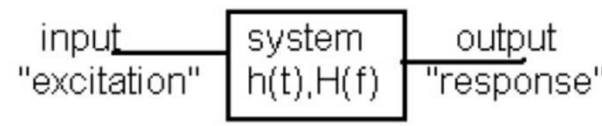
# Signal and Sytems

Part

# Transmission of Signals through Linear Systems

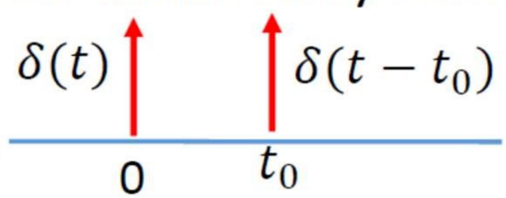
- ***Definition:*** A **system** refers to any physical device that produces an output signal in response to an input signal.
- ***Definition:*** A system is **linear** if the principle of superposition applies.
- If $x_1(t)$ produces output $y_1(t)$
- $x_2(t)$ produces output $y_2(t)$
- then $a_1x_1(t)+ a_2x_2(t)$ produces an output $a_1y_1(t)+ a_2y_2(t)$
- Also, a zero input should produce a zero output.

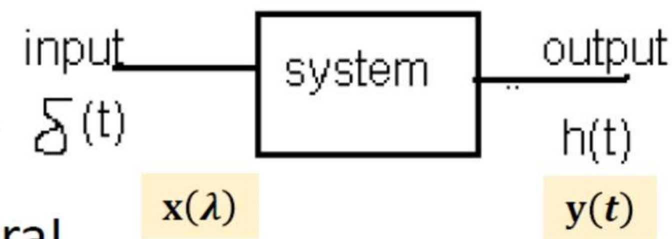



- Examples of linear systems include **filters and communication channels**.
- ***Definition:*** A **filter** refers to a frequency selective device that is used to limit the spectrum of a signal to some band of frequencies (will be discussed in detail in a later lecture)
- ***Definition:*** A **channel** refers to a transmission medium that connects the transmitter and receiver of a communication system.
- Time domain and frequency domain may be used to evaluate system performance.

# Basic Time-domain Definitions

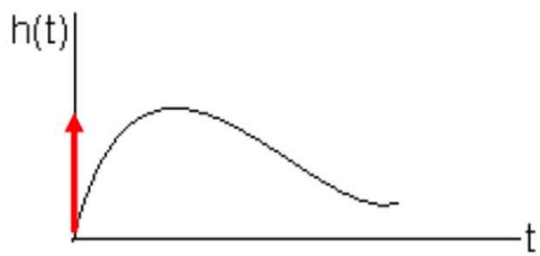
- ***Definition***: The **impulse response h(t)** is defined as the response of a system to an impulse $\delta(t)$ applied to the input at t=0 .
- ***Definition***: A system is **time-invariant** when the shape of the impulse response is the same no matter when the impulse is applied to the system.
- $\delta(t) \rightarrow h(t)$, then $\delta(t-t_0) \rightarrow h(t-t_0)$
- When the input to a linear time-invariant system in a signal x(t) , then the output is given by

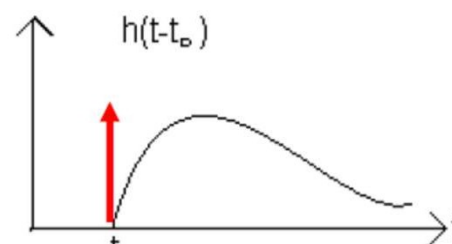



- $\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{x}(\lambda)\mathbf{h}(t-\lambda)\mathbf{d}\lambda$

  $= \int_{-\infty}^{\infty} \mathbf{h}(\lambda)\mathbf{x}(t-\lambda)d\lambda$; convolution integral



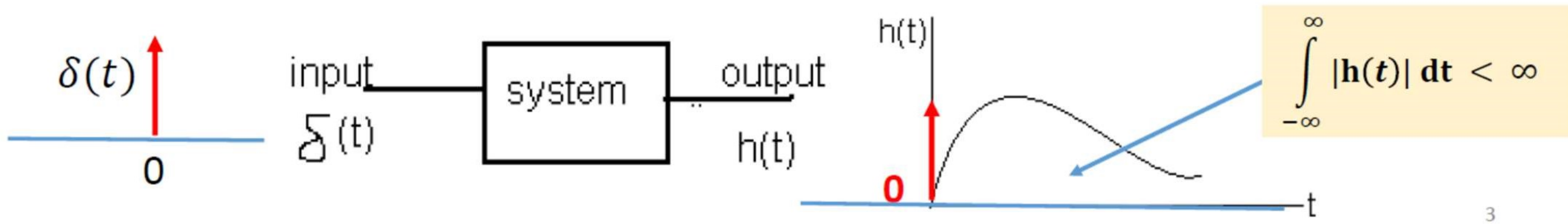



# Basic Time-domain Definitions

- ***Definition***: A system is said to be **causal** if it does not respond before the excitation is applied, i.e.,
- h(t) = 0 for t < 0 ; the causal system is physically realizable.
- ***Definition***: A system is said to be **stable** if the output signal is bounded for all bounded input signals.

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{h}(\tau)\mathbf{x}(t-\tau)\ \mathbf{d}\tau$$

- If $|x(t)| \leq M$ ; M is the maximum value of the input
- then $|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau = M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$
- Therefore, a necessary and sufficient condition for stability (a bounded output) is
- $\int_{-\infty}^{\infty} |\mathbf{h}(t)|\, \mathbf{dt} < \infty$ ; h(t) is absolutely integrable ( zero initial conditions assumed)



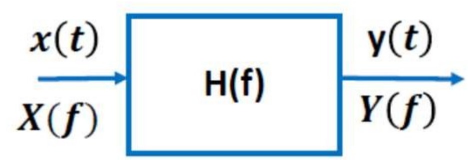
## Basic Frequency-domain Definitions

- ***Definition***: The **transfer function** of a linear time invariant system is defined as the Fourier transform of the impulse response h(t)

$$\boldsymbol{H(f) = \Im\{h(t)\}}$$

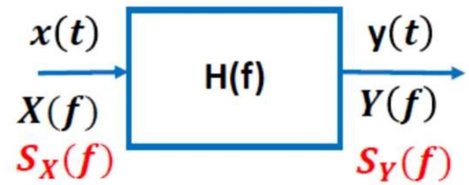
- Since $\boldsymbol{y(t) = x(t) * h(t)}$, then $\boldsymbol{Y(f) = H(f)X(f)}$.
- The system transfer function is thus the ratio of the Fourier transform of the output to that of the input $\boldsymbol{H(f) = \frac{Y(f)}{X(f)}}$
- The transfer function $H(f)$ is a complex function of frequency, which can be expressed as
- $H(f) = |H(f)|e^{j\,\theta(f)}$
- where, $|H(f)|$: Amplitude spectrum

  $\theta(f)$: Phase spectrum.



## System input–output energy spectral density

- Let $x(t)$ be applied to a LTI system, then the Fourier transform of the output is related to the Fourier transform of the input through the relation
- $Y(f) = H(f)X(f)$.
- Taking the absolute value and squaring both sides, we get
- $|Y(f)|^2 = |H(f)|^2|X(f)|^2$




$$\boldsymbol{S_Y(f) = |H(f)|^2 S_X(f)}$$

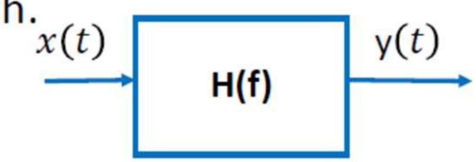
- $S_X(f)$, $S_Y(f)$: Input and output Energy Spectral Density

$output\ energy\ spectral\ densit = |H(f|^2\ (input\ energy\ spectral\ density)$

- Total input and output energies
- $E_x = \int_{-\infty}^{+\infty} S_x(f)df = \int_{-\infty}^{+\infty} |X(f)|^2 df$; **Recall Rayleigh Energy Theorem**
- $E_y = \int_{-\infty}^{+\infty} S_Y(f)df = \int_{-\infty}^{+\infty} |\mathrm{H(f)}|^2 S_X(f)df$

# Example: Response of a LPF filter to a sinusoidal input

- **Example**: The signal $x(t) = \cos(2\pi f_0 t)\,, -\infty < t < \infty$, is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$, $B$ is the 3-dB bandwidth. Find the filter output $y(t)$.
- **Solution**: Here, we will find the output using the frequency domain approach.




- $Y(f) = H(f)X(f),\;\; H(f) = \frac{1}{\sqrt{1+(\frac{f}{B})^2}} e^{-j\theta};\; \theta = \tan^{-1}\frac{f}{B};\; \theta_0 = \tan^{-1}\frac{f_0}{B}$
- $Y(f) = H(f)[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0),\; \Rightarrow Y(f) = \frac{1}{2}H(f_0)\delta(f-f_0) + \frac{1}{2}H(-f_0)\delta(f+f_0)$
- $Y(f) = \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{-j\theta_0}\delta(f-f_0) + \frac{1}{2}\frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} e^{j\theta_0}\delta(f+f_0)$

$\boldsymbol{g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0);}$

- Taking the inverse Fourier transform, we get
- $y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}}\frac{1}{2}[e^{j(2\pi f_0 t-\theta_0)} + e^{-j(2\pi f_0 t-\theta_0)}]$, $\quad \boldsymbol{y(t) = \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} cos(2\pi f_0 t - tan^{-1}\frac{f_0}{B})}$
- Note that in the last step we have made use of the Fourier transform pair $e^{j2\pi f_0 t} \leftrightarrow \delta(f-f_0)$
- **Remark**: Note that the amplitude of the output as well as its phase depend on the frequency of the input, $f_0$, and the bandwidth of the filter, $B$.

