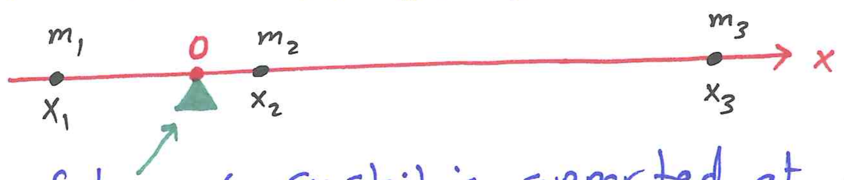
Moments and Centers of Mass

We will study the cases one, two and three-dimensional objects.

* One-dimension : Masses along a line

- Consider the masses m_1, m_2, m_3 arranged along x-axis



- Assume a fulcrum (نقطة ارتكاز) is supported at origin.
- We define the k^{th} torque (عزم الدوران) by multiplying the k^{th} force $m_k g$ due to gravity by the signed distance x_k:
- The sum of the torques is called the system torque

$$\text{system torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3$$

which measures the tendency (ميل) of a system to rotate about the origin.

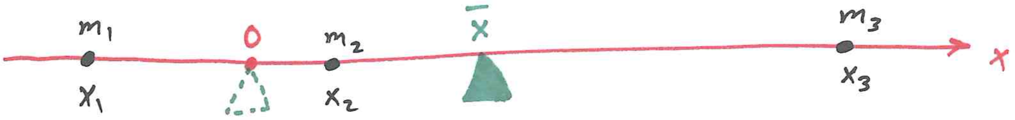
- The system is balance iff its torque is zero.
- Note that system torque $= g \, (m_1 x_1 + m_2 x_2 + m_3 x_3)$

$$= g \, M_0$$

- M_0 is the sum of the moments $m_1 x_1, m_2 x_2, m_3 x_3$ and called Moment of system about origin.

Question : Where to locate the fulcrum to make the system balance ?

- let $\bar{x}$ be the point that makes the system torque $= 0$



- The k^{th} torque about the fulcrum at $\bar{x}$ is $(x_k - \bar{x}) m_k g$
- The system torque is zero at $\bar{x}$ $\Longrightarrow$

$$\sum_{k=1}^{3} (x_k - \bar{x}) m_k g = 0 \iff \bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{M_0}{M}$$

- $\bar{x}$ is called the center of mass.

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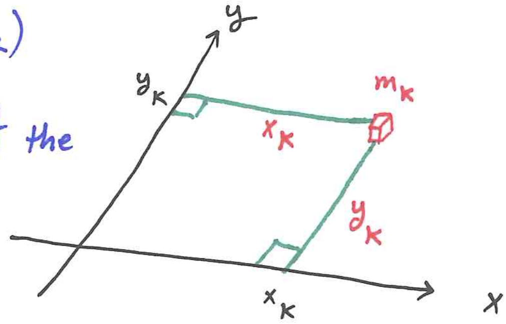
* Masses distributed over a plane (Two-dimensional object)

- Suppose there is a finite collection of masses located in the plane with mass m_k at point (x_k, y_k)



- Each mass m_k has a moment about the

  x-axis $= m_k y_k$ and

  y-axis $= m_k x_k$

- Hence, the moment about x-axis is $M_x = \sum m_k y_k$

  and the moment about y-axis is $M_y = \sum m_k x_k$

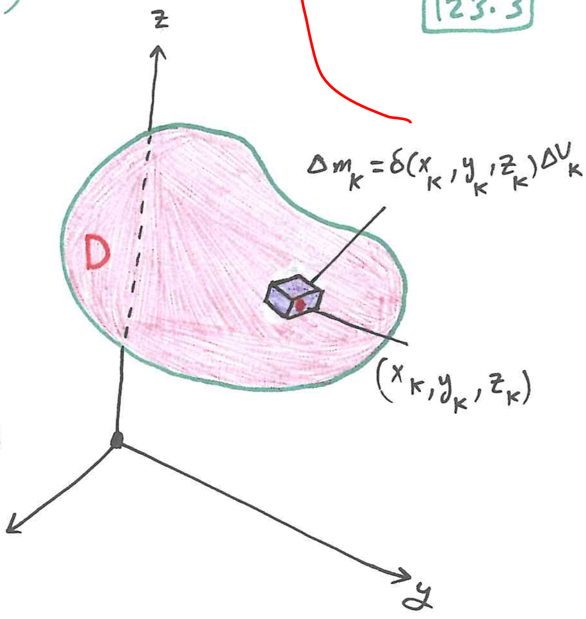
- As above, the system mass is $M = \sum m_k$
- The center of the mass is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}$$

$\delta = \frac{m}{V}$

123.3

* Three - dimensional object (solid)



- Let D be an object in space.
- Let $\delta(x,y,z)$ be the density mass per unit volume at (x,y,z).
- If we partion the object into n mass elements, then the mass of this object is the integral of δ over D :

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\, \Delta V_k = \iiint_D \delta(x,y,z)\, dV$$

