7.2 Natural Logarithms

* $\frac{d}{\partial x}(\ln u) = \frac{du/\partial x}{u}$

* Properties of the natural Logarithm:

1) $\ln bx = \ln b + \ln x$

2) $\ln \frac{b}{x} = \ln b - \ln x$

3) $\ln \frac{1}{x} = -\ln x$

4) $\ln x^r = r \ln x$

* $\int \frac{du}{u} = \ln |u| + c$

2 Express the following in terms of ln 5 and ln 7

a) $\ln\left(\frac{1}{125}\right) = \ln \frac{1}{5^3} = \ln 5^{-3} = -3 \ln 5$

b) $\ln 9.8 = \ln \frac{98}{10} = \ln \frac{49}{5} = \ln 49 - \ln 5$

$= \ln 7^2 - \ln 5 = 2 \ln 7 - \ln 5$

c) $\ln 7\sqrt{7} = \ln 7(7^{\frac{1}{2}}) = \ln 7^{3/2} = \frac{3}{2} \ln 7$

d) $\ln 1225 = \ln 35^2 = 2 \ln 35 = 2 \ln (7 * 5)$

$= 2 [\ln 7 + \ln 5]$

e) $\ln 0.056 = \ln \frac{56}{1000} = \ln \frac{7}{125} = \ln 7 - \ln 125$

$= \ln 7 - \ln 5^3 = \ln 7 - 3 \ln 5$

f) $\frac{\ln 35 + \ln\left(\frac{1}{7}\right)}{\ln (25)} = \frac{\ln 5 + \ln 7 + \ln 7^{-1}}{\ln 5^2}$

$= \frac{\ln 5 + \cancel{\ln 7} - \cancel{\ln 7}}{2 \ln 5} = \frac{\ln 5}{2 \ln 5} = \frac{1}{2}$

[4] Use the properties of logarithms to simplify the expressions.

a) $\ln \sec\theta + \ln \cos\theta = \ln \frac{1}{\cos\theta} + \ln \cos\theta$

$= \ln \cos^{-1}\theta + \ln \cos\theta = -\ln \cos\theta + \ln \cos\theta = 0$

b) $\ln(8x+4) - 2\ln 2$

$\ln 4(2x+1) - 2\ln 2 = \cancel{\ln 4} + \ln(2x+1) - \cancel{\ln 4}$

$= \ln(2x+1)$

c) $3\ln \sqrt[3]{t^2-1} - \ln(t+1)$

$= \ln\left(\sqrt[3]{t^2-1}\right)^3 - \ln(t+1) = \ln(t^2-1) - \ln(t+1)$

$= \ln(t-1)(t+1) - \ln(t+1)$

$= \ln(t-1) + \cancel{\ln(t+1)} - \cancel{\ln(t+1)}$

$= \ln(t-1)$

Question 10. page 376

Find the derivative of y with respect to x. $y = \ln\left(\frac{10}{x}\right)$

$$y' = \frac{\left(\frac{10}{x}\right)'}{\left(\frac{10}{x}\right)} = \frac{\frac{-10}{x^2}}{\frac{10}{x}} = -\frac{\cancel{10}}{x^2} \cdot \frac{x}{\cancel{10}} = -\frac{1}{x}$$

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Find the derivative of y with respect to x. $y = (\ln x)^3$

$$y' = 3(\ln x)^2 (\ln x)'$$

$$y' = 3(\ln x)^2 \cdot \frac{1}{x} = \frac{3(\ln x)^2}{x}$$

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Find the derivative of y with respect to t. $y = t\sqrt{\ln t}$

$$y' = t \cdot \left(\sqrt{\ln t}\right)' + \sqrt{\ln t} \cdot 1$$

$$= t \cdot \frac{1}{2}(\ln t)^{-\frac{1}{2}} \cdot 1 + \sqrt{\ln t}$$

$$y' = \frac{t}{2\sqrt{\ln t}} + \sqrt{\ln t}$$

Question 22. Find the derivative of y with respect
page 376 to x. $y = \dfrac{x \ln x}{1 + \ln x}$