## Response of a LPF to a sum of two sinusoidal signals

- **Example**: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.
- **Solution**: From the previous example, we have
- $\cos(2\pi f_0 t) \ \rightarrow \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \cos(2\pi f_0 t - \tan^{-1}\frac{f_0}{B})$
- Therefore, using linearity property
- $cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t \rightarrow$
- $\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}} \cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right) \quad - \frac{1}{\pi}\frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}} \cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right)$

## Example: Response of a LPF to a periodic square pulse

- **Example**: Consider the periodic rectangular signal $g(t)$ defined over one period $T_0$ as

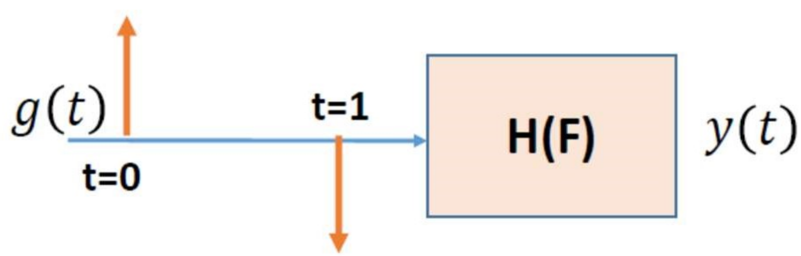
$$g(t) = \begin{cases} +A, & -T_0/4 \le t \le T_0/4 \\ 0, & otherwise \end{cases}.$$

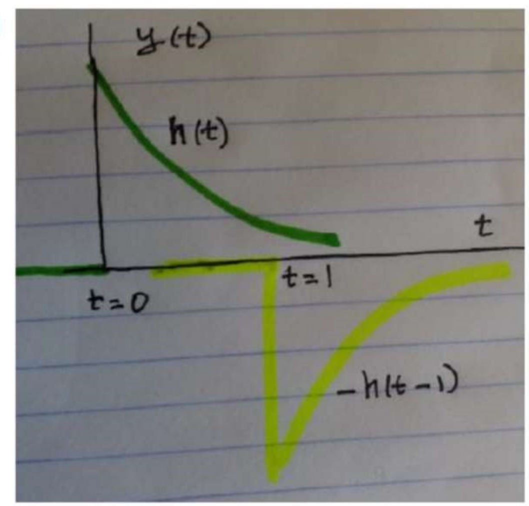
- If $g(t)$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. use the result of the previous example to find the filter output $y(t)$.
- **Solution**: The Fourier series of g(t) is:
- $g(t) = \frac{A}{2} + \frac{2A}{\pi}\{\cos(2\pi f_0 t) - \frac{1}{3}\cos(2\pi 3 f_0 t) + \frac{1}{5}\cos(2\pi 5 f_0 t) - \frac{1}{7}\cos(2\pi 7 f_0 t)$
- Using the result of the previous example:
- $y(t) = \frac{A}{2} + \frac{2A}{\pi}\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}}\cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right)$

$$-\frac{2A}{\pi}\frac{1}{3}\frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}}\cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right) \quad +\ldots$$

## Transmission of Signals through Linear Systems: A Convolution Example

- **Example:** The signal $g(t) = \delta(t) - \delta(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the convolution integral to find the channel output.
- **Solution:** The impulse response of the channel is obtained by taking the inverse Fourier transform of $H(f)$, which is $h(t) = 2\pi B e^{-2\pi Bt} u(t)$
- Using the linearity and time invariance property, the output can be obtained as
- $y(t) = h(t) * [\delta(t) - \delta(t-1)]; \quad y(t) = h(t) - h(t-1)$
- $y(t) = 2\pi B[e^{-2\pi Bt} u(t) - e^{-2\pi B(t-1)} u(t-1)]$



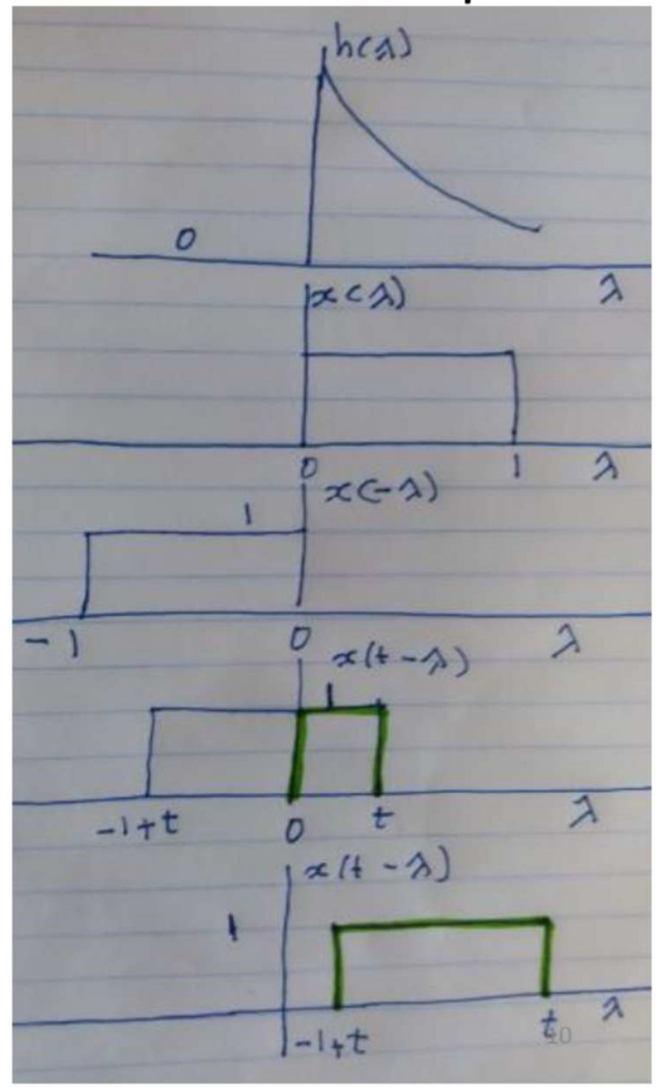



## Transmission of Signals through Linear Systems: A Convolution Example

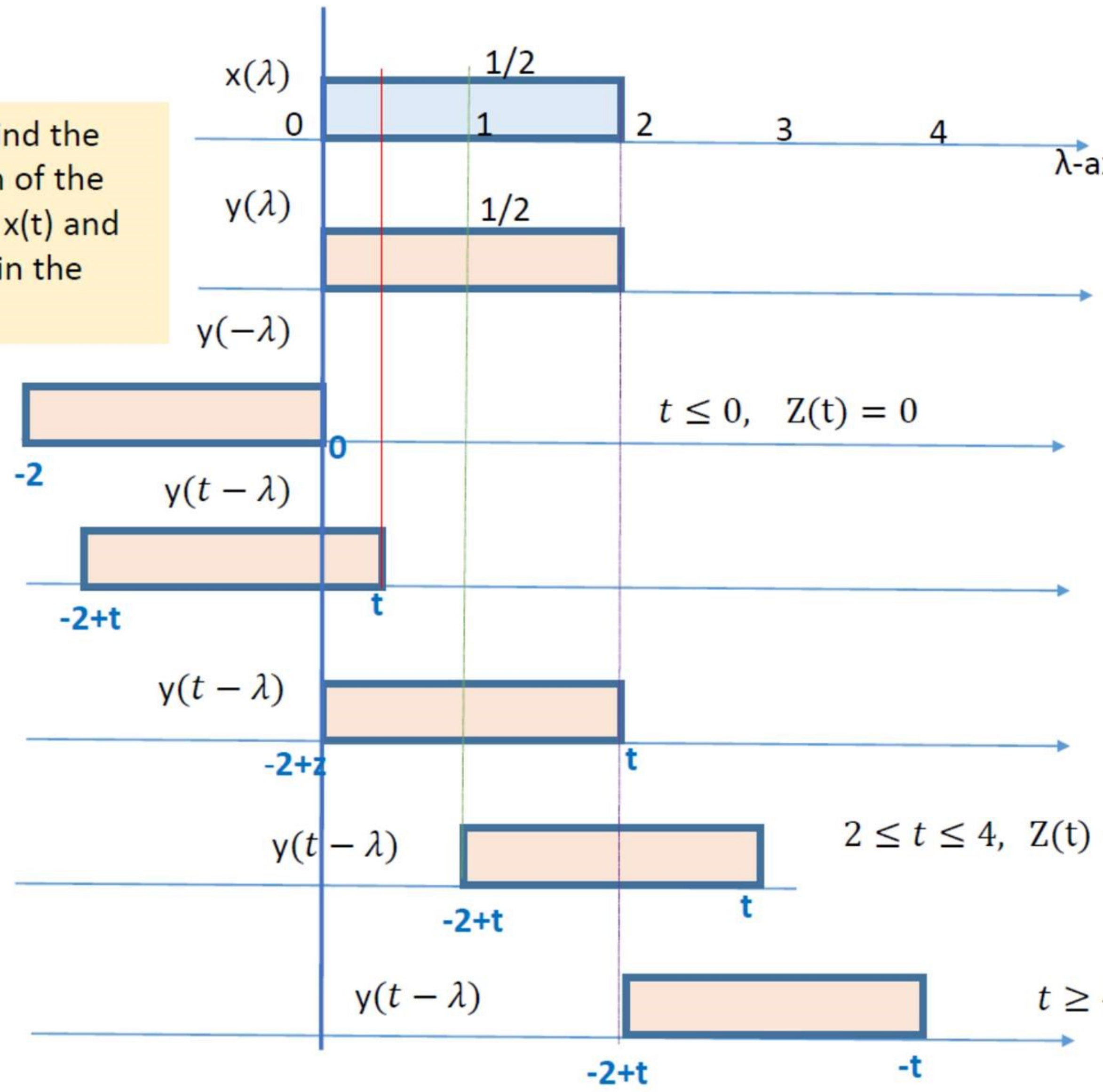
- **Example: channel response due to a rectangular pulse**
- The signal $x(t) = u(t) - u(t-1)$ is applied to a channel described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Find the channel output $y(t)$.
- **Solution:** The impulse response of the channel is:
- $h(t) = 2\pi B e^{-2\pi B t} u(t)$

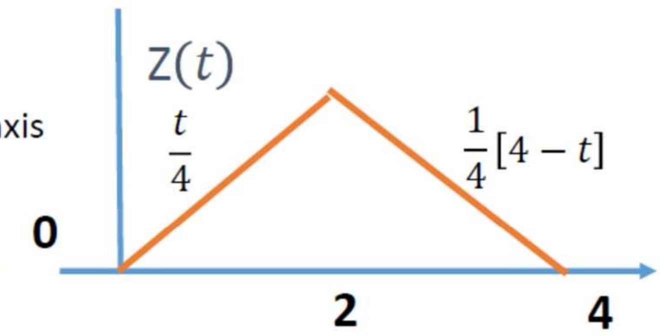
$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda$$

- The output is the convolution
- $y(t) = h(t) * [u(t) - u(t-1)]$. The answer is
- $y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda$
- $y(t) = 0 \; for \; t < 0$
- $y(t) = \int_{0}^{t} 2\pi B e^{-2\pi B\lambda}\, d\lambda = 1 - e^{-2\pi B t}, \;$ for $0 \le t < 1$
- $y(t) = \int_{-1+t}^{t} 2\pi B e^{-2\pi B\lambda}\, d\lambda = (e^{2\pi B} - 1)e^{-2\pi B t}$, for $t \ge 1$