- The first moment about the yz- plane is

$$M_{yz} = \iiint_D x\, \delta(x,y,z)\, dV$$

The first moment about the xz- plane is $M_{xz} = \iiint_D y\,\delta\, dV$

The first moment about the xy- plane is $M_{xy} = \iiint_D z\,\delta\, dV$

- The center of the mass is the point $(\bar{x}, \bar{y}, \bar{z})$ where

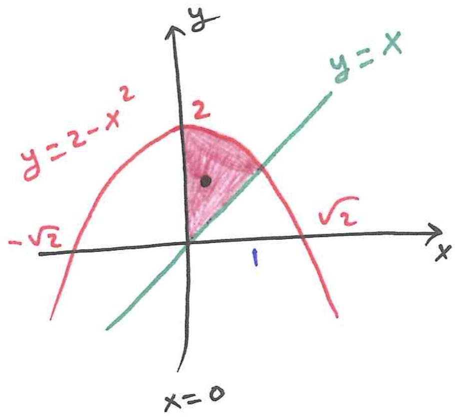
$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

For two- dimensional plate

- Mass: $M = \iint_R \delta\, dA$ where $\delta = \delta(x,y)$ is the density at (x,y).
- First Moments : $M_y = \iint_R x\,\delta\, dA$ and $M_x = \iint_R y\,\delta\, dA$
- Center of mass is $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{M_y}{M}$ and $\bar{y} = \frac{M_x}{M}$

(لوحة) طبق

Exp Find the center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$ and the parabola $y = 2 - x^2$ in the first quadrant. $\boxed{123.4}$



- $2 - x^2 = x \iff x^2 + x - 2 = 0$
  $\iff (x-1)(x+2) = 0$

- Mass $= M = \displaystyle\int_0^1 \int_x^{2-x^2} 3 \, dy \, dx = \frac{7}{2}$

- First Moments $M_y = \displaystyle\int_0^1 \int_x^{2-x^2} 3x \, dy \, dx = \frac{5}{4}$

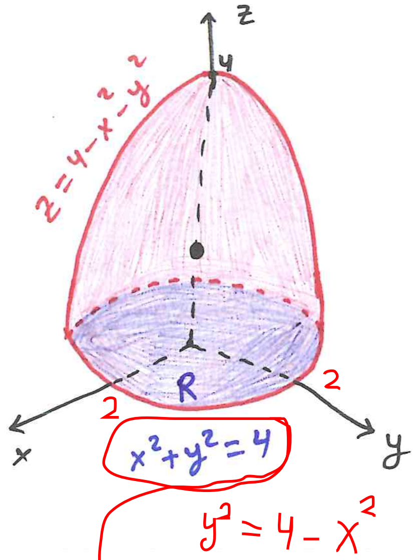
$$M_x = \int_0^1 \int_x^{2-x^2} 3y \, dy \, dx = \frac{19}{5}$$

- Center is $(\bar{x}, \bar{y}) = \left(\dfrac{M_y}{M}, \dfrac{M_x}{M} \right) = \left(\dfrac{5}{14}, \dfrac{38}{35} \right)$

Remark When the density of a solid object or plate is constant, we set $\delta = 1$ and in this case the center of mass is called centroid.

123.5

<u>Exp</u> Find the centroid of a solid bounded below by the disk $R: x^2+y^2 \le 4$ in the plane $z=0$ and above by the paraboloid $z = 4-x^2-y^2$

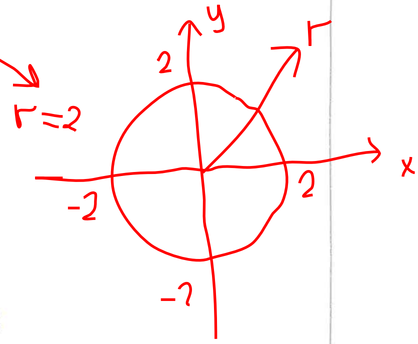


- By symmetry $\bar{x} = \bar{y} = 0$
- To find $\bar{z}$ $\Rightarrow$ The mass is

$$M = \iint_R \int_0^{4-x^2-y^2} dz\, dy\, dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} dz\, dy\, dx$$

$$= \iint_R (4-x^2-y^2)\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{2} (4-r^2)\, r\, dr\, d\theta = 8\pi$$



- First Moment $M_{xy} = \iint_R \int_0^{4-x^2-y^2} z\, dz\, dy\, dx$

$$= \frac{1}{2}\iint_R (4-x^2-y^2)^2\, dy\, dx$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{2} (4-r^2)^2\, r\, dr\, d\theta = \frac{32\pi}{3}$$

- $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{4}{3}$
- Hence, the centroid is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{4}{3})$

* Moments of Inertia (Second moment)

- Recall that first moment tell us about the balance and torque of object about different axes.
- If the object is a rotating shaft (محور دوران), then how much energy is stored in the shaft? That is where the moment of inertia comes in.
- what makes the shaft hard to start or stop is its moment of inertia. Hence, the moment of inertia depends not only on the mass of the shaft but also on its distribution. That is, mass farther away from the axis of rotation contributes more to the moment of inertia.

