\vec{v}(t)|} \cdot \left| \frac{d\vec{T}}{dt} \right|.$$

Example: Given A straight line vector

$\vec{r}(t) = \vec{C}_1 + t\vec{C}_2$. Find K.

$\vec{v}(t) = \vec{r}\,'(t) = \vec{C}_2$.

$\Rightarrow \vec{T}(t) = \dfrac{v(t)}{|\vec{v}(t)|} = \dfrac{\vec{C}_2}{|\vec{C}_2|}$ which is a constant vector.

$\dfrac{d\vec{T}}{dt} = \vec{0}$ & $\left|\dfrac{dT}{dt}\right| = 0$

$\Rightarrow K = \dfrac{1}{|v(t)|}\left|\dfrac{dT}{dt}\right| = 0.$ (No انحناء)

Example: Let $\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j}$. Find K.

$\vec{v}(t) = -(a\sin t)\vec{i} + (a\cos t)\vec{j}$.

$|\vec{v}(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = |a| = a.$

$\therefore \vec{T} = \dfrac{v(t)}{|\vec{v}(t)|} = (-\sin t)\vec{i} + (\cos t)\vec{j}$.

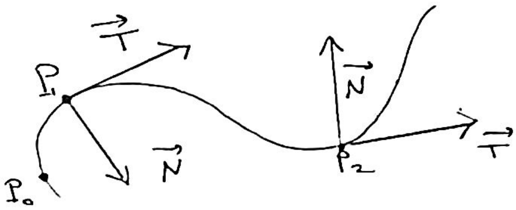
$\dfrac{d\vec{T}}{dt} = (-\cos t)\vec{i} - (\sin t)\vec{j}$.

$\left|\dfrac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1.$

$\therefore K = \dfrac{1}{a}\cdot(1) = \dfrac{1}{\text{radius}}$.

Def: At a point where $K \neq 0$, the principle Unit normal vector for a smooth curve in the plane is

$$\vec{N} = \frac{1}{K}\frac{d\vec{T}}{ds}$$



Formula for Calculating $\vec{N}$.

If $\vec{r}(t)$ is a smooth curve, then the principle Unit normal is

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}.$$

$\frac{ds}{dt} = |v(t)|$

proof:

$$\vec{N} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} = \frac{(d\vec{T}/dt)(dt/\cancel{ds})}{|d\vec{T}/dt||dt\cancel{/ds}|}$$

but $\frac{ds}{dt} = |\vec{v}(t)| \Rightarrow \left|\frac{ds}{dt}\right| = |\vec{v}(t)|$.

$$\therefore \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}.$$

Remarks:

1) $|\vec{N}| = \frac{1}{K}\left|\frac{d\vec{T}}{ds}\right| = \frac{1}{K}\cdot K = 1$

عدد

2) $\vec{N} \perp \vec{T}$ " $\vec{N}\cdot\vec{T} = 0$ "

3) $\vec{N}$ points the direction in which the curve turning.

Example: Find $\vec{T}$ & $\vec{N}$ for $\vec{r}(t) = (\cos 2t)\vec{i} + (\sin 2t)\vec{j}$

$$\vec{v}(t) = -(2\sin 2t)\vec{i} + (2\cos 2t)\vec{j}$$

$$|\vec{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2.$$

$$\therefore \ T = \frac{\vec{v}}{|\vec{v}|} = -(\sin 2t)\vec{i} + (\cos 2t)\vec{j}.$$

$$\frac{d\vec{T}}{dt} = -(2\cos 2t)\vec{i} - (2\sin 2t)\vec{j}.$$

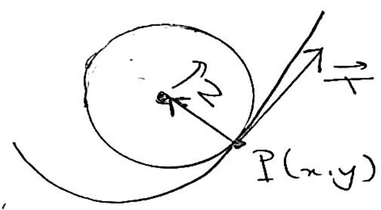
$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2.$$

$$\therefore \ \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -(\cos 2t)\vec{i} - (\sin 2t)\vec{j}.$$

Def: Circle of Curvature :

The Circle of curvature at a point P on a plane curve where $K \neq 0$ is the circle that :

1) is tangent to the curve at P.
2) has the same curvature the curve has at P
3) Lies toward the Concave or inner side of the Curve



Remark: 1) radius of Curvature is the radius of the Circle of curvature

2) Radius of Curvature: $\rho = \frac{1}{K}$.