**Example:** Find the convolution of the two signals x(t) and y(t) shown in the figure.




$$\mathbf{Z(t)} = \int_{-\infty}^{\infty} \boldsymbol{x}(\boldsymbol{\lambda})\, \mathbf{y}(\mathbf{t}-\boldsymbol{\lambda})\, \boldsymbol{d\lambda}$$

$$t \le 0, \quad \mathrm{Z(t)} = 0$$

$$0 \le t \le 2, \qquad \mathrm{Z(t)} = \int_0^t \frac{1}{2} \times \frac{1}{2}\ d\lambda = \frac{t}{4}$$

$$t = 2, \quad \mathrm{Z(t)} = \int_0^2 \frac{1}{2} \times \frac{1}{2}\ d\lambda = \frac{1}{2}$$

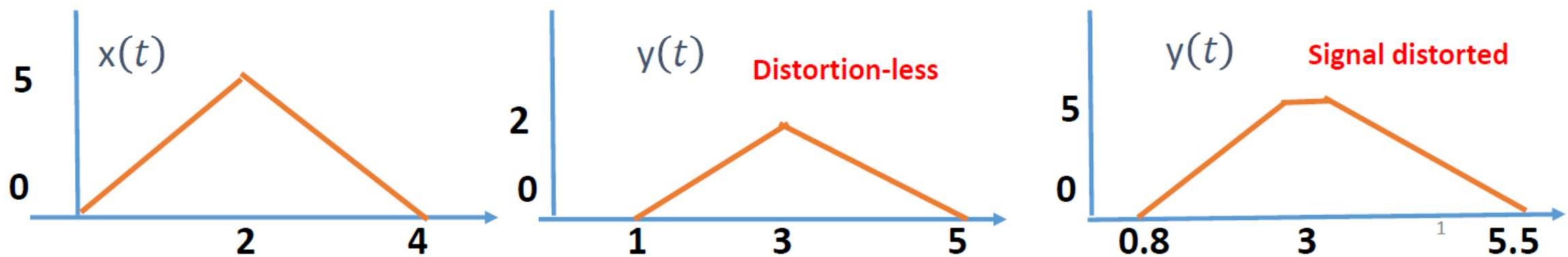
$$2 \le t \le 4, \quad \mathrm{Z(t)} = \int_{-2+t}^{2} \frac{1}{2} \times \frac{1}{2}\ d\lambda = \frac{\lambda}{4}\Big|_{-2+t}^{2} = \frac{1}{4}[4 - t]$$

$$t \ge 4, \quad \mathrm{Z(t)} = 0$$

# Signal Distortion in Transmission

- The objective of a communication system is to deliver to the receiver almost an exact copy of what the source generates.
- However, communication channels are not perfect in the sense that impairments on the channel will cause the received signal to differ from the transmitted one. During the course of transmission, the signal undergoes **attenuation**, **phase delay**, **interference** from other transmissions, **Doppler shift** in the carrier frequency, **AWGN**, and many other effects.
- **In this lecture, we consider the conditions for a distortion-less transmission over a channel. In addition, we consider linear and non-linear distortion**
- **Distortion-less Transmission**: A signal transmission is said to be *distortion-less* if the output signal y(t) is an exact replica of the input signal x(t) , i.e., y(t) has the same shape as the input, except for a constant amplification (or attenuation) and a constant time delay.



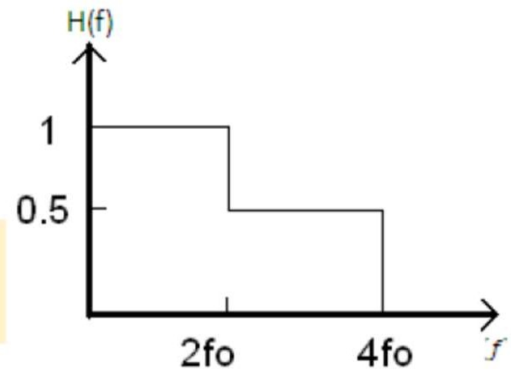
## Signal Distortion in Transmission

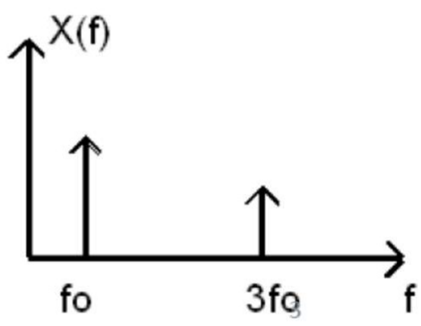
- Condition for **distortion-less transmission in the time-domain:**
- $\boldsymbol{y(t) = kx(t - t_d)}$; where k is a constant amplitude scaling, $t_d$ is a constant time delay.
- **In the frequency domain**, the condition for a distortion-less transmission becomes
- $Y(f) = kX(f)e^{-j2\pi f t_d}$ or $\boldsymbol{H(f) = \frac{Y(f)}{X(f)} = ke^{-j2\pi f t_d} = ke^{-j\theta(f)}}$
- That is, for a distortion-less transmission, the transfer function should satisfy two conditions:
- **|H(f)| = k** ; The magnitude of the transfer function is constant (gain or attenuation) over the frequency range of interest.
- $\boldsymbol{\theta(f) = -2\pi f t_d = -(2\pi t_d) f}$ ; The phase function is linear in frequency with a negative slope that passes through the origin (or multiples of π).
- When |H(f)| is not constant for all frequencies of interest, *amplitude distortion* results.
- When $\theta(f) \neq -2\pi f t_d \pm 180^0$, then we have *phase distortion* (or delay distortion).
- The following examples demonstrate the two types of distortion mentioned above.

# Example: amplitude distortion

- Consider the signal $x(t) = \cos w_0 t - \frac{1}{3} \cos 3w_0 t$. If this signal passes through a channel with zero time delay (i.e., $t_d = 0$) and amplitude spectrum as shown in the figure
- Find y(t)
- Is this a distortion-less transmission?
- **Solution:** x(t) consists of two frequency components, $f_0$ and $3f_0$ . Upon passing through the channel, each component will be scaled by a different factor.
- $y(t) = (\mathbf{1})\cos w_0 t - (\mathbf{\frac{1}{2}}) \cdot \frac{1}{3} \cos 3w_0 t$
- Since $y(t) = \left(\cos w_0 t - \frac{1}{2} \cdot \frac{1}{3} \cos 3w_0 t\right) \neq k\left(\cos w_0 t - \frac{1}{3} \cos 3w_0 t\right)$
- then this is not a distortion-less transmission.

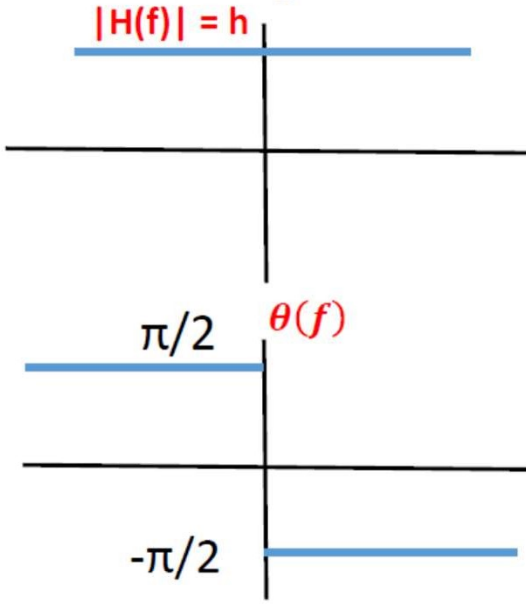
In this figure, only the positive part of the spectrum is shown



# Example: phase distortion

- Consider the signal $x(t) = \ cos\ w_0 t \ - \frac{1}{3}\ cos\ 3w_0 t$. If x(t) passes through a channel whose amplitude spectrum is a constant h. Each component in x(t) suffers a $-\frac{\pi}{2}$ phase shift.
- Find y(t).
- Is this a distortion-less transmission?
- **Solution:**
- $x(t) = \cos w_o t - \frac{1}{3} \cos 3w_o t$
- $y(t) = \text{h}\ \cos(w_o t - \frac{\pi}{2})\ -\ \frac{1}{3}\ h\ \cos\left(3w_o t - \frac{\pi}{2}\right)$
- $y(t) = \text{h}\ \cos w_o(t - \frac{\pi}{2w_o})\ -\ \frac{1}{3}\ h\ \cos\left(3w_o(t - \frac{\pi}{2x3w_o})\right)$
- $y(t) = \text{h}\ \cos w_o(t - t_{d1})\ -\ \frac{1}{3}\ h\ \cos(3w_o(t - t_{d2}))$
- Since $t_{d1} \neq t_{d2}$, we cannot write $y(t) = kx(t - t_d)$. Here, each component in $x(t)$ suffers from a different time delay. Hence, this transmission introduces phase (delay) distortion.



## Example: Amplitude and Phase Distortion

- **Example**: The signal $x(t) = \ cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t$ is applied to a filter described by the transfer function $H(f) = \frac{1}{1+jf/B}$. Use the result of the previous example to find the filter output $y(t)$.
- **Solution**: From the previous example, we have
- $\cos(2\pi f_0 t) \ \rightarrow \frac{1}{\sqrt{1+(\frac{f_0}{B})^2}} \cos(2\pi f_0 t - \tan^{-1}\frac{f_0}{B})$
- Therefore, using linearity property
- $cos\ w_0 t - \frac{1}{\pi} cos\ 3w_0 t \rightarrow$
- $\frac{1}{\sqrt{1+\left(\frac{f_0}{B}\right)^2}} \cos\left(2\pi f_0 t - \tan^{-1}\frac{f_0}{B}\right) \quad - \frac{1}{\pi}\frac{1}{\sqrt{1+\left(\frac{3f_0}{B}\right)^2}} \cos\left(2\pi 3 f_0 t - \tan^{-1}\frac{3f_0}{B}\right)$
- Note that we cannot write $y(t) = kx(t - t_d)$. Here, each component in $x(t)$ suffers from a different amplitude attenuation and a different time delay. Hence, this transmission introduces both amplitude and phase distortion.

# Nonlinear distortion

- When a system contains nonlinear elements, it is **not** described by a transfer function H(f), but rather by a transfer characteristic of the form
- $y(t) = a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \dots$ (time domain)
- In the frequency domain,
- $Y(f) = a_1 X(f) + a_2 X(f)*X(f) + a_3 X(f)*X(f)*X(f) + \dots$
- Here, the output contains new frequencies not originally present in the original signal. The nonlinearity produces undesirable frequency component for $|f| \le W$, in which W is the signal bandwidth.