$$y' = \frac{(1+\ln x) \cdot (x \ln x)' - x \ln x (1+\ln x)'}{(1+\ln x)^2}$$

$$y' = \frac{(1+\ln x)\left(x \cdot \frac{1}{x} + \ln x \cdot 1\right) - x \ln x \left(0 + \frac{1}{x}\right)}{(1+\ln x)^2}$$

$$y' = \frac{(1+\ln x)(1+\ln x) - x \ln x \cdot \frac{1}{x}}{(1+\ln x)^2}$$

$$y' = \frac{1 + 2\ln x + (\ln x)^2 - \ln x}{(1+\ln x)^2}$$

$$y' = \frac{(\ln x)^2 + \ln x + 1}{(1+\ln x)^2}$$

$\boxed{21}$ find the derivative of y

$$y = \frac{\ln x}{1+\ln x}$$

$$y' = \frac{(1+\ln x)\cdot\frac{1}{x} - \ln x\left(\frac{1}{x}\right)}{(1+\ln x)^2}$$

$$= \frac{\frac{1 + \cancel{\ln x} - \cancel{\ln x}}{x}}{(1+\ln x)^2} = \frac{\frac{1}{x}}{(1+\ln x)^2}$$

$$= \frac{1}{x(1+\ln x)^2}$$

$\boxed{24}$ find the derivative of $y = \ln(\ln(\ln x))$

$$y' = \frac{[\ln(\ln x)]'}{\ln(\ln x)} = \frac{1}{\ln(\ln x)} \cdot \frac{(\ln x)'}{\ln x}$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x\ln x(\ln(\ln x))}$$

Question 31. Find the derivative of y with respect to θ. $y = \ln(\sec(\ln\theta))$

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$$y' = \frac{(\sec(\ln\theta))'}{\sec(\ln\theta)} = \frac{\sec(\ln\theta)\tan(\ln\theta)(\ln\theta)'}{\sec(\ln\theta)}$$

$$y' = \frac{\sec(\ln\theta)\tan(\ln\theta)\cdot\frac{1}{\theta}}{\sec(\ln\theta)}$$

$$y' = \frac{\sec(\ln\theta)\tan(\ln\theta)}{\theta\sec(\ln\theta)}$$

Question 44. Evaluate the integral $\int_2^4 \frac{dx}{x\ln x}$

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$$\int_2^4 \frac{dx}{x\ln x} = \int \frac{1}{u}\cdot du$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$= \ln|u| = \ln|\ln x|$$

$$\int_2^4 \frac{dx}{x\ln x} = \ln|\ln x|\Big|_2^4$$

$$= \ln|\ln 4| - \ln|\ln 2| = \ln\left|\frac{\ln 4}{\ln 2}\right|$$

$$= \ln\left|\frac{2\ln 2}{\ln 2}\right| = \ln|2| = \ln 2$$

$\boxed{34}$ Find the derivative of $y = \ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

$$y = \ln \frac{(x+1)^{5/2}}{(x+2)^{10}} = \ln (x+1)^{5/2} - \ln (x+2)^{10}$$

$$y = \frac{5}{2} \ln (x+1) - 10 \ln (x+2)$$

$$y' = \frac{5}{2} \cdot \frac{1}{x+1} - 10 \cdot \frac{1}{x+2} = \frac{5}{2(x+1)} - \frac{10}{x+2}$$

$$y' = \frac{5(x+2) - 20(x+1)}{2(x+1)(x+2)} = \frac{-15x - 10}{2(x+1)(x+2)}$$

$\boxed{54}$ Evaluate the integral $\displaystyle\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}}\, dx$

let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx = \sec x(\tan x + \sec x)\, dx$

$$\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}}\, dx = \int \frac{1}{\sqrt{\ln u}} \cdot \frac{du}{u}$$