3) center of Curvature = center of the circle of Curvature

Example: Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

parametrization: Let $x = t$, then $y = t^2$.

$\therefore \quad \vec{r}(t) = t\vec{i} + t^2\vec{j}$.

$\vec{v}(t) = \vec{i} + 2t\vec{j}$.

$|\vec{v}(t)| = \sqrt{1+4t^2}$.

$\therefore \quad \vec{T} = \frac{1}{\sqrt{1+4t^2}}(\vec{i} + 2t\vec{j})$.

$$\frac{d\vec{T}}{dt} = \frac{-4t}{(1+4t^2)^{3/2}}\vec{i} + \left[\frac{2}{(1+4t^2)^{1/2}} - \frac{8t^2}{(1+4t^2)^{3/2}}\right]\vec{j}.$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0} = 2\vec{j}$$

$$\therefore K(0) = \frac{1}{|v(0)|} \cdot \left|\frac{d\vec{T}}{dt}(0)\right| = \frac{1}{\sqrt{1}}|2\vec{j}| = 2.$$

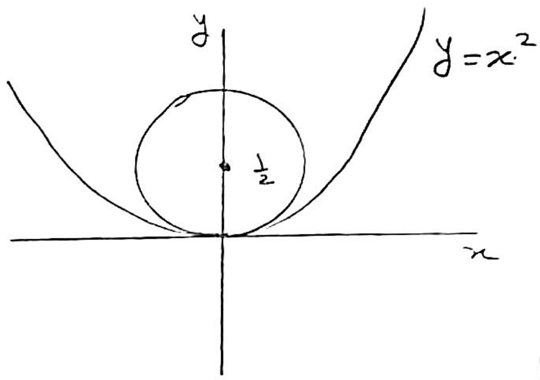
$\therefore \quad \rho = \frac{1}{K} = \frac{1}{2}$.

at $t=0$, $x=0$ & the radius $= \frac{1}{2}$.

$\therefore$ one point $(x_0, y_0) = (0, \frac{1}{2})$

$\therefore (x-x_0)^2 + (y-y_0)^2 = a^2$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$



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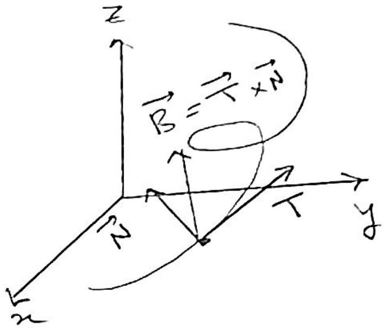
2) Find $\vec{T}$ at 0 & $\vec{N}$ at $\vec{0}$

$\vec{T}(0) = \vec{i}$ then $\vec{N} = \vec{j}$ since $\vec{N} \perp \vec{T}$

or $\vec{N} = \dfrac{\frac{dT}{dt}(0)}{\left|\frac{dT}{dt}(0)\right|} = \dfrac{2\vec{j}}{2} = \vec{j}$.

13.5 Tangent and Normal Component of Acceleration.

Assume a particle moves in space along a curve S.
Then the motion of this particle can be described in terms of:



1. Unit Tangent vector $\vec{T}$
   (forward or backward motion).
2. Unit Normal vector $\vec{N}$
   (tendency of motion)
3. Unit binomial ~~Unit~~ vector $\vec{B} = \vec{T} \times \vec{N}$.
   $\vec{B} \perp \vec{T}$ & $\vec{N}$.

Together $\vec{T}, \vec{N}, \vec{B}$ define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space.