# Harmonic distortion in nonlinear systems

- Let the input to a nonlinear system be the single tone signal $x(t) = \cos(2\pi f_0 t)$.
- This signal is applied to a channel with characteristic $y(t) = a_1x(t) + a_2x(t)^2 + a_3x(t)^3$ ;
- $y(t) = a_1 \cos(2\pi f_0 t) + a_2(\cos(2\pi f_0 t))^2 + a_3(\cos(2\pi f_0 t))^3$;
- upon substituting x(t) and arranging terms, we get
- $y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}\, a_2 \cos 4\pi f_0 t + \frac{1}{4}a_3 cos 6\pi\, f_0 t$
- Note that the output contains a component proportional to x(t), which is
- $\left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t$, in addition to a second and a third harmonic terms (terms at twice and three times the frequency of the input).
- These new terms are the result of the nonlinear characteristic and are, therefore, considered as harmonic distortion. The DC term does not constitute a distortion, for it can be removed using a blocking capacitor.
- Note: Use was made of the inequalities $Cos^2x = \frac{1}{2}\{1 + cos 2x\}$; $Cos^3x = \frac{1}{4}\{3 cos x + cos 3x\}$.

# Harmonic distortion in nonlinear systems

- Let the input to a nonlinear system be the single tone signal
- $y(t) = a_1x(t) + a_2x(t)^2 + a_3x(t)^3$; $x(t) = \cos(2\pi f_0 t)$;
- $y(t) = \frac{1}{2}a_2 + \left(a_1 + \frac{3}{4}a_3\right)\cos 2\pi f_0 t + \frac{1}{2}\, a_2 \cos 2(2\pi f_0 t) + \frac{1}{4}a_3 cos3(2\pi\, f_0 t)$

- Define the second harmonic distortion
- $D_2 = \frac{|amplitude\ of\ second\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$; $D_2 = \frac{|\frac{1}{2}a_2\ |}{|\left(a_1+\frac{3}{4}a_3\right)|}\ x\ 100$
- In a similar way, we can define the third harmonic distortion as:
- $D_3 = \frac{|amplitude\ of\ third\ harmonic\ |}{|amplitude\ of\ fundamental\ term|}$; $D_3 = \frac{|\frac{1}{4}a_3\ |}{|\left(a_1+\frac{3}{4}a_3\right)|}\ x\ 100\%.$

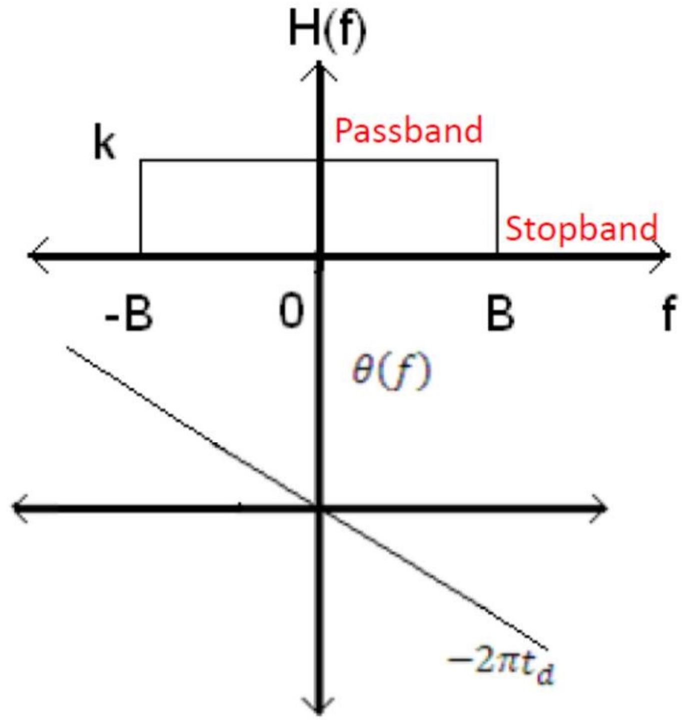
## Introduction to Filters

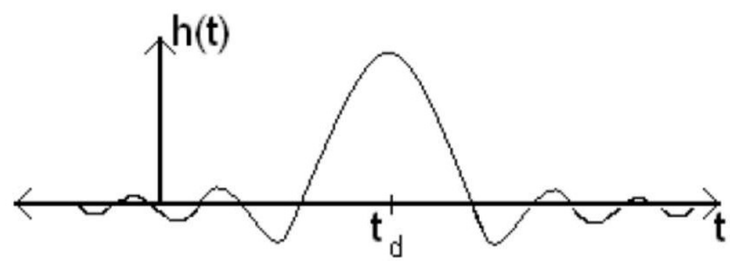
- A filter is a frequency selective device. It allows certain frequencies to pass almost without attenuation while it suppresses other frequencies.
- Filters are an integral part of any communication system

# Ideal Filters

### Ideal low pass filter

- $H(f) = \begin{cases} k\ e^{-j2\pi f t_d} & |f| < B \\ 0 & o.w \end{cases}$; B is the bandwidth
- The transfer function satisfies the condition for the distortion-less transmission (constant channel gain and linear phase shift with negative slope)
- $h(t) = 2Bk\ sinc2B(t - t_d)$
- Since h(t) is the response to an impulse applied at t=0, and because h(t) has nonzero values for t < 0 , the filter is *non-causal* (physically non realizable).



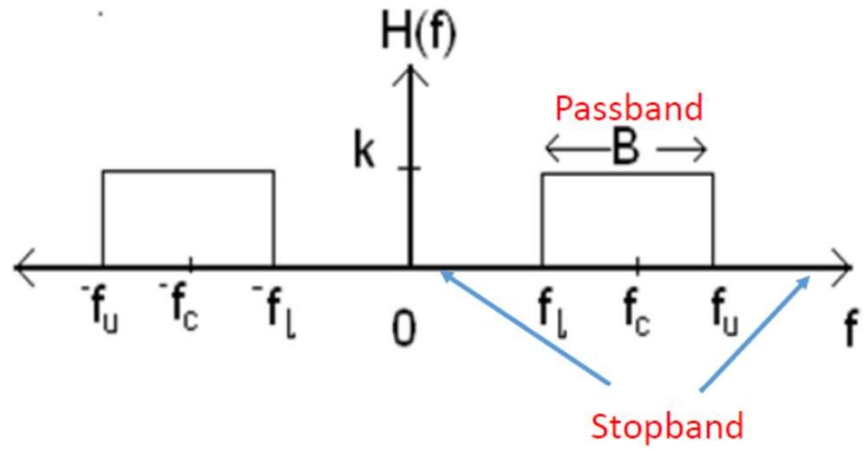
## Filters and Filtering

**Ideal band-pass filter**

- $H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & f_l < |f| < f_u \\ 0 & o.w \end{cases}$
- Filer bandwidth $B = f_u - f_l$; difference between upper and lower positive frequencies
- $f_c = \frac{f_u + f_l}{2}$; Center frequency of the filter

**impulse response**:

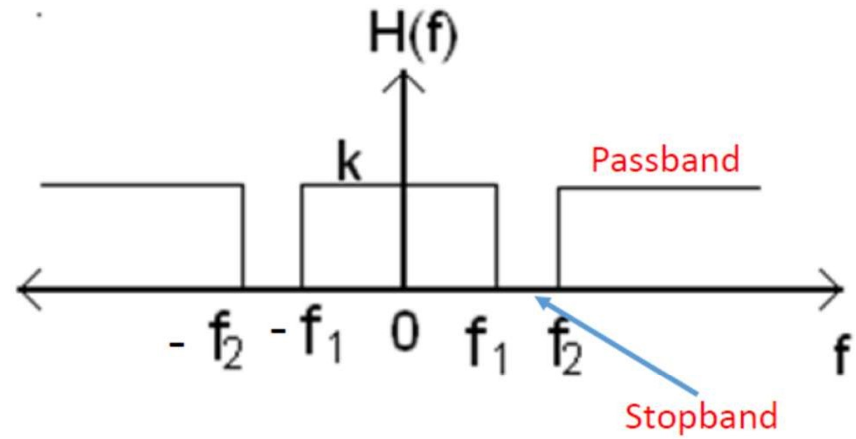
- $h(t) = 2Bk\ sincB(t - t_d)cos\, w_c(t - t_d)$



## Filters and Filtering

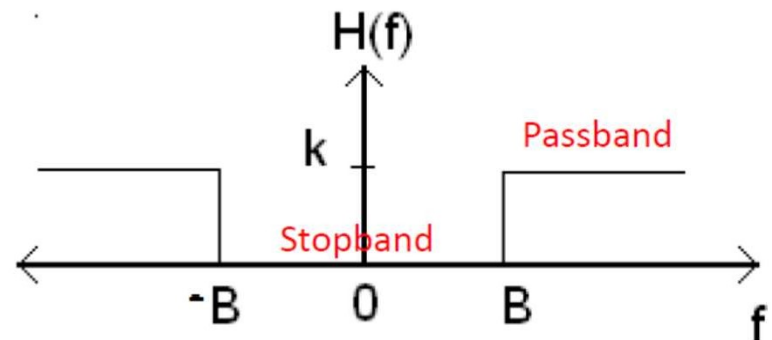
**Band rejection or notch filter**

- $H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & o.w \\ 0 & f_1 < |f| < f_2 \end{cases}.$



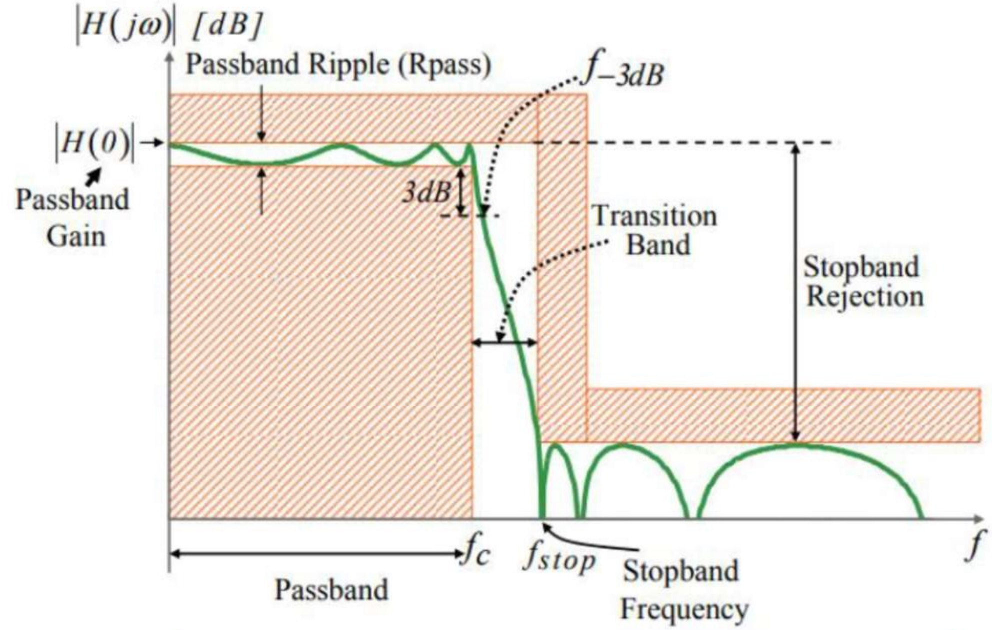

**High-pass filter**

- $H(f) = \begin{cases} k\, e^{-j2\pi f t_d} & |f| > B \\ 0 & o.w \end{cases}$



# Real filters

- Ideal filter do not exist in practice, but are used to simplify the analysis of the system
- For a real filter, there are three frequency bands
  - Passband
  - Transition band
  - Stopband (rejection)
- There are several specifications that dictate the filter order
  - The passband edge frequency and the maximum allowable attenuation (ripple) within the passband
  - The 3-dB cutoff frequency.
  - The minimum required attenuation at the edge of the stopband and the desired stopband frequency.