let $z = \ln u \rightarrow dz = \dfrac{du}{u}$

$$\rightarrow \int \frac{dz}{\sqrt{z}} = \int z^{-\frac{1}{2}}\, dz = \frac{z^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{z} + c$$

$$= 2\sqrt{\ln u} + c$$

$$= 2\sqrt{\ln(\sec x + \tan x)} + c$$

$\boxed{65}$ Use logarithmic differentiation to find y'

$$y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$$

$$\ln y = \ln \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} = \ln x + \ln\sqrt{x^2+1} - \ln(x+1)^{2/3}$$

$$\ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{x} + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}x}{x^2+1}\right) - \frac{2}{3}\cdot\frac{1}{x+1}$$

$$y' = y\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3x+3}\right)$$

$$y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3x+3}\right)$$

Question 64.
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Use logarithmic differentiation to find the derivative of y with respect to the independent variable θ.

$$y = \frac{\theta \sin\theta}{\sqrt{\sec\theta}}$$

$$\ln y = \ln \frac{\theta \sin\theta}{\sqrt{\sec\theta}} = \ln(\theta \sin\theta) - \ln\sqrt{\sec\theta}$$

$$\ln y = \ln\theta + \ln\sin\theta - \frac{1}{2}\ln\sec\theta$$

$$\frac{y'}{y} = \frac{(\theta)'}{\theta} + \frac{(\sin\theta)'}{\sin\theta} - \frac{1}{2}\cdot\frac{(\sec\theta)'}{\sec\theta}$$

$$\frac{y'}{y} = \frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{\sec\theta\tan\theta}{2\sec\theta}$$

$$y' = y\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]$$

$$y' = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left[\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right]$$

Question 66. Use logarithmic differentiation
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to find the derivative of y with respect of y with respect to x.

$$y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$$

$$\ln y = \ln \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} = \ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\ln \frac{(x+1)^{10}}{(2x+\$)^5}\right] =$$

$$\ln y = \frac{1}{2}\left[\ln (x+1)^{10} - \ln(2x+1)^5\right]$$

$$\ln y = \frac{1}{2}\left[10 \ln(x+1) - 5\ln(2x+1)\right]$$

$$\frac{y'}{y} = \frac{1}{2}\left[10\frac{(x+1)'}{(x+1)} - 5\frac{(2x+1)'}{(2x+1)}\right]$$

$$\frac{y'}{y} = \frac{1}{2}\left[10\left(\frac{1}{x+1}\right) - 5\left(\frac{2}{2x+1}\right)\right]$$

$$\frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1} \rightarrow y' = y\left[\frac{5}{x+1} - \frac{5}{2x+1}\right]$$

$$y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left[\frac{5}{x+1} - \frac{5}{2x+1}\right]$$

Question 70.

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a) Prove that $f(x) = x - \ln x$ is increasing for $x > 1$

$f'(x) = 1 - \frac{1}{x}$

if $x > 1 \rightarrow 1 > \frac{1}{x} \rightarrow -1 < -\frac{1}{x}$

$0 < 1 - \frac{1}{x}$

$0 < f'(x)$ so f is increasing.

b) Using part a, show that $\ln x < x$ if $x >$

Since $f(x)$ is increasing for $x > 1$

$f(x) > f(1)$ for $x > 1$

$f(1) = 1 - \frac{1}{1} = 1 - 1 = 0$

so $f(x) > 0$

$x - \ln x > 0$

$x > \ln x$ for $x > 1$.

[71] Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$

$$A = \int_1^5 \ln 2x - \ln x \, dx$$

$$= \int_1^5 \ln 2 + \cancel{\ln x} - \cancel{\ln x} \, dx$$

$$= \int_1^5 \ln 2 \, dx$$

$$= \ln 2 \cdot x \Big|_1^5 = \ln 2 \, [5 - 1] = 4 \ln 2$$

$$= \ln 2^4 = \ln 16 .$$