Def: If the acceleration vector is written as:

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

then: $a_T = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}|\vec{v}|$ is the tangent component

and $a_N = K \left(\dfrac{ds}{dt}\right)^2 = K|\vec{v}|^2$ is the Normal Component

Note V. I

Finding $a_{\vec{T}}$ & $a_{\vec{N}}$:

we know that: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{T}\frac{ds}{dt}$

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{dT}{dt}$

$= \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\left(\frac{dT}{ds}\cdot\frac{ds}{dt}\right)$ $\qquad \vec{N} = \frac{1}{k}\frac{d\vec{T}}{ds}$

$= \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 k\vec{N}$

$= a_T\vec{T} + a_N\vec{N}$.

<u>Remark</u>: $a_N := \sqrt{|\vec{a}|^2 - a_T^2}$

<u>proof</u>: $\vec{a} = a_T\vec{T} + a_N\vec{N}$ $\qquad$ $\vec{T}$ & $\vec{N}$ are Unit vectors

$|\vec{a}| = \sqrt{a_T^2 + a_N^2}$

$\therefore |\vec{a}|^2 = a_T^2 + a_N^2$

$\therefore a_N = \sqrt{|\vec{a}|^2 - a_T^2}$.

Example: Given $\vec{r}(t) = (1+3t)\vec{i} + (t-2)\vec{j} - 3t\vec{k}$

without finding $\vec{T}$ & $\vec{N}$ write $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$\vec{v}(t) = 3\vec{i} + \vec{j} - 3\vec{k} \Rightarrow |v(t)| = \sqrt{19}$.

$a_T = \frac{d}{dt}|\vec{v}(t)| = 0$

Now $\vec{a} = \vec{0} \Longrightarrow a_N = \sqrt{|\vec{a}|^2 - a_T^2} = 0$

Hence $\vec{a} = 0\vec{T} + 0\vec{N}$.

Example: without finding $\vec{T}$ & $\vec{N}$ write $\vec{a} = a_T\vec{T} + a_N\vec{N}$

where $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$, $t>0$

- $\vec{v}(t) = \frac{d\vec{r}}{dt} = (-\sin t + t\cos t + \sin t)\vec{i} + (\cos t + t\sin t - \cos t)\vec{j}$

$= (t\cos t)\vec{i} + (t\sin t)\vec{j}$.

$|\vec{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin t^2} = \sqrt{t^2} = |t| = t$. , $t>0$

$\therefore a_T = \frac{d}{dt}|\vec{v}(t)| = 1$.

- Now $\vec{a} = (\cos t - t\sin t)\vec{i} + (\sin t + t\cos t)\vec{j}$.
- $|\vec{a}|^2 = t^2+1$

$\therefore a_N = \sqrt{|\vec{a}|^2 - a_T^2} = \sqrt{t^2+1-1} = \sqrt{t^2} = t$.

$\therefore \vec{a} = a_T\vec{T} + a_N\vec{N} = (1)\vec{T} + t\vec{N} = \vec{T} + t\vec{N}$.

$\kappa > 0$ always

$\tau <, >, 0$ او صفر

Torsion: الفتل / الالتواء

Def: Let $\vec{B} = \vec{T} \times \vec{N}$. The torsion function of a smooth curve is

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}.$$

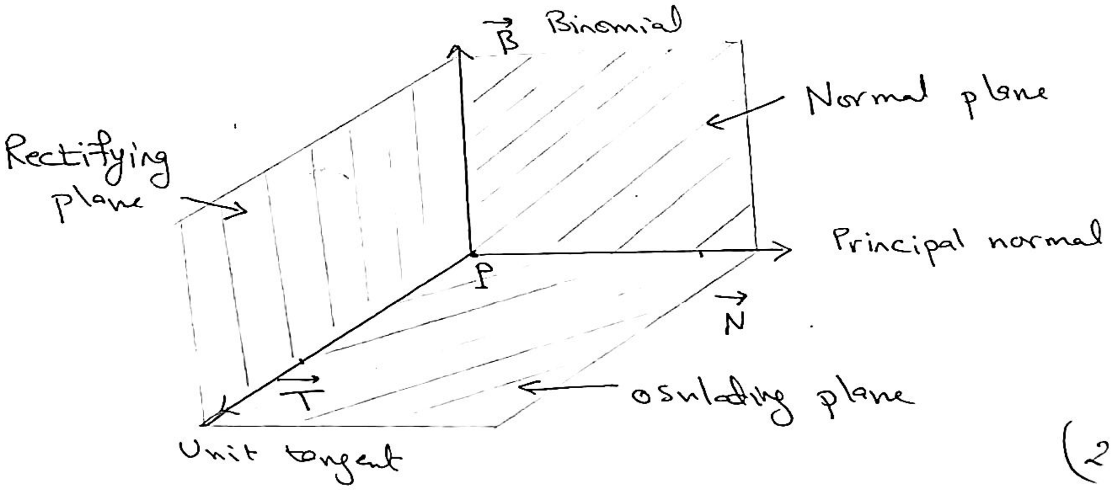
Computational formulas:

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}, \quad \text{if } (\vec{v} \times \vec{a} \neq \vec{0}).$$

$$\left(\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}, \quad \dddot{x} = \frac{d^3x}{dt^3} \right).$$

for $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$.

Recall:
$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$



To find the equation for a plane M we need:

1) point $(x_0, y_0, z_0) = P$.

2) Normal vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$.

Then Equation of plane is:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0.$$

Example: Given $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$.

1) Find $\vec{r}(0)$, $\vec{T}(0)$, $\vec{N}(0)$, $\vec{B}(0)$

2) Find torsion τ.

Sol:

1) $\vec{r}(0) = \vec{i} + 0\vec{j} + 0\vec{k} = \vec{i}$.

$\Rightarrow P_0 = (1, 0, 0)$.

$\vec{v}(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{k}$.

$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.

$$\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \left(\frac{-1}{\sqrt{2}}\sin t\right)\vec{i} + \left(\frac{1}{\sqrt{2}}\cos t\right)\vec{j} + \frac{1}{\sqrt{2}}\vec{k}.$$

$\therefore \vec{T}(0) = 0\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{K}$.

$= \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$.

$$\vec{N} = \frac{d\vec{T}/dt}{|dT/dt|}$$

$$\Rightarrow \quad \frac{d\vec{T}}{dt} = \left(\frac{-1}{\sqrt{2}}\cos t\right)\vec{i} - \left(\frac{1}{\sqrt{2}}\sin t\right)\vec{j}.$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}}$$

$$\therefore \vec{N} = (-\cos t)\vec{i} - (\sin t)\vec{j}.$$

$$\vec{N}(0) = -\vec{i}.$$

Now $\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{J} & \vec{k} \\ \frac{-1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$$= \left(\frac{1}{\sqrt{2}}\sin t\right)\vec{i} - \left(\frac{1}{\sqrt{2}}\cos t\right)\vec{j} + \frac{1}{\sqrt{2}}\vec{k}.$$

$$\therefore \vec{B}(0) = -\frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}.$$

2) $\vec{a} = (-\cos t)\vec{i} - (\sin t)\vec{j}$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{J} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (\sin t)\vec{i} - (\cos t)\vec{j} + \vec{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$|\vec{v} \times \vec{a}|^2 = 2.$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{u} \times \vec{a}|^2} = \frac{1}{2} \begin{vmatrix} -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$\therefore \tau = \frac{1}{2} \left[(-\sin t)(0) - \cos(t)(0) + (1)(1) \right] = \frac{1}{2}$$

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3) Find equation for Normal plane, osulating plane & rectifying plane.

when $t=0$, $P_0 = (1, 0, 0) = (x_0, y_0, z_0)$.

fo. the normal plane : $\vec{T} \perp$ Normal plane.

$$\therefore \vec{n} = \vec{T}(0) = \frac{1}{\sqrt{2}} \vec{j} + \frac{1}{\sqrt{2}} \vec{k}$$

$\therefore$ Normal plane : $0x + \frac{1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} z = 0(1) + \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0)$

$\Rightarrow$ Normal plane:

$$\frac{1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} z = 0$$

$$\Leftrightarrow \quad y + z = 0$$

Rectifying plane:

$\vec{n} = \vec{N}(0) = -i \quad , \quad P_0 = (1,0,0)$

$\Rightarrow \quad (-1)x + 0y + 0z = (-1)(1) + (0)(0) + (0)(0)$

$-x = -1 \quad \Rightarrow \quad \boxed{x = 1}$

Osulating plane:

$\vec{n} = \vec{B}(0) = \frac{-1}{\sqrt{2}} \vec{j} + \frac{1}{\sqrt{2}} \vec{k}$.