**Example: Let B be the 3-dB bandwidth**

- **- 1dB at f = 0.9 B,**
- **-30 dB at f = 1.6 B**
- **The designer then finds the order of the filter that meets these specifications and then realizes that filter.**

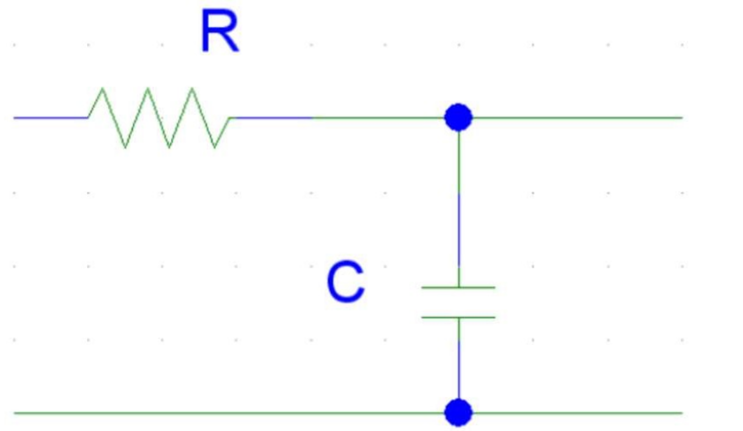
 

# Real filters

- Here, we consider Butterworth low pass filters. The transfer function of a low pass Butterworth filter is of the form:
- $H(f) = \frac{1}{P_n\left(\frac{jf}{B}\right)}$
- B is the 3-dB bandwidth of the filter and $P_n\left(\frac{jf}{B}\right)$ is a complex polynomial of order n. The family of Butterworth polynomials is defined by the property
- $\left(\left|P_n\left(\frac{jf}{B}\right)\right|\right)^2 = 1 + \left(\frac{f}{B}\right)^{2n}$ .
- Therefore, $|H(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^{2n}}}$
- The first few polynomials are:
- $P_1(x) = 1 + x;\; P_2(x) = 1 + \sqrt{2}x + x^2;\;\; P_3(x) = (1 + x)(1 + x + x^2)$

# Filters and Filtering

## A first order LPF

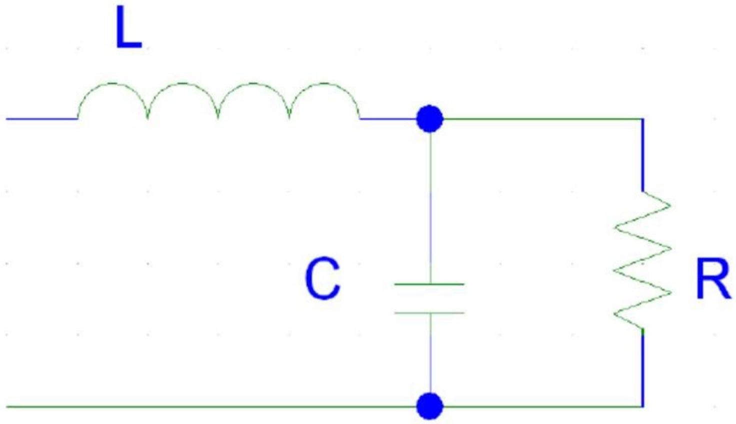



- $H(f) = \frac{1}{P_1\left(\frac{jf}{B}\right)} = \frac{1}{1+jf/B} = \frac{1}{P_1(jf/B)}$; $P_1(x) = 1 + x$
- $H(f) = \frac{\frac{1}{j2\pi f_C}}{R+\frac{1}{j2\pi f_C}}$
- Let $B = \frac{1}{2\pi RC}$ ; $H(f) = \frac{1}{1+jf/B}$
- Note: In this filter, there is only one energy storage element

Filters and Filtering

**A second order LPF**

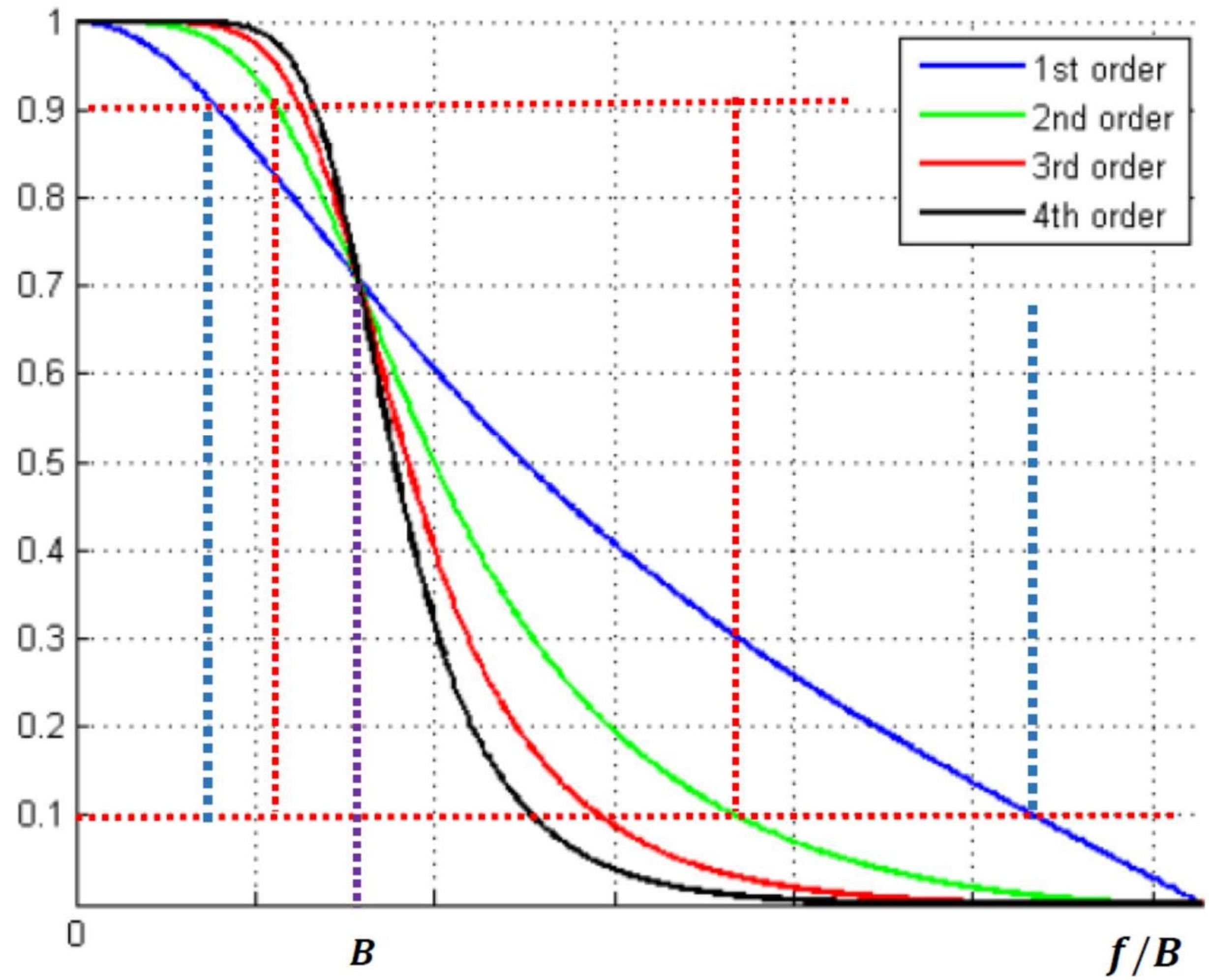



- $H(f) = \frac{1}{1+\frac{jwL}{R}-\left(2\pi\sqrt{LC}f\right)^2}$
- Let $R = \sqrt{\frac{L}{2C}}$ ; $\boldsymbol{B = \frac{1}{2\pi\sqrt{LC}}}$
- $H(f) = \frac{1}{1+\sqrt{2}(\frac{jf}{B})-(f/B)^2}$
- $H(f) = \frac{1}{P_2(jf/B)}$; $P_2(x) = 1 + \sqrt{2}x + x^2$
- Note: In this filter, there are two energy storage element

# Butterworth Low-pass Filters: Frequency Response and Filter Order

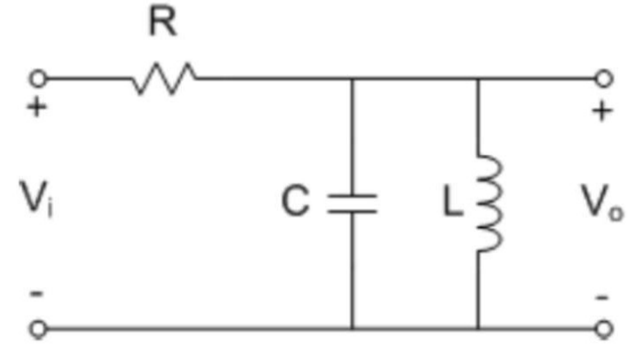
- B: is the 3-dB frequency at which the magnitude drops to 0.707 of the maximum value.
- Let the maximum allowable attenuation in the passband be 0.1 and the maximum gain within the stopband be 0.1.
- $H_1(f) = \frac{1}{1+jf/B}$
  - Passband:
  - Transition band:
- $H_2(f) = \frac{1}{1+j\sqrt{2}f/B-(f/B)^2}$
  - Passband:
  - Transition band:
- As the filter order increases, both of its pass-band and stop-band capabilities improve.