*** For three-dimensional solid, the moments of inertia formulas are:**

About x-axis : $I_x = \iiint (y^2+z^2)\,\delta\, dV$, $\delta = \delta(x,y,z)$

About y-axis : $I_y = \iiint (x^2+z^2)\,\delta\, dV$

About z-axis : $I_z = \iiint (x^2+y^2)\,\delta\, dV$

About line L : $I_L = \iiint r^2\,\delta\, dV$, $r(x,y,z)$ is distance from point (x,y,z) to line L

*** For two-dimensional plate, the moments of inertia formulas are:**

About x-axis : $I_x = \iint y^2\,\delta\, dA$, $\delta = \delta(x,y)$

About y-axis : $I_y = \iint x^2\,\delta\, dA$

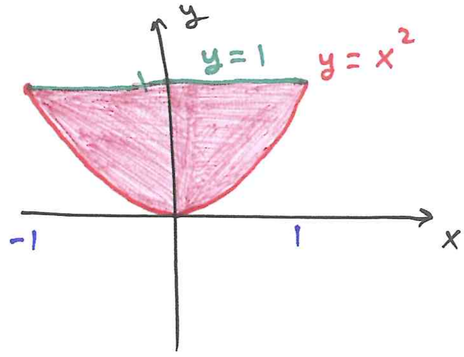
About line L : $I_L = \iint r^2\,\delta\, dA$, $r = r(x,y)$ is the distance from (x,y) to L

About the origin: (Polar moment)

$$I_0 = \iint (x^2+y^2)\,\delta\, dA = I_x + I_y$$

$\boxed{123.7}$

Exp Find the center of mass and the moment of inertia about the y-axis of a thin plate bounded by the line $y=1$ and the parabola $y=x^2$ if the density is $\delta(x,y)=y+1$.



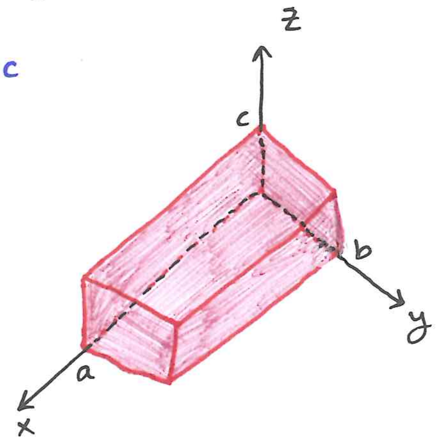
- $M=\int_{-1}^{1}\int_{x^2}^{1}(y+1)\,dy\,dx = \frac{32}{15}$

$M_x=\int_{-1}^{1}\int_{x^2}^{1} y(y+1)\,dy\,dx = \frac{48}{35}$

$M_y=\int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx = 0 \Rightarrow \bar{x}=0$ and $\bar{y}=\frac{M_x}{M}=\frac{9}{14}$

- $I_y=\int_{-1}^{1}\int_{x^2}^{1} x^2(y+1)\,dy\,dx = \frac{16}{35}$

Exp Find the moments of inertia of this rectangular solid w.r.t its edges.



- Note that $M=\int_0^a\int_0^b\int_0^c dz\,dy\,dx = abc$

- $I_x=\int_0^a\int_0^b\int_0^c (y^2+z^2)\,dz\,dy\,dx$

$=\int_0^a\int_0^b \left(cy^2+\frac{c^3}{3}\right)dy\,dx$

$=\int_0^a \left(\frac{cb^3}{3}+\frac{c^3b}{3}\right)dx = \frac{abc(b^2+c^2)}{3} = \frac{M}{3}(b^2+c^2)$

- $I_y=\frac{M}{3}(a^2+c^2)$ and $I_z=\frac{M}{3}(a^2+b^2)$ by symmetry.

123.21

Exp Find the fulcrum to make the following system balance :

The fulcrum $\bar{x}$ is the center of mass where the system torque $= 0$

$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{(-1)(2) + (2)(4) + (7)(3)}{2 + 4 + 3}$$

$$= \frac{27}{9} = 3$$

123.51

To see $M_{xz} = 0$ and $M_{yz} = 0 \Rightarrow$

$$M_{xz} = \iint_R \int_0^{4-x^2-y^2} y \, dz \, dy \, dx$$

$$= \iint_R y(4-x^2-y^2) \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^2 r \sin\theta (4-r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \sin\theta (4r^2 - r^4) \, dr \, d\theta$$

$$= \int_0^{2\pi} \sin\theta \left(\frac{4r^3}{3} - \frac{r^5}{5} \Big|_0^2 \right) d\theta$$

$$= \int_0^{2\pi} \sin\theta \, \underbrace{\left(\frac{32}{3} - \frac{32}{5} \right)}_{K} d\theta = K \cos\theta \Big|_0^{2\pi}$$

$$= K(1-1)$$

$$= 0$$