$\Rightarrow \quad (0)x + \left(\frac{-1}{\sqrt{2}}\right)y + \left(\frac{1}{\sqrt{2}}\right)z = (0)(1) + \left(\frac{-1}{\sqrt{2}}\right)(0) + \left(\frac{1}{\sqrt{2}}\right)(0)$

$\Rightarrow \quad \frac{-1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} z = 0$

$\therefore \quad \boxed{-y + z = 0}$

Important: 13.5 (21):

$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$.

13.1 (17) $\vec{r}(t) = (\ln(t^2+1))\vec{i} + (\tan^{-1}t)\vec{j} + \sqrt{t^2+1}\,\vec{k}$.

Find the angle between $\vec{v}$ & $\vec{a}$ when $t = 0$.

$$\vec{v}(t) = \frac{2t}{t^2+1}\vec{i} + \frac{1}{t^2+1}\vec{j} + \frac{t}{\sqrt{t^2+1}}\vec{k}$$

$$\vec{a}(t) = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\vec{i} - \left[\frac{2t}{(t^2+1)^2}\right]\vec{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\vec{k}$$

$\vec{v}(0) = \vec{j}$ & $\vec{a}(0) = 2\vec{i} + \vec{k}$.

$|v||\vec{a}|\cos\theta = \vec{v}(0)\vec{a}(0)$

$\vec{v}(0) \cdot \vec{a}(0) = 0 \Longrightarrow \cos\theta = 0 \Longrightarrow \boxed{\theta = \frac{\pi}{2}}$

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(20) Find parametric equations for the line that is tangent to the given curve at $t = t_0$.

$\vec{r}(t) = t^2\vec{i} + (2t-1)\vec{j} + t^3\vec{k}$, $t_0 = 2$.

To find the parametric equations, we need

1) a point (x_0, y_0, z_0) on the line

2) vector that is // to the line $[\vec{u} = \langle a, b, c\rangle]$.

• We know that $v(t_0)$ // line. tangent line.

$\Longrightarrow v(t) = 2t\vec{i} + 2\vec{j} + 3t^2\vec{k}$

$\therefore \vec{v}(2) = 4\vec{i} + 2\vec{j} + 12\vec{k} \Longrightarrow \vec{u} = \langle 4, 2, 12\rangle$

& $\vec{r}(2) = 4\vec{i} + 3\vec{j} + 8\vec{k} \Longrightarrow (4, 3, 8)$

$$\Longrightarrow \begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned} \Longrightarrow \begin{aligned} x &= 4 + 4t \\ y &= 3 + 2t \\ z &= 8 + 12t \end{aligned}$$

13.2 (15) Solve the following IVP:

$$\frac{d^2\vec{r}}{dt^2} = -32\,\vec{k} \quad , \quad \vec{r}(0) = 100\,\vec{k} \ \& \ \left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$$

$$\frac{d\vec{r}}{dt} = \int -32\vec{k}\,dt = -32t\,\vec{k} + \vec{C}_1$$

$$8\vec{i} + 8\vec{j} = \vec{C}_1 \Rightarrow \frac{d\vec{r}}{dt} = -32t\,\vec{k} + 8\vec{i} + 8\vec{j}$$

$$\int \frac{d\vec{r}}{dt} = \vec{r}(t) = -16t^2\,\vec{k} + 8t\vec{i} + 8t\vec{j} + \vec{C}_2$$

$$100\,\vec{k} = \vec{C}_2 \Rightarrow \vec{r}(t) = 8t\vec{i} + 8t\vec{j} - 16t^2\vec{k} + 100\,\vec{k}$$