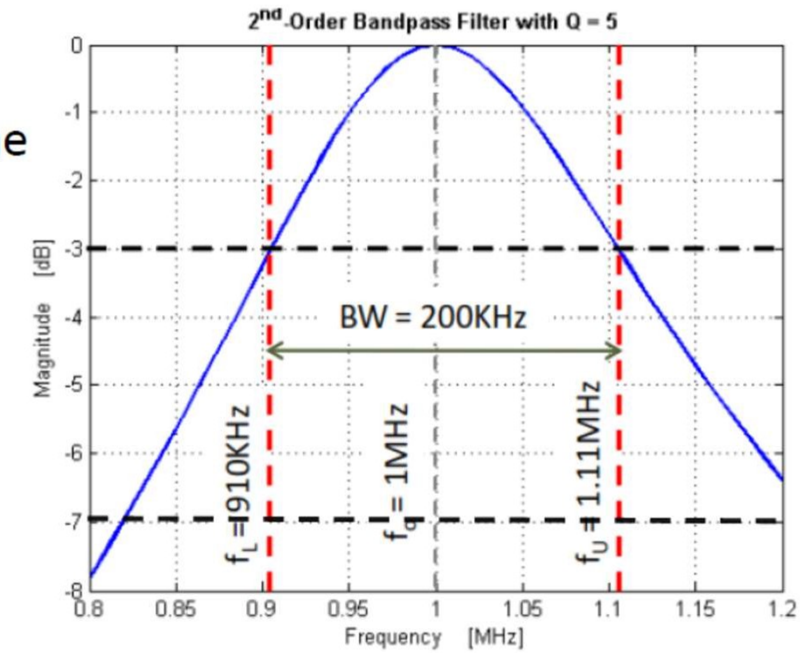
# A second order BPF

- The figure shows a band-pass filter. Its transfer function is

$$H(f) = \frac{\frac{j\omega}{RC}}{(j\omega)^2 + \frac{j\omega}{RC} + 1/LC} = \frac{\frac{\omega_0}{Q}(j\omega)}{(j\omega)^2 + \frac{\omega_0}{Q}(j\omega) + (\omega_0)^2}$$

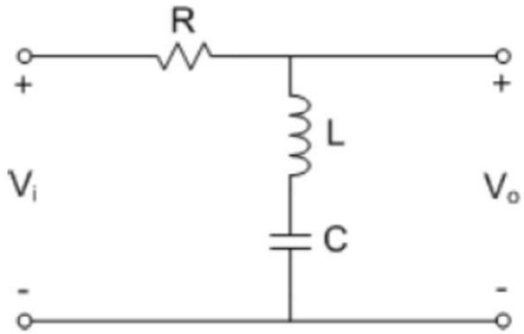



- $\omega_0 = 2\pi(\frac{1}{2\pi\sqrt{LC}})$ ; $f_0$: Resonance frequency
- $Q = \omega_0 RC = \frac{R}{\omega_0 C}$; Quality factor which determines the sharpness of the resonance.



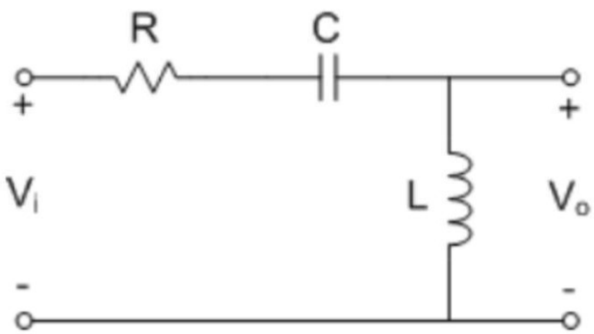

- Bandwidth is inversely proportional to Q
- $\mathbf{Q = \frac{f_0}{B.W}}$ ; ***Higher Q*** provides ***higher selectivity***
- For the shown characteristic, Q = 1MHz/200KHz = 5

# Other practical second order filters




$$H(\omega) = \frac{(j\omega)^2 + 1/LC}{(j\omega)^2 + j\omega R/L + 1/LC}$$

Second order band-stop filter




$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + j\omega R/L + 1/LC}$$

Second order high-pass filter

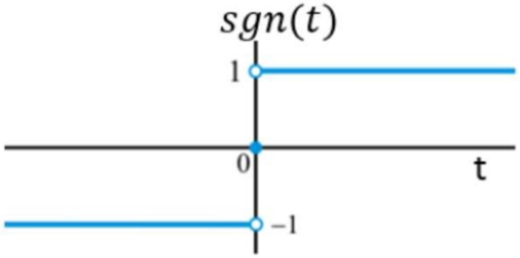
# Hilbert Transform

- **The quadrature filter** is an all pass filter that shifts the phase of positive frequency by (-90°) and negative frequency by (+90°).
- **The transfer function** of such a filter is
- $H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases} = -jsgn(f)$
- Note that $|H(f)| = 1$ for all f.
- Using the duality property of Fourier transform, the impulse response of the filter is $h(t) = \frac{1}{\pi t}$ $(\Im\{sgn(t)\} = \frac{1}{j\pi f})$
- The Hilbert transform is the output of the quadrature filter to the signal $g(t)$
- $\hat{g}(t) = \frac{1}{\pi t} * g(t)$ = $\int_{-\infty}^{\infty} \frac{g(\lambda)}{\pi(t-\lambda)} d\lambda$
- Note that the Hilbert transform of a signal is a function of time (not frequency as in the case of the Fourier transform). The Fourier transform of $\hat{g}(t)$
- $\hat{G}(f) = -jsgn(f)G(f)$
- Hilbert transform can be found using either the time domain approach or the frequency domain approach depending on the given problem. That is
- **Time-domain**: Perform the convolution

  $\frac{1}{\pi t} * g(t)$.
- **Frequency-domain**: Find the Fourier transform $\hat{G}(f)$, then find the inverse Fourier transform
- $\hat{g}(t) = \int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$

## Some properties of the Hilbert transform

- A signal $g(t)$ and its Hilbert transform $\hat{g}(t)$ have the same energy spectral density
- $\left|\hat{G}(f)\right|^2 = |-j\, sgn(f)|G(f)||^2 = |-j\, sgn(f)|^2|G(f)|^2$
- $= |G(f)|^2$



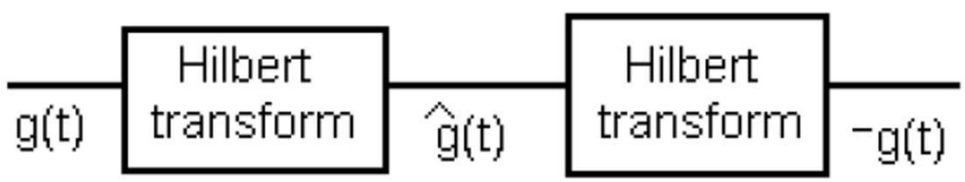

**The consequences of this property are:**

- If a signal g(t) is bandlimited to a bandwidth W Hz, then $\hat{g}(t)$ is bandlimited to the same bandwidth (note that $\left|\hat{G}(f)\right| = |G(f)|$)
- $\hat{g}(t)$ and $g(t)$ have the same total energy (or power). $E = \int_{-\infty}^{\infty} |G(f)|^2 df$
- $\hat{g}(t)$ and $g(t)$ have the same autocorrelation function (in the next lecture, we will see that the autocorrelation function and the energy spectral density form a Fourier transform pair $R_g(\tau) \leftrightarrow |G(f)|^2$)

## Some properties of the Hilbert transform

- A signal g(t) and $\hat{g}(t)$ are orthogonal, i.e., $\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t) dt = 0$
- This property can be verified using the general formula of Rayleigh energy theorem
- $\int_{-\infty}^{\infty} \mathrm{g(t)}\, \hat{g}(t) dt = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \hat{G}^{*}(f) df = \int_{-\infty}^{\infty} \mathrm{G(f)}\, \{-j sgn(f)\, G(f)\}^{*} df$

  $= \int_{-\infty}^{\infty} j sgn(f)\, |G(f)|^{2} df = 0.$
- The result above follows from the fact that $|G(f)|^2$ is an even function of $f$ while $sgn(f)$ is an odd function of $f$. Their product is odd. The integration of an odd function over a symmetrical interval is zero.
- If $\hat{g}(t)$ is a Hilbert transform of $g(t)$, then the Hilbert transform of $\hat{g}(t)$ is $-g(t)$ (each Hilbert transform introduces 90 degrees phase shift).



## Example on Hilbert transform

**Example**: Find the Hilbert transform of the impulse function $g(t) = \delta(t)$

**Solution**: Here, we use the convolution in the time domain

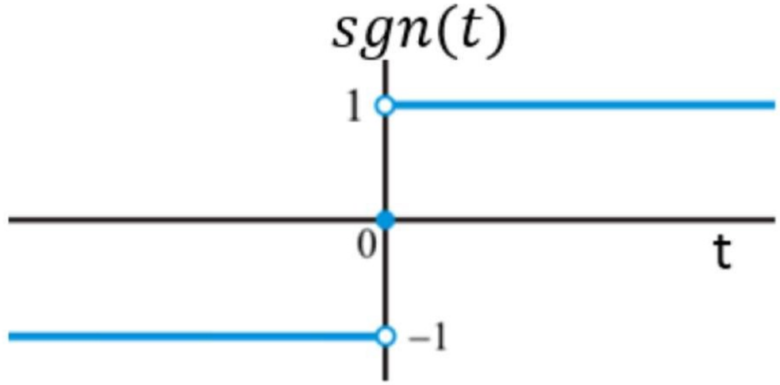
- $$\hat{g}(t) = \frac{1}{\pi t} * \delta(t)$$
- As we know, the convolution of the delta function with a continuous function is the function itself. Therefore,
- $$\hat{g}(t) = \frac{1}{\pi t}.$$

# Example on Hilbert transform

**Example**: Find the Hilbert transform of $g(t) = \cos(2\pi f_0 t)$

**Solution**: Here, we use the frequency domain approach

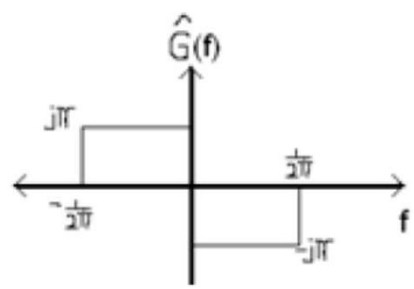
- $\hat{G}(f) = -jsgn(f)G(f) = -\frac{jsgn(f)\{\delta(f-f_0)+\delta(f+f_0)\}}{2}$
- $\hat{G}(f) = -jsgn(f)G(f) = \frac{sgn(f)\{\delta(f-f_0)+\delta(f+f_0)\}}{j2} = \frac{\{\delta(f-f_0)-\delta(f+f_0)\}}{j2}$
- $\hat{g}(t) = \sin(2\pi f_0 t)$