(18) A particle traveling in a straight line is located at the point $(1,-1,2)$ and has speed 2 at time $t=0$. The particle moves toward the point $(3,0,3)$ with constant acceleration $2\vec{i} + \vec{j} + \vec{k}$. Find its position vector $\vec{r}(t)$.

$$\vec{a} = 2\vec{i} + \vec{j} + \vec{k} \Rightarrow v(t) = 2t\vec{i} + t\vec{j} + t\vec{k} + \vec{C}_1$$

The particle travels in the direction:

$$(3-1)\vec{i} + (0-(-1))\vec{j} + (3-2)\vec{k} = 2\vec{i} + \vec{j} + \vec{k}$$

(since its a straight line)

$$\text{Direction} = \frac{v(t)}{|\vec{v}(t)|} \Rightarrow \text{Note Direction is a Unit vector}$$

Then at $t = 0$, we have:

$$\vec{v}(0) = \text{speed} \times \text{Direction}$$

$$= 2 \times \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|}$$

$$= 2 \times \frac{(2\vec{i} + \vec{j} + \vec{k})}{\sqrt{6}}$$

but $\vec{v}(t) = 2t\vec{i} + t\vec{j} + t\vec{k} + \vec{C}$.

$$\Rightarrow \vec{v}(0) = \frac{2}{\sqrt{6}}(2\vec{i} + \vec{j} + \vec{k}) = \vec{C}$$

There for:

$$\vec{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\vec{i} + \left(t + \frac{2}{\sqrt{6}}\right)\vec{j} + \left(t + \frac{2}{\sqrt{6}}\right)\vec{k}.$$

$$\Rightarrow \vec{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\vec{i} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t\right)\vec{j} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t\right)\vec{k} + \vec{C_2}$$

but $\vec{r}(0) = \vec{i} - \vec{j} + 2\vec{k}$.

$$\Rightarrow \vec{r}(0) = \vec{i} - \vec{j} + 2\vec{k} = C_2$$

$$\therefore \vec{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\vec{i} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t\right)\vec{j} + \left(\frac{t^2}{2} + \frac{2}{\sqrt{6}}t\right)\vec{k}$$

$$+ \vec{i} - \vec{j} + 2\vec{k}.$$

13.3 (16) Find the point on the curve

$$\vec{r}(t) = (12\sin t)\vec{i} - (12\cos t)\vec{j} + 5t\vec{k}.$$

at a distance 13π units along the curve from the point $(0,-12,0)$ in the direction opposite to the direction of increasing arc length.

Let $P(t_0)$ denote the point. Then:

$$\vec{v}(t) = (12\cos t)\vec{i} + (12\sin t)\vec{j} + 5\vec{k}.$$

It start from the point $(0,-12,0)$ in $\vec{r}(t)$ $\Rightarrow t = 0$, then:

$$-13\pi = \int_0^{t_0} \sqrt{(12\cos t)^2 + (12\sin t)^2 + 5^2}\, dt$$

opposite

$$= \int_0^{t_0} \sqrt{169}\, dt = \int_0^{t_0} 13\, dt = 13 \Big|_0^{t_0} = 13t_0$$

$$\Rightarrow \boxed{t_0 = -\pi}$$

even or odd ??

Then: $P(-\pi) = (12\sin(-\pi))\vec{i} - 12\cos(-\pi)\vec{j} + 5(-\pi)\vec{k}$

$$= -12\vec{j} - 5\pi\vec{k}.$$

135 (26) The torsion of a helix.

Show that the torsion of the helix

$$\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + bt\vec{k}, \quad a, b \geq 0$$

is $\tau = \dfrac{b}{a^2+b^2}$.

What is the largest value of τ can have for given value of a?

$$\vec{v}(t) = (-a\sin t)\vec{i} + (a\cos t)\vec{j} + b\vec{k}.$$

$$\vec{a}(t) = (-a\cos t)\vec{i} - (a\sin t)\vec{j}.$$

$$\tau = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2+b^2}\right)^2} = \frac{b\left(a^2\cos^2 t + a^2\sin^2 t\right)}{a^2(a^2+b^2)}.$$

$$= \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}$$