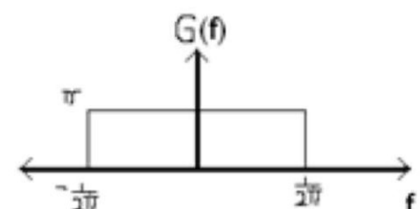


# Example on Hilbert transform

- Find the Hilbert transform of $g(t) = \frac{\sin t}{t}$
- **Solution:** Here, we will first find the Fourier transform of $g(t)$, find $\hat{G}(f)$, and then find $\hat{g}(t)$:
- $A\, rect\left(\frac{t}{\tau}\right) \leftrightarrow A\tau\, sinc\, f\tau\ ;\ \tau = \frac{1}{\pi}$
- $A\,\mathrm{rect}\left(\frac{t}{1/\pi}\right) \leftrightarrow A\ \frac{1}{\pi}\ \frac{\sin \pi f \tau}{\pi f \tau} = \frac{1}{\pi}\frac{\sin f}{f}$
- $\pi\, rect\left(\frac{t}{1/\pi}\right) \leftrightarrow \frac{\sin f}{f}$
- So, by the duality property, we get the pair
- $\pi\, rect\left(\frac{f}{1/\pi}\right) \leftrightarrow \frac{\sin t}{t}$
- i.e. $G(f) = \pi\, rect(\frac{f}{1/\pi})$ , (See figure next)

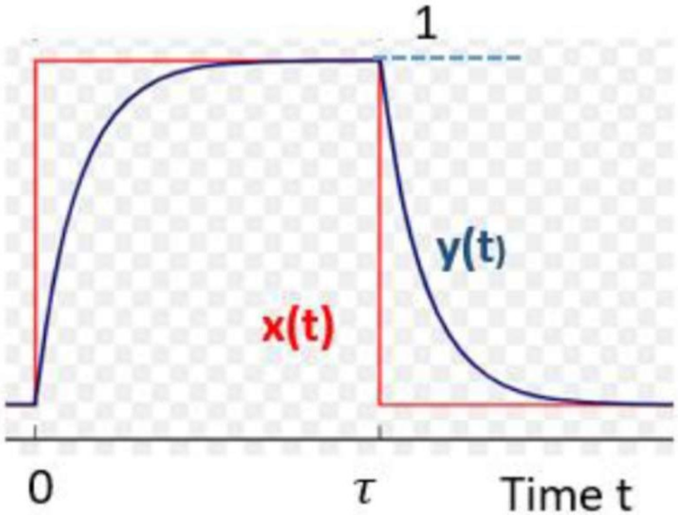
- $\hat{G}(f) = -jsgn(f)G(f) = \begin{cases} -j\pi & 0 < f < 1/2\pi \\ j\pi & -1/2\pi < f < 0 \end{cases}$
- $\hat{g}(t)$ = $\int_{-\infty}^{\infty} \hat{G}(f)\, e^{j2\pi f t}\, df$
- = $\int_{-1/2\pi}^{0} j\pi\ \ e^{j2\pi f t}\, df\ -$ $\int_{0}^{1/2\pi} j\pi\ \ e^{j2\pi f t}\, df$
- = $\frac{1}{2t}\left(1 - e^{-jt}\right) - \frac{1}{2t}\left(e^{jt} - 1\right)$
- = $\frac{1}{t} - \frac{1}{t}\,\frac{\left(e^{jt}+e^{-jt}\right)}{2}$ = $\frac{1-\cos t}{t}$



## Pulse Response of a First Order System

- It is the response of the circuit to a pulse of duration τ. For the same RC circuit, considered above, let us apply the pulse
- $x(t) = u(t) - u(t - \tau)$
- Using the linearity and time invariance properties, the output due to the pulse can be obtained from the step response g(t) as



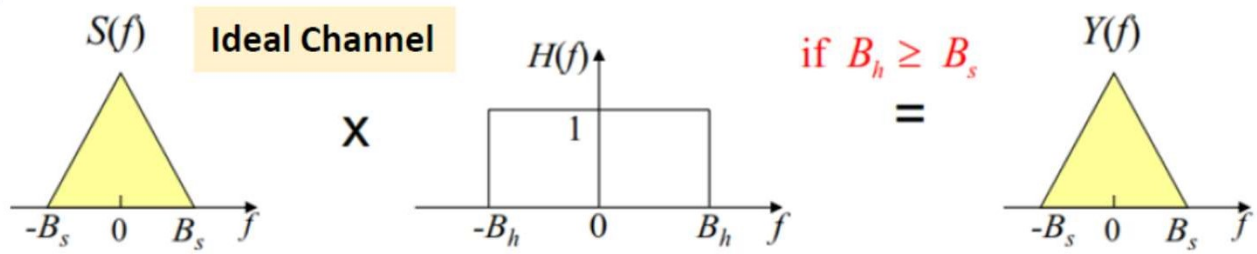

- $y(t) = g(t) - g(t - \tau)$
- $y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-2\pi B_{ch} t} & 0 < t < \tau \\ (1 - e^{-2\pi B_{ch}\tau})e^{-2\pi B_{ch}(t-\tau)} & t > \tau \end{cases}$
- This response is sketched in the figure below.
- From the equation above, we observe that the output $y(t)$ approximates the input $x(t)$ provided that ( $y(\tau) > 0.99$)

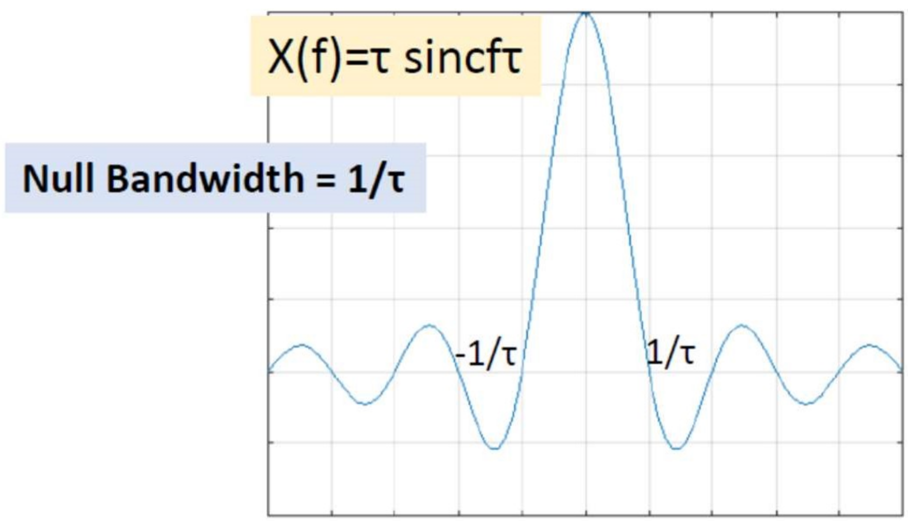
$$\boldsymbol{B_{ch}\tau \geq 1} \quad \textbf{or} \quad \boldsymbol{B_{ch} \geq \frac{1}{\tau}}$$

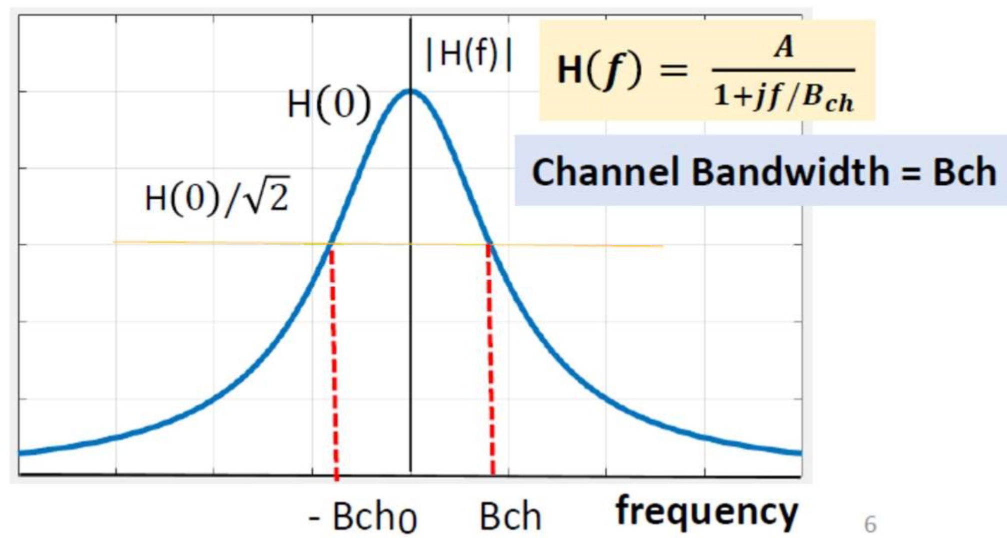
# Pulse Response of a First Order System

- The figure below shows the Fourier transform of the input and the channel.
- To reproduce the input, the channel bandwidth should be wider than the message bandwidth
- Y (f) = X(f) H(f)
- $Y(f) \simeq X(f)$

- If the channel bandwidth is much wider than the message bandwidth, then



## Relationship to data transmission

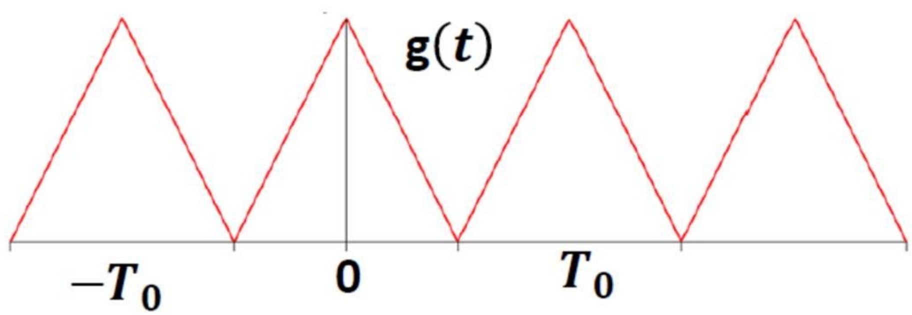
- In digital communication systems, data are transmitted at a rate of $R_b$ bits/sec. The time allocated for each bit is $\tau = \frac{1}{R_b}$. To enable the receiver to recognize the transmitted bit within its allocated slot and to prevent cross talk between neighboring time slots, we require that

$$\boldsymbol{B_{ch} \geq \frac{1}{\tau} = R_b}$$

- **Result**: the channel bandwidth in binary digital communication systems should be larger than the rate of the data sent over the channel.

# Autocorrelation and Spectral Density

- In this lecture, we define the autocorrelation function of a signal. Also, we present the relationship between the autocorrelation function and the power/energy spectral density.
- In this discussion, we restrict our attention to real signals. First, we consider power signals and then energy signals.
- **Definition**: The autocorrelation function of a signal $g(t)$ is a measure of similarity between $g(t)$ and a delayed version of $g(t)$.



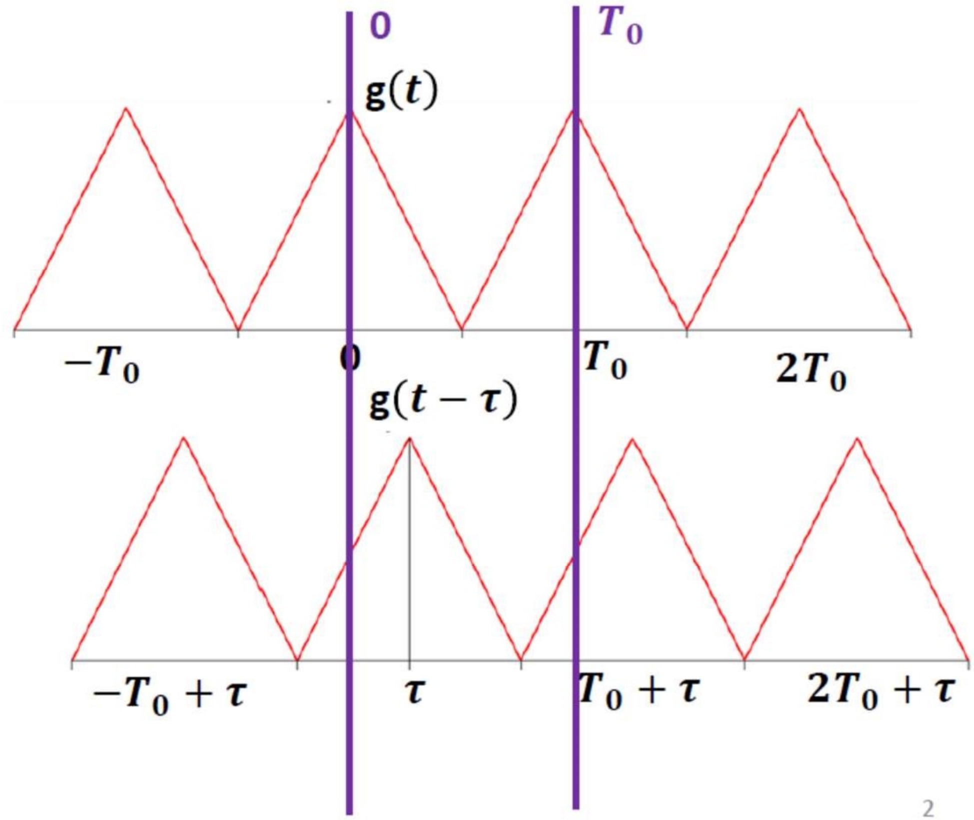
## Correlation and Spectral Density

- **Autocorrelation function of a periodic power signal**
- The autocorrelation function of a periodic power signal $g(t)$ with period $T_0$ is

$$R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$$

**Properties of $R_g(\tau)$**

- $R_g(\tau = 0) = \frac{1}{T_0}\int_0^{T_0} g(t)^2 dt;$ is the total average signal power.
- $R_g(\tau)$ is an even function of $\tau$, i.e., $R_g(\tau) = R_g(-\tau)$.



## Correlation and Spectral Density

**Properties of $R_g(\tau)$:** $R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$

- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \leq R_g(0)$.

**Proof of this property:**

Consider the quadratic quantity

$$[\, g(t) \pm g(t+\tau)]^2 \geq 0$$

Taking the time average ( $< \mathrm{y(t)} > \frac{1}{T_0}\int_0^{T_0} y(t)dt$ ) of both sides, and expanding, we get

$$< \{[g(t) \pm g(t+\tau)]^2\} > \; \geq 0$$

$$< \{\, g(t)^2\} > \; + < \{\, g(t+\tau)^2\} > \pm\, 2 < \{\, g(t)g(t+\tau)\} > \geq 0$$

But, $< \{\, g(t)^2\} > \; = R_g(0)$ and $R_g(0) = < \{\, g(t+\tau)^2\} >$ as well.

Combining these results, we get: $\boldsymbol{-R_g(0) < \; R_g(\tau) < R_g(0).}$

## Correlation and Spectral Density

- **Theorem**: The autocorrelation function $R_g(\tau)$ of a periodic signal g$(t)$ is also periodic with the same period $T_0$.
- **Proof:** $R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$
  - Expand $g(t)$ in a complex Fourier series $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$.
  - Form the delayed signal $g(t-\tau) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0(t-\tau)}$
  - Perform the integration over a complete period $T_0$ , making use of orthogonality. The result is:
- $\boldsymbol{R_g(\tau) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0\tau} = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0\tau}}$ ; Fourier series expansion of $R_g(\tau)$.
- $D_n = |C_n|^2$ Fourier coefficients of $R_g(\tau)$; $C_n$ Fourier coefficients of $g(t)$.
- Note that the real Fourier coefficients $D_n$ of $R_g(\tau)$ are related to the complex Fourier coefficients $C_n$ of $g(t)$ by the relation $D_n = |C_n|^2$.
- The Fourier transform of the autocorrelation function is
- $\boldsymbol{S_g(f) = \Im\{R_g(\tau)\} = \sum_{n=-\infty}^{\infty}|C_n|^2\delta(f-nf_0)}$ ; Discrete spectrum
- This is, of course, the power spectral density of $g(t)$ , which we considered earlier.

## Autocorrelation of a periodic sinusoidal signal

- **Example**: Find the auto-correlation function and power spectral density of the sine signal $g(t) = A\cos(2\pi f_0 t + \theta)$, where $A$ and $\theta$ are constants.
- **Solution**: As we know, $g(t)$ is a periodic signal. Therefore, we find $R_g(\tau)$ using the definition
- $R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} g(t)g(t-\tau)dt$
- $R_g(\tau) = \frac{1}{T_0}\int_0^{T_0} A\cos(2\pi f_0 t + \theta)A\cos(2\pi f_0 t - 2\pi f_0 \tau + \theta)dt$
- $R_g(\tau) = \frac{A^2}{2T_0}\int_0^{T_0}[\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\theta) + \cos(2\pi f_0 \tau)]dt$
- $R_g(\tau) = \frac{A^2}{2T_0}[0 + \cos(2\pi f_0 \tau)T_0]$
- $R_g(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$; Periodic with period $T_0$.
- $S_g(f) = \frac{A^2}{4}\{\delta(f - f_0) + \delta(f + f_0)\}$; power spectral density

# Correlation and Spectral Density

**Autocorrelation function of energy signals**

- When g(t) is an energy signal, $R_g(\tau)$ is defined as

$$\boldsymbol{R_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt}$$

**Properties of R(τ)**

- $R_g(\tau = 0) = \int_{-\infty}^{\infty} g(t)^2 dt;$ is the total signal energy.
- $R_g(\tau)$ is an even function of $\tau$, i.e., $R_g(\tau) = R_g(-\tau)$.
- $R_g(\tau)$ has a maximum (positive) magnitude at τ = 0 , i.e. $|R_g(\tau)| \leq R_g(0)$.

# Correlation and Spectral Density

**Theorem**: The autocorrelation function of an energy signal and its energy spectral density (a continuous function of frequency) are **Fourier transform pairs**, i.e.,

- $S_g(f) = \Im\{R_g(\tau)\} = \int_{-\infty}^{\infty} R_g(\tau) e^{-j2\pi f\tau} d\tau;$
- $R_g(\tau) = \int_{-\infty}^{\infty} S_g(f) e^{j2\pi f\tau} df.$

**Proof:** The autocorrelation function is defined as:

- $R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda) g(\lambda - \tau) d\lambda$
- In this integral we have replaced t by $\lambda$ (both are dummy variables of integration). With this substitution, we can rewrite the integral as
- $R_g(\tau) = \int_{-\infty}^{\infty} g(\lambda) g(-(\tau - \lambda)) d\lambda$
- One can realize that $R_g(\tau)$ is nothing but the convolution of $g(\tau)$ and $g(-\tau)$. That is,
- $R_g(\tau) = g(\tau) * g(-\tau)$
- Taking the Fourier transform of both sides, we get
- $F\{R_g(\tau)\} = G(f)G^*(f)$, Therefore $\boldsymbol{S_g(f) = \Im\{R_g(\tau)\} = |G(f)|^2}$.

## Example: Autocorrelation of a non-periodic signal

- **Example**: Determine the autocorrelation function of the sinc pulse

  $g(t) = A sinc 2Wt.$

- **Solution:** Using the duality property of the Fourier transform, we deduce that
- $$G(f) = \frac{A}{2W} rect(\frac{f}{2W})$$
- The energy spectral density of $g(t)$ is
- $$S_g(f) = |G(f)|^2 = (\frac{A}{2W})^2 rect(\frac{f}{2W})$$
- Taking the inverse Fourier transform, we get the autocorrelation function
- $$R_g(\tau) = \frac{A^2}{2W} sinc 2W\tau$$

$$rect\left(\frac{t}{T}\right) \leftrightarrow T sinf T$$

$$sinc 2Wt \leftrightarrow \frac{1}{2W} rect(\frac{f}{2W})$$

## Autocorrelation function of the rectangular pulse

- **Example:** Find the autocorrelation function of the pulse $g(t) = rect\left(\frac{t-0.5T}{T}\right),\ T = 1.$
- **Solution**: As we saw earlier, this pulse is an energy signal and therefore, we can find its $R_g(\tau)$ as: $R_g(\tau) = \int_\tau^1 (A)(A)dt = A^2(1-\tau)$ ; $0 < \tau < 1$
- Using the even symmetry property of the autocorrelation function, we can find $R_g(\tau)$ for -ve values of τ as:
- $R_g(\tau) = A^2(1+\tau)$ ; $-1 < \tau < 0$
- This function is sketched below. Note that that the maximum value occurs at τ = 0 and that g(t) and g(t-τ) become uncorrelated for τ = 1 sec, which is the duration of the pulse.
- The energy spectral density is $S_g(\text{f}) = \Im\{R_g(\tau)\} = A^2\left((sincf)^2\right)$